A general analytical solution for flow to a single horizontal well by Fourier and Laplace transforms

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ARTICLE INFO
Article history:
Received 26 August 2010
Received in revised form 23 February 2011
Accepted 23 February 2011
Available online 1 March 2011

Keywords:
Free surface equation
Unconfined aquifer
Stream depletion rate

ABSTRACT
The objective of this paper is to present an analytical solution for describing the head distribution in an unconfined aquifer with a single pumping horizontal well parallel to a fully penetrating stream. The Laplace-domain solution is developed by applying Fourier sine, Fourier and Laplace transforms to the governing equation as well as the associated initial and boundary conditions. The time-domain solution is obtained after taking the inverse Laplace transform along with the Bromwich integral method and inverse Fourier and Fourier sine transforms. The upper boundary condition of the aquifer is represented by the free surface equation in which the second-order slope terms are neglected. Based on the solution and Darcy’s law, the equation representing the stream depletion rate is then derived. The solution can simulate head distributions in an aquifer infinitely extending in horizontal direction if the well is located far away from the stream. In addition, the solution can also simulate head distributions in confined aquifers if specific yield is set zero. It is shown that the solution can be applied practically to evaluate flow to a horizontal well.

1. Introduction

The stream depletion rate (SDR) is herein defined as the ratio of infiltrative water from the stream (or lake, stream, etc.) to the total amount of water produced from the pumping well. The stream depletion rate increases gradually with pumping time and finally approaches one, which reflects most of water comes from the stream, after a certain period of pumping time. Many analytical models treat the stream as a constant-head boundary to estimate stream depletion rate [e.g., 1–3, 6, 17, 21]. Some researchers proposed to treat the stream as a variable stream stage represented by a periodic function for seasonal variations or a function changed in space and time for flood wave [7]. The articles mentioned above investigated the behavior of stream depletion rate induced from a vertical well. In addition, their mathematical model neglects vertical flow to unconfined aquifers.

Recently, the horizontal well becomes popular due to the advanced technique in drilling the wellbore. There are some advantages to use the horizontal well. For example, the cost from operating horizontal wells is half of that from vertical wells [8]. Horizontal wells have better contact with aquifers and are appropriate to install in thin aquifers. A shallow drawdown cone will be produced if adopting a long and deep horizontal well. Some semi-analytical or numerical solutions are developed to investigate the behaviors of the groundwater flow induced from horizontal wells [e.g., 9–12, 20]. Zhan et al. [18] provided a method to solve the boundary problem of groundwater flow to the horizontal well. Their method is first to consider a pumping point source and then to integrate the point source solution along the well axis. Based on this approach, they developed an analytical solution for describing the groundwater flow induced from horizontal wells in a confined aquifer. Zhan and Zlotnik [19] developed a semi-analytical solution requiring numerical inversion to investigate the drawdown due to slanted wells in an unconfined aquifer. They indicated that the type curve of drawdown had three stages, including rapid increase at early time, middle flat stage and rapid increase again at late time. Sun and Zhan [13] presented semi-analytical solutions to describe the groundwater flow induced from horizontal wells in an aquitard-aquifer system beneath a water reservoir. The flow in aquitard overlying the confined aquifer is represented by a transient equation with a term accounting for the specific storage. These articles involved in horizontal wells consider the aquifer extending infinitely in horizontal direction. Thus they did not investigate the behavior of stream depletion rate induced from a horizontal well.

Tsou et al. [14] presented an analytical solution developed by Fourier transforms to describe the stream depletion rate induced from the slanted well in confined aquifers. They found that the horizontal well parallel to the stream had less time to reach quasi-steady stream depletion rate in comparison with various directions of well axis. Therefore, it is better to install the horizontal well...
and the depth of the well is \( H \). The origin of coordinate system is located at the middle of the well. The top of the stream is situated at the interface between the stream and aquifer and the origin of coordinate system is located far away from the stream. Based on Darcy’s law, this solution can also be derived to simulate head distribution in a confined aquifer when the specific yield is set zero or in an aquifer extending infinitely in horizontal direction if the well is located far away from the stream. The equation describing the change in water table with the absence of a surface recharge can be written by neglecting the second-order terms as [16]

\[
S_y \frac{\partial h}{\partial t} = -K_z \frac{\partial h}{\partial z} \quad \text{at} \quad z = h,
\]

where \( S_y \) is specific yield. Note that the stream provides water to the well and consequently avoids significant deformation of the free surface due to pumping. The largest drawdown due to pumping is much smaller than the aquifer thickness. The domain in \( z \) direction \((-H < z < 0)\) is hence approximated from \(-H\) to 0, where the head distribution at \( z = 0 \) is the water table. Eq. (7) is further simplified as [19]

\[
S_y \frac{\partial h}{\partial t} = -K_z \frac{\partial h}{\partial z} \quad \text{at} \quad z = 0.
\]

Except Eq. (3), the mathematical formulation herein is identical to Zhan and Zlotnik [19] where they considered an unconfined aquifer extending infinitely in the horizontal direction.

