On spanning connected graphs

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Abstract

A \textit{k}-\textit{container} \(C(u, v)\) of \(G\) between \(u\) and \(v\) is a set of \(k\) internally disjoint paths between \(u\) and \(v\). A \textit{k}-\textit{container} \(C(u, v)\) of \(G\) is a \(k\)-\textit{container} if the set of the vertices of all the paths in \(C(u, v)\) contains all the vertices of \(G\). A graph \(G\) is \(k\)-\textit{connected} if there exists a \(k\)-container between any two distinct vertices. Therefore, a graph is \(2\)-\textit{connected} (respectively, \(2\)-\textit{connected}) if and only if it is Hamiltonian connected (respectively, Hamiltonian). In this paper, a classical theorem of Ore, providing sufficient conditional for a graph to be Hamiltonian (respectively, Hamiltonian connected), is generalized to \(k\)-\textit{connected} graphs.

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1. Introduction and definitions

For the graph definition and notation we follow [3]. \(G = (V, E)\) is a graph if \(V\) is a finite set and \(E\) is a subset of \(\{(u, v) \mid \{u, v\}\ \text{is an unordered pair of } V\}\). We say that \(V\) is the vertex set and \(E\) is the edge set. We use \(n(G)\) to denote \(|V|\). A graph \(H\) is called a subgraph of \(G\) if \(V(H) \subseteq V(G)\) and \(E(H) \subseteq E(G)\). The induced subgraph \(G[H]\) is a subgraph of \(G\) where \(V(G[H]) = V(H)\) and \(E(G[H]) = \{(u, v) \mid (u, v)\ \in E(G)\text{ and }u, v \in V(H)\}\). Two vertices \(u\) and \(v\) are adjacent if \((u, v)\) is an edge of \(G\). Let \(v\) be a vertex of \(G\) and \(H\) be a subgraph of \(G\). The neighborhood of \(u\) respective to \(H\), denoted by \(N_H(u)\), is \(\{v \in V(H) \mid (u, v) \in E(G)\}\). The degree \(d_H(u)\) of a vertex \(u\) respective to \(H\) is the number of edges between \(u\) and \(V(H)\). The minimum degree of \(G\), written \(\delta(G)\), is \(\min\{d_G(x) \mid x \in V\}\). A path is a sequence of vertices represented by \(\langle v_0, v_1, \ldots, v_k \rangle\) with no repeated vertex, and \((v_i, v_{i+1})\) is an edge of \(G\) for all \(0 \leq i \leq k - 1\). We also write the path \(\langle v_0, v_1, \ldots, v_k \rangle\) as \(\langle v_0, v_1, Q, v_j, \ldots, v_k \rangle\), where \(Q\) is a path form \(v_i\) to \(v_j\). A path is a Hamiltonian path if it contains all the vertices of \(G\). A graph \(G\) is Hamiltonian connected if, for any two distinct vertices of \(G\), there exists a Hamiltonian path joining those two vertices. A cycle is a path with at least three vertices such that the first vertex is the same as the last one. A Hamiltonian cycle of \(G\) is a cycle that traverses every vertex of \(G\). A graph is Hamiltonian if it has a Hamiltonian cycle. We use \(G \cup H\) to denote the disjoint union of graph \(G\) and graph \(H\). Moreover, we use \(G \lor H\) to denote the graph obtained from \(G \cup H\) by joining all the edges with one vertex in \(G\) and the other vertex in \(H\). Let \(u\) and \(v\) be two nonadjacent vertices of \(G\), we use \(G + uv\) to denote the graph obtained from \(G\) by adding the edge \((u, v)\).
A k-container $C(u, v)$ of $G$ between $u$ and $v$ is a set of $k$ internally disjoint paths between $u$ and $v$. In other words, $C(u, v)$ consists of paths $P_1, P_2, \ldots, P_k$ such that $E(P_i) \cap E(P_j) = \emptyset$ and $V(P_i) \cap V(P_j) = \{u, v\}$ for $1 \leq i \neq j \leq k$. The concept of container is proposed by Hsu [5] to evaluate the performance of communication of an interconnection network. The connectivity of $G$, $\kappa(G)$, is the minimum number of vertices whose removal leaves the remaining graph disconnected or trivial. It follows from Menger’s Theorem [7] that there is a $k$-container between any two distinct vertices of $G$ if and only if $G$ is $k$-connected.

In this paper, we are interested in a special type of container. A $k$-container $C(u, v)$ of $G$ is a $k^*$-container if the set of the vertices of all the paths in $C(u, v)$ contains all the vertices of $G$. A graph $G$ is $k^*$-connected if there exists a $k^*$-container between any two distinct vertices. A $1^*$-connected graph except $K_1$ and $K_2$ is $2^*$-connected. A $1^*$-connected graph is actually a hamiltonian connected graph. Moreover, a $2^*$-connected graph is a hamiltonian graph. Thus, the concept of $k^*$-connected graph is a hybrid concept of connectivity and hamiltonicity. The study of $k^*$-connected graph is motivated by the globally $3^*$-connected graphs proposed by Albert et al. [1]. A globally $3^*$-connected graph is a cubic graph that is $w^*$-connected for all $1 \leq w \leq 3$. Recently, Lin et al. [6] proved that the pancake graph $P_n$ is $w^*$-connected for any $w$ with $1 \leq w \leq n - 1$ and if only if $n \neq 3$. Thus, we defined the spanning connectivity $\kappa^*(G)$ of a graph $G$ to be the largest integer $k$ such that $G$ is $w^*$-connected for all $1 \leq w \leq k$ if $G$ is $1^*$-connected graph and undefined otherwise.

