A Cubic Analogue of the Jacobsthal Identity

Heng Huat Chan, Ling Long, and YiFan Yang

Abstract. It is well known that if \( p \) is a prime such that \( p \equiv 1 \pmod{4} \), then \( p \) can be expressed as a sum of two squares. Several proofs of this fact are known and one of them, due to E. Jacobsthal, involves the identity \( p = x^2 + y^2 \), with \( x \) and \( y \) expressed explicitly in terms of sums involving the Legendre symbol. These sums are now known as the Jacobsthal sums.

In this short note, we prove that if \( p \equiv 1 \pmod{6} \), then \( 3p = u^2 + uv + v^2 \) for some integers \( u \) and \( v \) using an analogue of Jacobsthal’s identity.

1. INTRODUCTION. The following theorem is well known:

**Theorem 1.1.** If \( p \) is a prime such that \( p \equiv 1 \pmod{4} \) then

\[
p = x^2 + y^2
\]

for some integers \( x \) and \( y \).

Theorem 1.1 was first observed independently by A. Girard (1595–1632) and P. de Fermat (1601–1665) (see [5, p. 14]). A complete proof of Theorem 1.1 appears to have been first obtained by L. Euler (1707–1783) (see [3, pp. 7–12]). Since then, many different proofs of this result have been discovered, one of which is due to E. Jacobsthal.

To describe Jacobsthal’s proof, we introduce the Legendre symbol. Let \( p \) be an odd prime. An integer \( a \) relatively prime to \( p \) is said to be a quadratic residue modulo \( p \) if the congruence \( x^2 \equiv a \pmod{p} \) is solvable in integers; otherwise, it is called a quadratic nonresidue. The Legendre symbol is defined by

\[
\left( \frac{a}{p} \right) = \begin{cases} 
0 & \text{if } p \mid a, \\
1 & \text{if } a \text{ is a quadratic residue modulo } p, \\
-1 & \text{if } a \text{ is a quadratic nonresidue modulo } p.
\end{cases}
\]

Around 1907, E. Jacobsthal (see [8]) proved Theorem 1.1 using the identity [1]:

\[
p = \left\{ \frac{1}{2} \sum_{\alpha=1}^{p-1} \left( \frac{\alpha^3 + a}{p} \right) \right\}^2 + \left\{ \frac{1}{2} \sum_{\alpha=1}^{p-1} \left( \frac{\alpha^3 + a\alpha}{p} \right) \right\}^2,
\]

where the integer \( a \) is any quadratic nonresidue modulo \( p \).

The relation (1.1) can be interpreted as the factorization of the number \( p \) as a product of elements \( (x + iy) \) and \( (x - iy) \) in \( \mathbb{Z}[i] \), where \( i = \sqrt{-1} \). If we replace \( i \) and \( \alpha, a, \beta \) by \( \omega = e^{\pi i/3} \) and \( a \alpha, \beta \) by \( a, \beta \), respectively, then it is known that \( p \) is a product of \( (x + \omega y) \) and \( (x + \bar{\omega} y) \), where \( x, y \in \mathbb{Z} \) and \( \bar{\omega} \) denotes the complex conjugate of \( \omega \). Equivalently, we have the following theorem.

\[
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\]

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Theorem 1.2. If $p$ is a prime such that $p \equiv 1 \pmod{6}$ then

$$p = x^2 + xy + y^2$$

(1.3)

for some integers $x$ and $y$.

Theorem 1.2 is clearly a cubic analogue\(^1\) of Theorem 1.1 and it can be proved using Euler’s ideas in his proof of Theorem 1.1. A natural question is to ask for a cubic analogue of (1.2). After several attempts, we were led to a possible generalization of (1.2), which we now describe.