### 2. Method

#### 2.1. Mathematical model

Fig. 1 shows the three-dimensional (3D) conceptual model for an anisotropic unconfined aquifer with a horizontal well parallel to a fully penetrating stream. The origin of coordinate system is located at the interface between the stream and aquifer and the \( x \) axis passes through the middle of the well. The top of the stream is considered as reference datum. The thickness of the aquifer is \( H \) and the depth of the well is \( D \) as shown in Fig. 1(b). In addition, the distance measured from the stream to the well is \( d \) and the well length is \( L \) as indicated in Fig. 1(a).

Three assumptions introduced for the model are: (1) The aquifer is homogeneous. (2) The stream stage does not change during the pumping period, implying that the stream has a large quantity of water in comparison with that drawn from the well. (3) The stream bed has the same hydraulic conductivity as aquifers. Note that Intaraprasong and Zhan [7] had addressed the effects of variable stream stage and low-permeable stream bed on the stream depletion rate.

To acquire the solution for pumping from a horizontal well, we start with the development of a point source solution of the model [e.g., 14,19]. The governing equation for describing 3D transient hydraulic head distribution \( h(x,y,z,t) \) in the unconfined aquifer with a point source can be expressed as

\[
K_x \frac{\partial^2 h}{\partial x^2} + K_y \frac{\partial^2 h}{\partial y^2} + K_z \frac{\partial^2 h}{\partial z^2} = S_y \frac{\partial h}{\partial t} + Q \delta(x - x_0) \delta(y - y_0) \delta(z + z_0)
\]

where \( K_x, K_y, \) and \( K_z \) are hydraulic conductivities in the \( x, y \) and \( z \) direction, respectively; \( S_y \) is specific storage; \( Q \) is a positive constant pumping rate of the point source; \( \delta(\cdot) \) represents the Dirac delta function and \((x_0,y_0,z_0)\) is the location of the point source.

The water table before the pumping is considered to be horizontal and thus the initial condition is formulated as

\[
h = 0 \quad \text{at} \quad t = 0.
\]

The stream is hydraulic contact with the aquifer and considered as a constant-head boundary. The boundary condition at \( x = 0 \) is therefore expressed as

\[
h = 0 \quad \text{at} \quad x = 0.
\]

The remote boundary conditions in \( x \) and \( y \) directions are considered as constant-head boundaries

\[
\lim_{x \to -\infty} h = 0, \quad \lim_{y \to -\infty} h = 0,
\]

where the groundwater is not affected by the pumping. Consider the aquifer lies on an impermeable layer such that

\[
\frac{\partial h}{\partial z} = 0 \quad \text{at} \quad z = -H.
\]

The equation describing the change in water table with the absence of a surface recharge can be written by neglecting the second-order terms as [16]

\[
S_y \frac{\partial h}{\partial t} = -K_z \frac{\partial h}{\partial z} \quad \text{at} \quad z = h,
\]

where \( S_y \) is specific yield. Note that the stream provides water to the well and consequently avoids significant deformation of the free surface due to pumping. The largest drawdown due to pumping is much smaller than the aquifer thickness. The domain in \( z \) direction \((-H < z < 0)\) is hence approximated from \(-H\) to 0, where the head distribution at \( z = 0 \) is the water table. Eq. (7) is further simplified as [19]

\[
S_y \frac{\partial h}{\partial t} = -K_z \frac{\partial h}{\partial z} \quad \text{at} \quad z = 0.
\]

### Fig. 1. Schematic diagrams of an unconfined aquifer with a horizontal well near a stream. (a) Top view, (b) cross section view.
Based on the following dimensionless variables [14]:

\[
x_0 = \frac{x}{H}, \quad y_0 = \frac{y}{y}, \quad z_0 = \frac{z}{H}, \quad x_{OD} = \frac{x}{H}, \quad y_{OD} = \frac{y}{H}, \quad z_{OD} = \frac{z}{H},
\]
\[
d_0 = \frac{d}{H}, \quad D_0 = \frac{D}{H}, \quad t_0 = \frac{KxH}{H^2S}, \quad h_0 = \frac{\pi KxH}{Q}.
\]

where the subscript \( D \) denotes a dimensionless symbol, Eq. (1) can be expressed as

\[
\frac{\partial^2 h_D}{\partial x_D^2} + \frac{\partial^2 h_D}{\partial y_D^2} + \frac{\partial^2 h_D}{\partial z_D^2} + \frac{\partial h_D}{\partial t_D} + \pi \delta(x_D - x_{OD})\delta(y_D - y_{OD}) - \delta(z_D - z_{OD}),
\]

where \( K_x = K/K_x \) and \( K_z = K/K_z \).