A graph $G$ is super spanning connected if $\kappa^*(G) = \kappa(G)$. The complete graph $K_n$ is super spanning connected, and the pancake graph $P_n$ is super spanning connected if and only if $n \neq 3$.

Let $k$ be a positive integer. In this paper, we have the following results. If there exist two nonadjacent vertices $u$ and $v$ with $d_G(u) + d_G(v) \geq n(G) + k$ then $G$ is $(k+2)^*$-connected if and only if $G + uv$ is $(k+2)^*$-connected. Moreover, if there exist two nonadjacent vertices $u$ and $v$ with $d_G(u) + d_G(v) \geq n(G) + k$, then $G$ is $i^*$-connected if and only if $G + uv$ is $i^*$-connected for $1 \leq i \leq k + 2$. Assume that $d_G(u) + d_G(v) \geq n + k$ for all nonadjacent vertices $u$ and $v$, then $G$ is $r^*$-connected for every $r \in \{1, 2, \ldots, k + 2\}$.

2. Sufficient condition for spanning connected graphs

Ore [8,9], and Bondy and Chvátal [2] proved the following theorem:

**Theorem 1** (Bondy and Chvátal [2], Ore [8,9]). Assume that there exist two nonadjacent vertices $u$ and $v$ with $d_G(u) + d_G(v) \geq n(G) + 1$ then $G$ is $1^*$-connected if and only if $G + uv$ is $1^*$-connected.

**Lemma 1.** Let $k$ be a positive integer. Suppose that there exist two nonadjacent vertices $u$ and $v$ with $d_G(u) + d_G(v) \geq n(G) + k$. Then, for any two distinct vertices $x$ and $y$, $G$ has a $(k+2)^*$-container between $x$ and $y$ if and only if $G + uv$ has a $(k+2)^*$-container between $x$ and $y$.

**Proof.** If $G$ has a $(k+2)^*$-container between $x$ and $y$, then clearly $G + uv$ has a $(k+2)^*$-container between $x$ and $y$. For the other direction, let $C(x, y) = \{P_1, P_2, \ldots, P_k\}$ be a $(k+2)^*$-container of $G + uv$ between $x$ and $y$. Suppose that the edge $(u, v) \notin C(x, y)$. Then $C(x, y)$ forms a desired $(k+2)^*$-container of $G$. Thus, we suppose that $(u, v) \in E$. We write $P_i$ as $(x, H_i, u, v, H_i^{-1}, y)$ and write $P_i$ as $(x, P_i', y)$ for $2 \leq i \leq k + 2$. (Note that $l(H_1) = 0$ if $x = u$, and $l(H_2) = 0$ if $y = v$.) We set $C_i = \langle x, P_i', y, H_2^{-1}, v, u, H_1^{-1}, x \rangle$ for $2 \leq i \leq k + 2$.

**Case 1:** $d_G[C_1](u) + d_G[C_1](v) \geq n(C_1)$ for some $2 \leq i \leq k + 2$. Without loss of generality, we may assume that $d_G[C_2](u) + d_G[C_2](v) \geq n(C_2)$. By Theorem 1, there is a hamiltonian cycle $C$ of the induced subgraph $G[C_2]$. Let $C = (x, R_1, y, R_2, x)$. We set $Q_i = (x, R_1, y), Q_2 = (x, R_2^{-1}, y)$, and $Q_i = P_i$ for $3 \leq i \leq k + 2$. Then $\{Q_1, Q_2, \ldots, Q_{k+2}\}$ forms a $(k+2)^*$-container of $G$ between $x$ and $y$.

**Case 2:** $d_G[C_1](u) + d_G[C_1](v) \leq n(C_1) - 1$ for all $2 \leq i \leq k + 2$. Since

$$
\sum_{i=2}^{k+2} (d_G[C_1](u) + d_G[C_1](v)) = \sum_{i=2}^{k+2} (d_G[P_i'](u) + d_G[P_i](u) + d_G[P_i](v) + d_G[P_i](v))
$$

$$
= \sum_{i=2}^{k+2} (d_G[P_i'](u) + d_G[P_i'](v)) + (k+1)(d_G[P_i](u) + d_G[P_i](v))
$$

This concludes the proof.
Theorem 2. Assume that $k$ is any positive integer and there exist two nonadjacent vertices $u$ and $v$ with $d_G(u) + d_G(v) \geq n(G) + k$. Then $G$ is $(k + 2)^*$-connected if and only if $G + u v$ is $(k + 2)^*$-connected. Moreover, $G$ is $i^*$-connected if and only if $G + u v$ is $i^*$-connected for $1 \leq i \leq k + 2$. 