Let $\Delta(A, B, C) = B^2 - 4AC$ be the discriminant of the binary quadratic form

$$f(x, y) = Ax^2 + Bxy + Cy^2.$$ 

We first observe that (1.2) can be expressed as

$$|\Delta(1, 0, 1)|p = 4p = \left\{ \sum_{\alpha=1}^{p} \left( \alpha^3 + \alpha \right) \right\}^2 + \left\{ \sum_{\alpha=1}^{p} \left( \alpha^3 + a \alpha \right) \right\}^2. \quad (1.4)$$

With this interpretation of (1.2), we have the following analogue:

Theorem 1.3. Let $p \equiv 1 \pmod{6}$. Suppose that $a$ is any integer such that $x^3 \equiv a \pmod{p}$ is not solvable. Then

$$|\Delta(1, 1, 1)|p = 3p = x^2 + xy + y^2,$$ 

(1.5)

with

$$x = \sum_{\alpha=1}^{p} \left( \frac{\alpha^3 + 1}{p} \right) \quad \text{and} \quad y = \left( \frac{a}{p} \right) \sum_{\alpha=1}^{p} \left( \frac{\alpha^3 + a}{p} \right).$$

2. THE GAUSS SUMS. Let $p$ be a prime number and $\mathbf{F}_p = \mathbf{Z}/p\mathbf{Z}$ be the finite field of $p$ elements. In particular, $\mathbf{F}_p^*$, the set of invertible elements in $\mathbf{F}_p$, is a cyclic group of order $p - 1$ under multiplication. An integer $g$ is a primitive root modulo $p$ if $g$ generates the cyclic group $\mathbf{F}_p^*$. A (multiplicative) character $\chi$ of $\mathbf{F}_p^*$ (or modulo $p$) is a group homomorphism from $\mathbf{F}_p^*$ to $\mathbb{C}^*$, the set of nonzero complex numbers. Namely, for any nonzero elements $\alpha, \beta \in \mathbf{F}_p$, $\chi(\alpha\beta) = \chi(\alpha) \chi(\beta)$. The image $\chi(\mathbf{F}_p^*)$ is a finite cyclic subgroup of $\mathbb{C}^*$ whose cardinality is called the order of the character $\chi$. By convention, $\chi(0) = 0$.

Example 2.1. The assignment $\chi(\alpha) = 1$ for all $\alpha \in \mathbf{F}_p^*$ is an order-1 character, called the principal character modulo $p$.

Example 2.2. The Legendre symbol defined on $\mathbf{Z}$ in Section 1 is periodic with period $p$ and can be viewed as an order-2 character (or a quadratic character) on $\mathbf{F}_p^*$ when $p$ is an odd prime. Note that this character is the only character of order 2 on $\mathbf{F}_p^*$ because a character of order 2 must take value $-1$ at a primitive root $g$, and the assignment $g \mapsto -1$ completely determines this character.

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\(^1\)The word “cubic” refers to the fact that the binary quadratic form $x^2 + xy + y^2$ factors over the field generated by the cube root of unity.
For convenience, we shall adopt the following notation.

**Notation 2.1.** When we write \((\alpha^{-1}/p)\), we view \(\alpha^{-1}\) as the inverse of \(\alpha\) in \(\mathbb{F}_p^*\). Furthermore, we will replace

\[
\sum_{\alpha=1}^{p-1} \text{ by } \sum_{\alpha \in \mathbb{F}_p^*} \quad \text{(or simply } \sum^* \text{)}
\]

and

\[
\sum_{\alpha=1}^{p} \text{ by } \sum_{\alpha \in \mathbb{F}_p} \quad \text{(or simply } \sum \text{)}
\]

The following well-known lemma about characters will be used later in this article.

**Lemma 2.3.** If \(\chi\) is a nonprincipal character of \(\mathbb{F}_p^*\), then

\[
\sum_{\alpha} \chi(\alpha) = 0.
\]

**Proof.** As the character \(\chi\) is nonprincipal, there exists \(\beta \in \mathbb{F}_p^*\) such that \(\chi(\beta) \neq 1\). Also, the map that sends \(\alpha \mapsto \alpha \beta\) is one-to-one on \(\mathbb{F}_p\), and is therefore a bijection since \(\mathbb{F}_p\) is finite. Consequently,

\[
\sum_{\alpha} \chi(\alpha) = \sum_{\alpha} \chi(\alpha \beta) = \left(\sum_{\alpha} \chi(\alpha)\right) \chi(\beta).
\]