Accordingly, the initial and boundary conditions are denoted as

\[
\begin{align*}
    h_D &= 0 \quad \text{at} \ t_0 = 0, \\
    h_D &= 0 \quad \text{at} \ x_0 = 0, \\
    \lim_{t_0 \to \infty} h_D &= 0, \\
    \lim_{y_0 \to \pm \infty} h_D &= 0, \\
    \frac{\partial h_D}{\partial z_D} &= 0 \quad \text{at} \ z_D = -1, \\
    \frac{\partial^2 h_D}{\partial z_D^2} &= -\frac{\partial h_D}{\partial z_D} \quad \text{at} \ z_D = 0,
\end{align*}
\]

where \( \kappa = S/(S.H) \).

2.2. Head distribution solutions

Applying Fourier and Laplace transforms to Eqs. (10)-(16) and then inverting the result yield the point source solution shown below. Readers can refer to Appendix A for the detailed derivation.

\[
h(x, y, z, t) = \int_0^\infty \int_0^\infty \left( \Phi_a(\xi) + \sum_{n=1}^{\infty} \psi_a(\xi) \right) \sin(\omega_0 \xi) \cos(\eta_0 \xi) dz d\xi
\]

with

\[
\Phi_a = \frac{\gamma_0 \cos(\beta_0(1 - z_D))}{\lambda_0 \eta_0},
\]

\[
\psi_a = \frac{\gamma_0 \sin(\beta_0(1 - z_D))}{\lambda_0 \eta_0},
\]

\[
\Phi_b = \frac{\gamma_0 \cos(1 + z_D) \beta_0}{\lambda_0 \eta_0},
\]

\[
\psi_b = \frac{\gamma_0 \sin(1 + z_D) \beta_0}{\lambda_0 \eta_0},
\]

\[
\lambda_0 = w^2 + K_x \kappa^2 - K_z \kappa^2,
\]

\[
\lambda_0 = w^2 + K_x \kappa^2 + K_z \kappa^2,
\]

\[
\eta_0 = K_x \beta_0(1 + 2z_D) \cos(\beta_0) + [K_z - \kappa \beta_0] \sin(\beta_0),
\]

\[
\eta_0 = K_z \beta_0(1 + 2z_D) \cos(\beta_0) + [K_z - \kappa \beta_0] \sin(\beta_0),
\]

\[
\gamma_0 = \sin(\omega_0 \xi)[K_x \beta_0 \cos(\beta_0 z_0) + \kappa \alpha_0 \sin(\beta_0 z_0)],
\]

\[
\gamma_0 = \sin(\omega_0 \xi)[K_x \beta_0 \cos(\beta_0 z_0) + \kappa \alpha_0 \sin(\beta_0 z_0)],
\]

where \( w \) and \( \xi \) are the dummy variables for Fourier sine and Fourier transform, respectively; the subscripts \( a \) and \( b \) represent the solution for the aquifer above and below the horizontal well, respectively; \( \beta_0 \) and \( \beta_0 \) are respectively the roots of following two equations

\[
e^{\alpha_0 \xi} = \frac{K_x \beta_0 - \kappa \xi}{K_z \beta_0 + \kappa \xi},
\]

\[
\tan \beta_0 = -\frac{\kappa \xi}{K_z \beta_0}
\]

Note that Eq. (30) has infinite roots due to the periodical function \( \tan \beta_0 \). The roots of these two equations are easily found by Newton’s method [15]. Only the positive roots of these two equations are chosen for evaluation. The suggested initial guesses for \( \beta_0 \) and \( \beta_0 \) are \( [K_z + \sqrt{K_z^2 + 4K_x \kappa^2 w^2}] / (2K_x \kappa) \) and \( (2n - 1)\pi/2 \), respectively, where \( n \) is an integer from 1, 2, 3, …, \( \infty \). For the reason of choosing the positive roots, please refer to Appendix C.