\[ n(G) + k + k(d_{G[P_1]}(u) + d_{G[P_1]}(v)) \leq n(G) + k(n(P_1)) - (k + 1) \]
and
\[ n(G) + k + d_{G[P_1]}(u) + d_{G[P_1]}(v) \leq n(G) + k(n(P_1)) - (k + 1) \]

Without loss of generality, we may assume that $Q_{i+2} \subseteq \{Q_1, Q_2, \ldots, Q_{k+1} \}$ forms a $k^*$-container of $G$ between $x$ and $y$. See Fig. 1 for an illustration. \hfill $\Box$
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We have

Theorem 3 (Ore [9]). Assume that \( d_G(u) + d_G(v) \geq n(G) + 1 \) for all nonadjacent vertices \( u \) and \( v \) of \( G \). Then \( G \) is 1*-connected.

Theorem 4. Let \( k \) be a positive integer. Assume that \( d_G(u) + d_G(v) \geq n(G) + k \) for all nonadjacent vertices \( u \) and \( v \) of \( G \), then \( G \) is \( r^* \)-connected for every \( 1 \leq r \leq k + 2 \).

Proof. By Theorem 3, \( G \) is 1*-connected and 2*-connected. Let \( x \) and \( y \) be two distinct vertices in \( G \). Suppose there exists an \( r^* \)-container \( \{P_1, P_2, \ldots, P_r\} \) of \( G \) between \( x \) and \( y \) for some \( 2 \leq r \leq k + 1 \). We only need to construct an \((r+1)^*\)-container of \( G \) between \( x \) and \( y \). We have \( d_G(y) \geq k + 2 \), for otherwise let \( w \notin N_G(y) \) then \( d_G(y) + d_G(w) \leq (k + 1) + (n - 2) = n + k - 1 \), which is a contradiction. We can choose a vertex \( u \) in \( N_G(y) \) such that \( (u, y) \notin E(P_r) \) for all \( 1 \leq i \leq r \). Without loss of generality, assume that \( u \in P_r \) and we write \( P_r \) as \( \langle x, H_1, u, v, H_2, y \rangle \). We set \( Q_i = P_i \) for all \( 1 \leq i \leq r - 1 \), \( Q_r = \langle x, H_1, u, y \rangle \), and \( Q_{r+1} = \langle x, v, H_2, y \rangle \). Suppose that \( (x, v) \in E(G) \). Then \( \{Q_1, Q_2, \ldots, Q_{r+1}\} \) forms an \((r+1)^*\)-container of \( G \) between \( x \) and \( y \). Suppose that \( (x, v) \notin E(G) \). Then, \( \{Q_1, Q_2, \ldots, Q_{r+1}\} \) forms an \((r+1)^*\)-container of \( G + xu \) between \( x \) and \( y \). By Lemma 1, there exists an \((r+1)^*\)-container of \( G \) between \( x \) and \( y \). □

We give an example to show that the above result may not hold for \( r = k + 3 \). Therefore, our result is optimal. Let \( K_n \) be a complete graph with \( n \) vertices. We set \( G = (K_1 \cup K_b) \cup K_a \) where \( a \geq 3 \) and \( b \geq 2 \). Obviously, \( \delta(G) = a \) and \( d_G(u) + d_G(v) \geq 2a + b - 1 \) for any two distinct vertices \( u \) and \( v \). Thus, \( G \) is not \( r^* \)-connected for any \( r > a \).

Dirac [4] proved that any graph \( G \) with at least three vertices and \( \delta(G) \geq n(G)/2 \) is \( 2^* \)-connected. Any graph \( G \) with at least four vertices and \( \delta(G) \geq n(G)/2 + 1 \) is \( 1^* \)-connected. Obviously, if \( G \) is a complete graph then it is super spanning connected. Thus, we consider incomplete graphs.

Theorem 5. Assume that \( G \) is a graph with \( n(G)/2 + 1 \leq \delta(G) \leq n(G) - 2 \). Then \( G \) is \( r^* \)-connected for \( 1 \leq r \leq 2(\delta(G) - n(G))/2 + 1 \).

Proof. Since \( n(G)/2 + 1 \leq \delta(G) \leq n(G) - 2 \), \( n(G) \geq 6 \). Let \( k \) be a positive integer and \( m \geq 3 \). Suppose that \( n(G) = 2m \) and \( \delta(G) = m + k \) for some \( m \geq 3 \) and \( 1 \leq k \leq m - 2 \). Then \( d_G(u) + d_G(v) \geq 2\delta(G) = 2m + 2k \). By Theorem 4, \( G \) is \( r^* \)-connected for \( 1 \leq r \leq 2k + 2 \). Suppose that \( n(G) = 2m + 1 \) and \( \delta(G) = m + 1 + k \) for some \( m \geq 3 \) and \( 1 \leq k \leq m - 2 \). We have \( d_G(u) + d_G(v) \geq 2\delta(G) = 2m + 2 + 2k \). By Theorem 4, \( G \) is \( r^* \)-connected for \( 1 \leq r \leq 2k + 3 \). □

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