This implies \((\sum_{\alpha} \chi(\alpha))(1 - \chi(\beta)) = 0\). By our choice of \(\beta\), \((1 - \chi(\beta)) \neq 0\). Hence,

\[
\sum_{\alpha} \chi(\alpha) = 0.
\]

For any character \(\chi\) on \(\mathbb{F}_p^*\) and \(\beta \in \mathbb{F}_p\), we define the Gauss sum

\[
G(\beta, \chi) = \sum_{\alpha} \chi(\alpha)e^{2\pi i \alpha \beta/p}.
\]

When \(\beta = 1\), we write

\[
G(\chi) = G(1, \chi).
\]

Note that for \(\beta \neq 0\),

\[
G(\beta, \chi) = \sum_{\alpha} \chi(\alpha)e^{2\pi i \alpha \beta/p} = \chi^{-1}(\beta) \sum_{\alpha} \chi(\alpha \beta)e^{2\pi i \alpha \beta/p} = \chi(\beta^{-1})G(\chi). \tag{2.1}
\]

Another basic property of Gauss sums that we need is the following:

**Lemma 2.4 ([2, Theorem 1.1.4(c)]).** Let \(\chi\) be a nonprincipal character of \(\mathbb{F}_p^*\). Then, for \(\beta \neq 0\),

\[
|G(\beta, \chi)|^2 = G(\beta, \chi)\overline{G(\beta, \chi)} = p.
\]
Proof. By (2.1), we have for $\beta \neq 0$,

$$\chi(\beta^{-1})G(\chi) = G(\beta, \chi) = \sum_{\alpha} \chi(\alpha)e^{2\pi i \alpha \beta/p}. \quad (2.2)$$

Notice that in the case $\beta = 0$, (2.2) still holds in view of Lemma 2.3. Multiplying the two sides of (2.2) by $e^{-2\pi i \beta/p}$ and summing over all $\beta$ in $F_p$, we get

$$\overline{G(\chi)}G(\chi) = \sum_{\alpha} \chi(\alpha) \sum_{\beta} e^{2\pi i \beta(\alpha - 1)/p}.$$

The inner sum is nonzero only when $\alpha = 1$, in which case the inner sum is equal to $p$. This implies that $G(\chi)\overline{G(\chi)} = p$. Hence, we conclude that

$$|G(\beta, \chi)|^2 = |\chi(\beta^{-1})|^2 \cdot |G(\chi)|^2 = 1 \cdot p = p. \quad \square$$

3. THE JACOBI SUMS. Let $\chi$ and $\xi$ be characters of $F_p^*$. The Jacobi sum is defined as

$$J(\chi, \xi) = \sum_{\alpha} \chi(\alpha)\xi(1 - \alpha).$$

The following lemma gives a relation between the Gauss sum and the Jacobi sum.

Lemma 3.1 ([2, Theorem 2.1.3]). Let $\chi$ and $\xi$ be two characters of $F_p^*$ such that $\chi\xi$ is nonprincipal. Then

$$J(\chi, \xi) = \frac{G(\chi)G(\xi)}{G(\chi\xi)}.$$

Proof. We observe that

$$G(\chi)G(\xi) = \sum_{\alpha} \sum_{\beta} \chi(\alpha)\xi(\beta)e^{2\pi i (\alpha + \beta)/p} = \sum_{\gamma} \sum_{\alpha + \beta = \gamma} \chi(\alpha)\xi(\beta)e^{2\pi i \gamma/p}$$

$$= \sum_{\alpha + \beta = 0} \chi(\alpha)\xi(\beta) + \sum_{\gamma \neq 0} e^{2\pi i \gamma/p} \sum_{\alpha} \chi(\alpha)\xi(\gamma - \alpha)$$

$$= \xi(-1) \sum_{\alpha} \chi(\alpha) + \sum_{\gamma \neq 0} e^{2\pi i \gamma/p} \sum_{\alpha} \chi(\gamma\alpha)\xi(\gamma - \gamma\alpha)$$

$$= 0 + \sum_{\gamma \neq 0} \chi(\gamma)\xi(e^{2\pi i \gamma/p}) J(\chi, \xi) = G(\chi\xi)J(\chi, \xi),$$

where we have used Lemma 2.3 in the second-to-last equality. \quad \square

The next lemma is useful in computing Jacobi sums of the type $J(\chi, \rho)$ where $\rho$ is the quadratic character of $F_p^*$.