The solution of the head distribution for pumping from a horizontal well can then be acquired by integrating the point source solution, Eqs. (17) and (18), along the well axis and then multiplying a reciprocal of well length [e.g., 13, 18, 19]. The point source therefore becomes a line sink representing a horizontal well along which the uniform pumping rate is \( Q/L_D \). The solution for such a horizontal well is expressed as

\[
h_{ub} = \int_0^{\infty} h_D dy_D \quad \text{for} \quad -D_0 \leq z_D \leq 0.
\]

\[
h_{ub} = \int_0^{\infty} h_D dy_D \quad \text{for} \quad -1 \leq z_D \leq -D_0.
\]

Eq. (31) developed based on Eq. (17) implicitly includes \( \Phi_a \) and \( \Psi_a \) while Eq. (32) developed from Eq. (18) implicitly includes \( \Phi_b \) and \( \Psi_b \). Note that \( x_D \) and \( z_D \) in \( \Phi_a \), \( \Psi_a \), \( \Phi_b \) and \( \Psi_b \) should be replaced by \( d_0 \) and \( D_0 \), respectively.

2.3. Stream depletion rate

Based on Darcy’s law, the infiltration rate from the stream can be written as

\[
q = \int_{x_D} \int_{y_D} K_idz dy_D \quad \text{at} \ x_D = 0.
\]

where \( i \) represents hydraulic gradient in \( x \) direction; \( \Omega_x \) and \( \Omega_z \) are the whole domain of the stream in \( y \) and \( z \) direction, respectively. In the case that the stream fully penetrates the aquifer, substituting \( i = \partial h_D/\partial x_D \) for \( -1 \leq z_D \leq D_0 \) and \( i = \partial h_D/\partial x_D \) for \( D_0 \leq z_D \leq 0 \) into Eq. (33) yields the stream depletion rate as

\[
\text{SDR} = \frac{q}{Q} = \frac{1}{\pi} \int_{-\infty}^{\infty} \left( \int_{-1}^{D_0} \frac{\partial h_D}{\partial x_D} dz_D + \int_{D_0}^{0} \frac{\partial h_D}{\partial x_D} dz_D \right) dy_D \quad \text{at} \ x_D = 0.
\]

Eq. (34) implicitly includes \( D_0 \) in \( \Phi_a \), \( \Psi_a \), \( \Phi_b \) and \( \Psi_b \), indicating that the stream depletion rate in unconfined aquifers is dependent of the depth of the horizontal well.

2.4. A special case: Tsou et al.’s solution [14]

If \( S_p = 0 \) (i.e., \( \kappa = 0 \)), the present solution, Eqs. (17) and (18), reduces to Tsou et al.’s solution [14] which describes groundwater flow induced from a point source in confined aquifers. For the detailed derivation, readers can refer to Appendix B.

Based on Darcy’s law, the stream depletion rate from a horizontal well for Tsou et al.’s solution [14] is expressed as

\[
\text{SDR} = \frac{2}{\pi L_D} \int_{x_D}^{1/2} \int_{-1}^{D_0} \int_{-\infty}^{\infty} \int_{0}^{\infty} w \sin(wx_D) \cos(\eta_D) \left( e^{(m^2 + k_2^2)z_D} - 1 \right) dz_D dw dy_D dy_D.
\]
Note that the stream depletion rate in confined aquifers is independent of the well depth because of no $D_D$ in Eq. (35).

3. Results and discussion

The numerical integrations for Eqs. (31), (32) and (34) can be done by Gaussian quadrature [5, p. 301]. The integrands in these three equations have oscillatory patterns with consecutive roots determined by $\cos(y_D - y_0) = 0$. Each area between two consecutive roots is evaluated by the 16-term Gaussian quadrature formula. The result of the numerical integration is equal to the total areas which is the sum of each area along the $\zeta$ axis. Accordingly, the total areas can be expressed in terms of an infinite series. The series converges quite fast and usually takes only few terms to achieve accuracy to centimeter.

3.1. Effect of specific yield on stream depletion rate

The unconfined aquifer has an effect of instantaneous drainage from water table on the stream depletion rate. Consider that the aquifer has anisotropic hydraulic conductivity falling in the range of 0.01–100 m/day if the aquifer consists of silty sand [4, p.604]. The other parameter values used for evaluation are shown in Table 1. Fig. 2 shows the stream depletion rate obtained from Eq. (34) versus dimensionless time $t_D$ for various $S_y$ (= 0.1 and 0.3). The curves exhibit three segments: rapid increase at early time, middle flat stage, and rapid increase again at late time. The curve with larger $S_y$ has longer flat stage. The first and third segments indicate that the flow from the stream increases with time. However, in the second segment the stream depletion rate maintains constant, indicating that the pumping well obtains the water from the pore

| Table 1 |
The default dimensional value for each parameter.|
<table>
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Fig. 2. The temporal stream depletion rate for $S_y = 0.1$ and 0.3.