Lemma 3.2 ([2, Theorem 2.1.4]). Let $\chi$ be a character of $F_p^*$, where $p$ is an odd prime, and $\rho$ be the quadratic character of $F_p^*$. Then

$$J(\chi, \rho) = \chi(4)J(\chi, \chi).$$

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Proof. Let \( \beta \) be a fixed element of \( F_p \). Consider the number of solutions to the equation

\[
\alpha(1 - \alpha) = \beta.
\]

Since 2 is invertible in \( F_p \), the above equation is equivalent to

\[
(2\alpha - 1)^2 = 1 - 4\beta,
\]

whose number of solutions is given by

\[1 + \rho(1 - 4\beta).
\]

Therefore

\[
J(\chi, \chi) = \sum_\alpha \chi(\alpha(1 - \alpha)) = \sum_\beta \chi(\beta)(1 + \rho(1 - 4\beta))
\]

\[
= \frac{1}{\chi(4)} \sum_\beta \chi(4\beta) \rho(1 - 4\beta) = \frac{1}{\chi(4)} J(\chi, \rho).
\]

Lemma 3.3. Let \( p \equiv 1 \pmod{6} \) and \( \chi \) be a character of order 6 of \( F_p \). Then

\[
J(\chi, \rho) = c + d \sqrt{-3},
\]

where \( c, d \in \mathbb{Z} \) such that

\[c^2 + 3d^2 = p.
\]

Proof. Pairing \( \alpha \) with \( 1 - \alpha \) and noting that \( 4\alpha(1 - \alpha) = 1 \) when \( \alpha = (p + 1)/2 \), we find that

\[
J(\chi, \rho) = \chi(4)J(\chi, \chi) = \sum_\alpha \chi(4\alpha(1 - \alpha))
\]

\[
= 1 + 2 \sum_{m=2}^{(p-1)/2} \chi(4m(1 - m)).
\]

Note that \( \chi(w) \) is a 6th root of unity since \( \chi \) has order 6. Therefore,

\[\chi(w) \in \mathbb{Z} \left[ 1 + \frac{\sqrt{-3}}{2} \right],
\]

and we conclude that \( J(\chi, \rho) = c + d \sqrt{-3} \), with \( c, d \in \mathbb{Z} \). Now, by Lemmas 2.4, 3.1, and 3.2,

\[|J(\chi, \rho)|^2 = |J(\chi, \chi)|^2 = \frac{|G^2(\chi)|^2}{|G(\chi^2)|^2} = p.
\]

Therefore, the integers \( c \) and \( d \) satisfy

\[c^2 + 3d^2 = |J(\chi, \rho)|^2 = p.
\]
4. THE JACOBSTHAL SUMS. Let \( p \) be a prime. Let \( a \) be an integer not divisible by \( p \) and \( k \) be a positive integer. The Jacobsthal sums \( \phi_k(a) \) and \( \psi_k(a) \) are defined by

\[
\phi_k(a) = \sum_{\alpha} \left( \frac{\alpha}{p} \right) \left( \frac{\alpha^k + a}{p} \right)
\]

and

\[
\psi_k(a) = \sum_{\alpha}^{*} \left( \frac{\alpha^k + a}{p} \right) = \sum_{\alpha} \left( \frac{\alpha^k + a}{p} \right) - \left( \frac{a}{p} \right).
\]

In this note, we will concentrate on evaluating \( \phi_k(a) \) and \( \psi_k(a) \) when \( k = 3 \). To shorten our notation, we let \( \phi = \phi_3 \) and \( \psi = \psi_3 \).