Fig. 3. The temporal stream depletion rate for $D_D = 0.2$ and 0.8.

Fig. 4. The temporal stream depletion rate for isotropic and anisotropic aquifers.
drainage. When the dimensionless time is larger than $3 \times 10^5$, the stream depletion rate curve of $S_y = 0.1$ approaches one, implying that the water all comes from the stream. In addition, the figure indicates that a larger specific yield ($S_y = 0.3$) leads to a less stream depletion rate after the second segment.

3.2. Effect of well depth on stream depletion rate and head distribution

The stream depletion rate for unconfined aquifers is dependent of the depth of the horizontal well. Fig. 3 shows stream depletion rate versus dimensionless time $t_D$ for shallow well ($D_y = 0.2$) and deep well ($D_y = 0.8$), indicating that the deep well has a larger stream depletion rate. On the other hand, the shallow well requires longer time to acquire the same amount of stream water as the deep one.

3.3. Effect of anisotropy on stream depletion rate and head distribution

Generally speaking, an anisotropic aquifer has smaller vertical hydraulic conductivity than horizontal one. The value of $K_z$ has an effect on stream depletion rate and head distribution. Consider that an isotropic aquifer has $K_x = K_y = K_z = 1$ m/day and an anisotropic aquifer has $K_x = K_y = 1$ m/day and $K_z = 0.1$ m/day. The comparison of stream depletion rate between these two aquifers shown in Fig. 4 indicates that the anisotropic aquifer has larger stream depletion rate than the isotropic one. This is because smaller $K_z$ makes less water form pore drainage and more water from the stream. A shallow drawdown cone can therefore be expected for the anisotropic aquifer as shown in Fig. 5.

3.4. Spatial head distribution for various depths

Figs. 6 and 7 demonstrate spatial head distribution predicted from Eq. (31) for $z_D = 0$ and $z_D = -D_y$, respectively. For fixed $x_D$ and $y_D$, the head at $z_D = -D_y$ is smaller than that at $z_D = 0$, indicating that the aquifer has downward flow induced from a pumping...
horizontal well. The minimum head occurs at the center of the horizontal well \((x_D = 4, y_D = 0, z_D = -D_D)\). It is interesting to note that the head distribution shown in Fig. 7 can reflect a line sink (horizontal well) obviously where the head changes dramatically.

### 3.5. Potential applications of the present solution

The present solutions can be applied to various types of groundwater problems associated with pumping in unconfined aquifers. For example, the present point source solution can also be used to develop the solutions for different type of wells such as slanted well and vertical well with full or partial penetration when integrating Eqs. (17) and (18) along the well axis. In addition, if the stream is located very far away from the well so that the drawdown cone never reaches the stream, the result evaluated from the present solution reduces to that of Zhan and Zlotnik’s solution [19, Eq. (23)] describing the head distribution for the aquifer of infinite extent in the horizontal direction. Fig. 8 shows a comparison of spatial head distributions at \(y_D = 0\) predicted by the present solution and Zhan and Zlotnik’s solution [19]. This figure indicates that the drawdown cone has not reached the stream and the present solution has a good agreement with Zhan and Zlotnik’s solution [19].

### 3.6. Comparison of present solution with observed field data

Mohamed and Rushton [12] carried out a field experiment with a horizontal well in a shallow aquifer in Sarawak, Malaysia. The aquifer can be considered to extend infinitely in the horizontal direction because the drawdown cone never reaches the boundary of the aquifer during early pumping period. The measured pumping rates are 230 m³/day at 1.25 day, 160 m³/day at 3.875 day, and 280 m³/day at 4.5 day. In fact, the designed pumping rate is 240 m³/day for long-term water requirement. The other field data and aquifer parameters are listed in Table 2. Fig. 9 shows the observed field data taken from Sarawak [12] and the predicted drawdown from the present solution based on the designed pumping rate and data given in Table 2. The figure shows that the predicted drawdown from present solution has a good agreement with the observed drawdown at \(t = 6\) days except at the middle and ends of the well \((y = -150, 0, 150\) m). This discrepancy may mainly arise from the energy loss at the caisson (middle) and the entrance loss at the ends of the field well. However, the predicted drawdown from present solution is obviously smaller than the observed drawdown at \(t = 3.5\) days. The differences may come from the fact that the present solution is evaluated based on the designed pumping rate of 240 m³/day which is larger than the measured early pumping rates given above.

### 4. Concluding remarks

A general analytical solution is developed for describing flow to a horizontal well. The stream bed is assumed to have the same hydraulic conductivity as the aquifer and the stream is considered to maintain a constant stage during the pumping period. The present solution reduces to Tsou et al.’s solution [14] if the specific yield equals zero. Moreover, the present solution will give good approximation to Zhan and Zlotnik’s solution [19] if the pumping well is located far away from the stream. It is found that a larger specific yield of an unconfined aquifer results in a longer middle flat stage and a less stream depletion rate to the pumping horizontal well. An unconfined aquifer with a deeper horizontal well has more stream depletion rate than those with a shallower one. An

![Fig. 8. Comparison between the predicted head by the present solution and Zhan and Zlotnik’s solution [19].](image)

![Fig. 9. The predicted drawdown from present solution and observed drawdown from Mohamed and Rushton [12].](image)

<table>
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aquifer with a smaller vertical hydraulic conductivity produces more stream depletion rate and shallower drawdown cone to the aquifer. In addition, the predicted drawdown from the present solution agrees with the observed drawdown in a horizontal well reported in Mohamed and Rushton [12].

Acknowledgements

This study was supported by the Taiwan National Science Council under the grants NSC 99–NU–E–009–001 and NSC 99–2221–E–009–062–MY3 as well as the “Aim for the Top University Plan” of the National Chiao Tung University and Ministry of Education, Taiwan. The authors would like to thank four anonymous reviewers for their valuable and constructive comments.

Appendix A. The derivation of Eqs. (17) and (18)

Applying Fourier sine, Fourier and Laplace transforms to Eqs. (10)–(16) results in an ordinary differential equation (ODE) and boundary conditions in terms of (10)–(16) results in an ordinary differential equation (ODE) and boundary conditions in terms of $z_0$ as

$$
\frac{\partial^2 h}{\partial z^2} - (w^2 + K_\nu^2) \frac{\partial h}{\partial z} = \frac{1}{p} e^{i\omega z} \sin(w\omega z) \delta(z_0 + z_0).
$$

(A.1)

$$
\frac{\partial h}{\partial z} = 0 \text{ at } z_0 = -1,
$$

(A.2)

$$
\frac{\partial h}{\partial z} = \frac{1}{p} e^{i\omega z} \sin(w\omega z) \text{ at } z_0 = 0,
$$

(A.3)

where $w, \xi, p$ are the variables of the Fourier sine, Fourier and Laplace transform, respectively. Due to Dirac delta function, Eq. (A.1) is divided into two homogeneous ODEs as

$$
\frac{\partial^2 h}{\partial z^2} - (w^2 + K_\nu^2) \frac{\partial h}{\partial z} = 0 \text{ for } -z_0 < z_0 < 0.
$$

(A.4)

$$
\frac{\partial^2 h}{\partial z^2} - (w^2 + K_\nu^2) \frac{\partial h}{\partial z} = 0 \text{ for } -1 < z_0 < -z_0.
$$

(A.5)

There are two continuity requirements at $z_0 = -z_0$. One is the continuity of the hydraulic head expressed as

$$
\hat{h}_{u0} = \hat{h}_{u0} \text{ at } z_0 = -z_0.
$$

(A.6)

Integrating Eq. (A.1) from $z_0 = -z_0$ to $z_0 = -z_0$ obtains the other continuity requirement which reflects hydraulic gradient discontinuity as

$$
\frac{\partial h}{\partial z} \bigg|_{z_0} \frac{\partial h}{\partial z} \bigg|_{z_0} = \frac{1}{p} e^{i\omega z} \sin(w\omega z) \text{ at } z_0 = -z_0.
$$

(A.7)

Solving Eqs. (A.4) and (A.5) with boundary conditions, Eqs. (A.2) and (A.3), as well as continuity requirements, Eqs. (A.6) and (A.7), simultaneously results in

$$
\hat{h}_{u0} = G(p) \cdot K_a(p) \text{ for } -z_0 \leq z_0 \leq 0,
$$

(A.8)

$$
\hat{h}_{u0} = G(p) \cdot K_b(p) \text{ for } -1 \leq z_0 \leq -z_0,
$$

(A.9)

where

$$
G(p) = \frac{1}{p}.
$$

(A.10)

Taking inverse Laplace transform to Eq. (A.10) yields

$$
g(t_0) = L^{-1}\{G(p)\} = 1,
$$

(A.14)

where $L^{-1}\{\}$ represents inverse Laplace transform.