**Lemma 4.1** ([2, Proposition 6.1.5]). Let \( g \) be a primitive root modulo \( p \). Let \( n \) and \( j \) be integers. Then

\[
\phi(g^{3n+j}) = \phi(g^j).
\]

**Proof.** We observe that

\[
\phi(g^{3n+j}) = \sum_{\alpha} \left( \frac{\alpha}{p} \right) \left( \frac{\alpha^3 + g^{3n+j}}{p} \right)
= \sum_{\alpha} \left( \frac{a}{p} \right) \left( \frac{g^{3n}}{p} \right) \left( \frac{(\alpha g^{-n})^3 + g^j}{p} \right)
= \sum_{\alpha} \left( \frac{\alpha}{p} \right) \left( \frac{g^{-3n}}{p} \right) \left( \frac{(\alpha g^{-n})^3 + g^j}{p} \right)
= \sum_{\alpha} \left( \frac{(\alpha g^{-n})^3}{p} \right) \left( \frac{(\alpha g^{-n})^3 + g^j}{p} \right) = \phi(g^j),
\]

where we have used \((g^{3n}/p) = (g^{-3n}/p)\) in the third equality and \((\alpha/p) = (\alpha^3/p)\) in the fourth equality.

**Lemma 4.2** ([2, Proposition 6.1.7]). Let \( p \) be an odd prime and \( a \) be an integer not divisible by \( p \). Then

\[
\phi(a) = \left( \frac{a}{p} \right) \psi(a^{-1}).
\]

**Proof.**

\[
\phi(a) = \sum_{\alpha} \left( \frac{\alpha}{p} \right) \left( \frac{\alpha^3 + a}{p} \right) = \sum_{\alpha}^{*} \left( \frac{\alpha}{p} \right) \left( \frac{\alpha^3 a}{p} \right) \left( \frac{\alpha^{-3} + a^{-1}}{p} \right)
= \left( \frac{a}{p} \right) \sum_{\alpha}^{*} \left( \frac{\alpha^{-3} + a^{-1}}{p} \right) = \left( \frac{a}{p} \right) \psi(a^{-1}).
\]

5. THE MAIN LEMMA.

Lemma 5.1 ([2, Proposition 6.2.2]). Let $p$ be a prime such that $p \equiv 1 \pmod{6}$, $g$ be a primitive root $\pmod{p}$, and $a$ be an integer such that $p \nmid a$. Let $\chi$ be the character of order 6 on $\mathbb{F}_p^*$ such that $\chi(g) = e^{2\pi i/6}$. If $a \equiv g^N \pmod{p}$, then

$$
\phi(a) = \begin{cases} 
-1 + 2 \left( \frac{-1}{p} \right) c & \text{if } N \equiv 0 \pmod{3} \\
-1 - \left( \frac{-1}{p} \right) (c - 3d) & \text{if } N \equiv 1 \pmod{3} \\
-1 - \left( \frac{-1}{p} \right) (c + 3d) & \text{if } N \equiv 2 \pmod{3},
\end{cases}
$$

where $c$ and $d$ are the integers in Lemma 3.3 such that $J(\chi, \rho) = c + d\sqrt{-3}$ with the property $c^2 + 3d^2 = p$.

Proof: By our assumption, the Legendre symbol can be identified with $\chi^3$. Hence,

$$
\phi(a) = \sum_{\alpha} \left( \frac{\alpha}{p} \right) \left( \frac{\alpha^3 + a}{p} \right) = \sum_{\alpha} \chi(\alpha^3) \chi^3(\alpha^3 + a).
$$

Now, observe that

$$
1 + \chi^2(g^s) + \chi^4(g^s) = \begin{cases} 
0 & \text{if } 3 \nmid s, \\
3 & \text{otherwise}.
\end{cases}
$$

Therefore, we may write

$$
\phi(a) = \sum_{m=0}^{p-1} \chi(g^{3m}) \chi^3(g^{3m} + a)
= \sum_{n=0}^{3p-1} \chi(g^n) \chi^3(g^n + a) \sum_{j=0}^{2} \chi^{2j}(g^n)
= \sum_{n=0}^{p-1} \chi(g^n) \chi^3(g^n + a) \sum_{j=0}^{2} \chi^{2j}(g^n).
$$