Both Eqs. (A.11) and (A.12) are a single value function. Based on Bromwich integral, the results of inverse Laplace transform for these two equations can therefore be, respectively, expressed as

$$
k_a(t_0) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} K_a(p)e^{pt_0}dp,
$$

(A.15)

$$
k_b(t_0) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} K_b(p)e^{pt_0}dp,
$$

(A.16)

where $i$ is an imaginary unit and $r$ is a real constant which is so large that all of the real parts of the singularities are smaller than it. The pathway of Bromwich integral contains a close contour with a semicircle and straight line parallel to the imaginary axis. According to Jordan’s Lemma, the value of the integration for the semicircle tends to zero if its radius approaches infinity. Based on the residue theory, the result of the integration for Eqs. (A.15) and (A.16) can be expressed as summation of the residue for each pole. Therefore, $k_a(t_0)$ and $k_b(t_0)$ can respectively be further expressed as

$$
k_a(t_0) = \sum_{\gamma=1}^{\infty} \text{Res}_{p=p_n} G(p) e^{pt_0},
$$

(A.17)

$$
k_b(t_0) = \sum_{\gamma=1}^{\infty} \text{Res}_{p=p_n} G(p) e^{pt_0},
$$

(A.18)

where $p_n$ is the pole in complex plane and Res represents the residue for each pole.

Let the denominator of Eq. (A.11) or (A.12) equal zero and the roots of this equation represent the location of poles in complex plane. Note that the poles exist only at the real axis. Only one pole, $p_0$, exists between $p = 0$ and $p = -w^2 - K_\nu^2$, while infinite poles, $p_n$, happen behind $p = -w^2 - K_\nu^2$.

For the residue without imaginary unit, the pole at $p = p_0$ should be expressed as $p_0 = K_\nu j_n - K_\nu^2 z - w^2$ obtained from letting $\lambda = 0$. Similarly, the poles at $p = p_n$ is expressed as $p_n = -K_\nu j_n - K_\nu^2 z - w^2$ obtained from letting $\lambda = j_n$. Substituting $\lambda = 0$ and $p = p_0 = K_\nu j_n - K_\nu^2 z - w^2$ into the denominator of Eq. (A.11) or (A.12) results in Eq. (29). Similarly, substituting $\lambda = j_n$ and $p = p_n = -K_\nu j_n - K_\nu^2 z - w^2$ into denominator of Eq. (A.11) or (A.12) yields Eq. (30).

The residues of Eqs. (A.17) and (A.18) can respectively be determined from the following formulas

$$
\text{Res}_{p=p_0} = \text{lim}_{p \rightarrow p_0} G(p) e^{pt_0} (p - p_n),
$$

(A.19)

$$
\text{Res}_{p=p_n} = \text{lim}_{p \rightarrow p_n} G(p) e^{pt_0} (p - p_n).
$$

(A.20)
Substituting Eqs. (A.11) into Eq. (A.19) and applying L'Hopital’s rule results in

\[
\text{Re} pN = \lim_{pN \to p_0} \frac{\sin(wx_0) \cosh[\lambda(1 - z_0)] - K_\lambda \cosh(z_0) + px \sinh(z_0) \lambda d^{(w + p)0}}{K_\lambda \left( \lambda d^{(w + p)0} \right)}.
\]

Similarly, substituting Eq. (A.12) into Eq. (A.20) and applying L'Hopital’s rule yields

\[
\text{Re} pN = \lim_{pN \to p_0} \frac{\sin(wx_0) \cosh[\lambda(1 - z_0)] - K_\lambda \cosh(z_0) - px \sinh(z_0) \lambda d^{(w + p)0}}{K_\lambda \left( \lambda d^{(w + p)0} \right)}.
\]

With \( pN = p_0 \) and \( \lambda = \beta_0 \), Eq. (A.21) leads to

\[
\text{Re} pN = -2 \gamma_0 \frac{\cosh[\beta_0(1 - z_0)]}{\eta_0} \text{ e}^{\xi (w + i\beta_0)}.
\]

Similarly, substituting \( pN = p_0 \) and \( \lambda = \gamma_0 \), into Eq. (A.21) results in

\[
\text{Re} pN = -2 \gamma_0 \frac{\cosh[\beta_0(1 - z_0)]}{\eta_0} \text{ e}^{\xi (w + i\beta_0)}.
\]

Note that the sum of Eqs. (A.23) and (A.24) is the result of integration of Eq. (A.15). With \( pN = p_0 \) and \( \lambda = \beta_0 \), Eq. (A.22) yields

\[
\text{Re} pN = -2 \gamma_0 \frac{\cosh[\beta_0(1 - z_0)]}{\eta_0} \text{ e}^{\xi (w + i\beta_0)}.
\]

Similarly, substituting \( pN = p_0 \) and \( \lambda = \gamma_0 \) into Eq. (A.22) yields

\[
\text{Re} pN = -2 \gamma_0 \frac{\cosh[\beta_0(1 - z_0)]}{\eta_0} \text{ e}^{\xi (w + i\beta_0)}.
\]

The sum of Eqs. (A.25) and (A.26) is the result of integration of Eq. (A.16).