The last equality follows from the fact that if

$$
F(k) = \sum_{n=(k-1)p}^{kp-1} \chi(g^n) \chi^3(g^n + a) \sum_{j=0}^{2} \chi^{2j}(g^n),
$$

then for any integer $k$,

$$
F(k + 1) = F(k).
$$
Therefore, we may rewrite

\[ \phi(a) = \sum_{\alpha} \chi(\alpha) \chi^3(\alpha + a) \sum_{j=0}^{2} \chi^{2j}(\alpha) \]

\[ = \chi(-1) \sum_{\alpha} \chi(-\alpha) \chi^3((-\alpha)(-1) + a) \sum_{j=0}^{2} \chi^{2j}(-\alpha) \]

\[ = \chi(-1) \sum_{\alpha} \chi(\alpha) \chi^3(a - \alpha) \sum_{j=0}^{2} \chi^{2j}(\alpha) \]

\[ = \chi(-1) \chi^4(a) \sum_{j=0}^{2} \chi^{2j}(a) J(\chi^{2j+1}, \chi^3). \]

Simplifying the above, we conclude that

\[ \phi(a) = \left( \frac{-1}{p} \right) \chi^{-2}(a) J(\chi, \chi^3) + \left( \frac{-1}{p} \right) J(\chi^3, \chi^3) + \left( \frac{-1}{p} \right) \chi^2(a) J(\chi^5, \chi^3). \]

(5.1)

The middle term of (5.1) is

\[ \left( \frac{-1}{p} \right) J(\chi^3, \chi^3) = \left( \frac{-1}{p} \right) \sum_{\alpha} \left( \frac{\alpha(1 - \alpha)}{p} \right). \]

Note that

\[ \sum_{\alpha} \left( \frac{\alpha(1 - \alpha)}{p} \right) = \sum^{*}_{\alpha} \left( \frac{\alpha(1 - \alpha)}{p} \right) \]

\[ = \sum^{*}_{\alpha} \left( \frac{\alpha}{p} \right) \left( \frac{\alpha}{p} \right) \left( \frac{\alpha^{-1} - 1}{p} \right) \]

\[ = \sum^{*}_{\alpha} \left( \frac{\alpha - 1}{p} \right) = - \left( \frac{-1}{p} \right) \]

by Lemma 2.3. Hence, the middle term of (5.1) is \(-1\).

Therefore, if we write \(J(\chi, \chi^3) = c + \sqrt{-3}d\) with \(c, d \in \mathbb{Z}\) as in Lemma 3.3, then

\[ \phi(a) = -1 + \left( \frac{-1}{p} \right) \left( 2 \text{Re}(\chi^{-2}(a))c - 2 \text{Im}(\chi^{-2}(a))\sqrt{3}d \right). \]

Now, if \(N \equiv 1 \pmod{3}\), then by Lemma 4.1, we may set \(a = g\) and observe that

\[ 2 \text{Re}(\chi^{-2}(g)) = -1 \quad \text{and} \quad 2 \text{Im}(\chi^{-2}(g)) = -\sqrt{3}. \]
If $N \equiv 2 \pmod{3}$, then by Lemma 4.1, we may set $a = g^2$ and observe that

$$2 \text{Re}(\chi^{-2}(g^2)) = -1 \quad \text{and} \quad 2 \text{Im}(\chi^{-2}(g^2)) = \sqrt{3}.$$ 

Finally, when $N \equiv 0 \pmod{3}$ then we only need to compute

$$\phi(1) = -1 + 2 \left( \frac{-1}{p} \right) c.$$ 

This completes the proof of the lemma. 

**Completion of the proof of Theorem 1.3.** If $a$ is a quadratic residue modulo $p$ and $a \equiv g^N \pmod{p}$ with $N \equiv 1 \pmod{3}$, then

$$\sum_{\alpha} \left( \frac{\alpha^3 + a}{p} \right) - 1 = \psi(a) = \phi(a^{-1}) = -1 - \left( \frac{-1}{p} \right) (c + 3d),$$

since $a^{-1} \equiv g^{-N} \pmod{3}$ and $-N \equiv 2 \pmod{3}$. This shows that

$$y = \sum_{\alpha} \left( \frac{\alpha^3 + a}{p} \right) = - \left( \frac{-1}{p} \right) (c + 3d).$$

Now,

$$x = \sum_{\alpha} \left( \frac{\alpha^3 + 1}{p} \right) = 2 \left( \frac{-1}{p} \right) c.$$ 

Hence,

$$x^2 + xy + y^2 = 3p.$$ 

Similarly, we conclude the identity in the case when $a$ is a quadratic residue modulo $p$ such that the integer $N$ in $a \equiv g^N \pmod{p}$ satisfies $N \equiv 2 \pmod{3}$.

The case when $a$ is not a quadratic residue can be treated in a similar way. 

6. **CONCLUDING REMARKS.**

1. The proof of Theorem 1.3 given here is a slight modification of the proof due to R. Evans. This result can also be obtained by counting points on elliptic curves over finite fields. For more details, see [7, p. 305, Theorem 4].

2. Evans informed us that using the same idea illustrated here, one can obtain similar results for other quadratic forms. For example, from [2, Theorem 6.2.3], one can obtain

$$\left( \frac{1}{4} \sum_{\alpha} \left( \frac{\alpha^5 + \alpha}{p} \right) \right)^2 + 2 \left( \frac{1}{4} \sum_{\alpha} \left( \frac{\alpha^5 + g\alpha}{p} \right) \right)^2 = p$$

where $p \equiv 1 \pmod{8}$ and $g$ is a primitive root modulo $p$.

3. There are other proofs of Theorem 1.3 using eigenforms associated with Hecke Grössencharacters. A subset of the authors are working in this direction and they
succeeded in deriving solutions to equations such as

\[ |\Delta(1, 0, 2)|p = 8p = A^2 + 2B^2 \]


4. Another cubic generalization of the Jacobsthal identity (1.2) was given by D. Zagier in [4, p. 92]. There, a solution was given to

\[ 4p = A^2 + 3B^2 \]

when \( p \equiv 1 \pmod{6} \) as follows: Let \( \chi \) be an order-6 character of \( \mathbb{F}_p^* \) such that \( \chi(a) \) is a primitive cubic root of unity and \( \chi(b) = \chi(a)^{-1} \). Then one can take

\[
A = \sum_{x=0}^{p-1} \left( \frac{x^3 + 1}{p} \right), \quad B = \frac{1}{3} \sum_{x=0}^{p-1} \left( \frac{x^3 + a}{p} \right) - \frac{1}{3} \sum_{x=0}^{p-1} \left( \frac{x^3 + b}{p} \right).
\]

Zagier mentioned a septic analogue for

\[ 4p = A^2 + 7B^2 \]

and encouraged readers to investigate the solutions of

\[ 4p = A^2 + dB^2 \]

whenever \( \mathbb{Q}(\sqrt{-d}) \) has class number 1.

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A Calculus Proof for the Pythagorean Theorem

Consider a right triangle with legs of length $a$ and $b$ and hypotenuse of length $c$. Construct a rectangular coordinate system so that the vertices of the triangle have coordinates $(0, 0)$, $(a, 0)$, and $(a, b)$. Draw the semicircle in the upper half-plane centered at the origin with radius $c$. This semicircle passes through the points $(a, b)$ and $(0, c)$.

Euclid showed in Book III, Proposition 16 that if $P$ is any point on the semicircle, then the line through $P$ that is perpendicular to the radius from the origin to $P$ is tangent to the semicircle, in the sense that it does not intersect the semicircle at any point other than $P$. We leave it as an exercise for the reader to verify that this line also satisfies the calculus definition of the tangent line to the semicircle at $P$. It follows that the semicircle must be the graph of a solution to the differential equation $\frac{dy}{dx} = -\frac{x}{y}$. Solving this differential equation, we find that $\int y \, dy = -\int x \, dx$, and therefore $x^2 + y^2 = C$ for some constant $C$. Since the circle passes through the point $(0, c)$, we must have $C = c^2$. But then since the circle also passes through $(a, b)$, we can conclude that $a^2 + b^2 = c^2$.

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