The results of inverse Laplace transform for Eqs. (A.8) and (A.9) can be obtained, respectively, by the convolution theorem.

\[
h_{\beta_0} = \int_0^t g(t - \tau) \cdot \kappa_1(\tau) d\tau,
\]

\[
h_{\gamma_0} = \int_0^t g(t - \tau) \cdot \kappa_2(\tau) d\tau.
\]

The result of integration of Eq. (A.27) leads to the sum of Eqs. (19) and (20) and the result of integration of Eq. (A.28) leads to the sum of Eqs. (21) and (22). The final solutions expressed as Eqs. (17) and (18) can then be obtained after taking inverse Fourier and Fourier sine transform.

**Appendix B. The derivation of reducing to Tsou et al.’s solution [14]**

Substituting \( z = 0 \) into Eqs. (29) and (30) yields \( e^{\xi_0} = 1 \) and \( \tan \theta_0 = 0 \), respectively. Obviously, the roots of these two equations are \( \beta_0 = 0 \) and \( \beta_0 = \pi \) where \( n \) is an integer from 1, 2, 3, ..., \( \infty \). Multiplying the numerator and denominator of Eq. (19) by \( 1/\beta_0 \) and substituting \( z = 0 \) into Eq. (19) leads to

\[
\phi_0 = \frac{\sin(wx_0) \cos(\beta_0 z_0) \cosh[\beta_0(1 - z_0)] - e^{\xi_0} \cos(\beta_0 z_0) \cosh[\beta_0(1 - z_0)] - 1}{(w^2 + K_\xi \xi^2 - K_\beta \beta_0^2) \left( \cosh[\beta_0(1 - z_0)] - 1 \right)}.
\]

Alternatively, Eq. (B.2) can be obtained from substituting \( \alpha = 0 \) and \( \beta_0 = 0 \) into Eq. (21).

With \( \alpha = 0 \) and \( \beta_0 = \pi \), Eq. (20) leads to

\[
\psi_0 = \frac{\sin(wx_0)(\pi \cos(\pi z_0) \cos(\pi(1 - z_0))) - 1}{(w^2 + K_\xi \xi^2 + K_\beta \beta_0^2) \left( \pi \cos(\pi) - \sin(\pi) \right)}.
\]

Substituting \( \pi \cos(\pi(1 - z_0)) = \cos(\pi(1 - z_0)) \), \( \pi \sin(\pi z_0) = \sin(\pi z_0) \), \( \pi \sin(\pi(1 - z_0)) = \sin(\pi(1 - z_0)) \) into Eq. (B.3) yields

\[
\psi_0 = \frac{\sin(wx_0)(\pi \cos(\pi z_0) \cos(\pi(1 - z_0))) - 1}{(w^2 + K_\xi \xi^2 + K_\beta \beta_0^2) \left( \pi \cos(\pi(1 - z_0)) \right)}.
\]

Alternatively, Eq. (B.4) can be obtained after substituting \( \alpha = 0 \) and \( \beta_0 = \pi \) into Eq. (22). Accordingly, Tsou et al.’s solution [14, Eq. (16)] with a point source can be obtained from substituting Eqs. (B.2) and (B.4) into Eq. (17) as

\[
\psi_\text{ch} = \frac{2}{\pi} \int_0^\infty \int_0^\infty \frac{\sin(wx_0) \cos(z_0 - y_0) \xi}{\left( w^2 + K_\xi \xi^2 \right) \left( w^2 + K_\gamma \gamma^2 \right) \left( w^2 + K_\beta \beta_0^2 \right) \left( e^{(w^2 + K_\xi \xi^2 + K_\beta \beta_0^2) z_0} - 1 \right)} d\lambda d\eta.
\]

**Appendix C. The reason of choosing positive roots \( \beta_0 \) and \( \beta_0 \)**

Eqs. (29) and (30) for the roots of \( \beta_0 \) and \( \beta_0 \) are derived, respectively, based on setting \( \sqrt{(w^2 + K_\xi \xi^2 + p) / K_\xi} = \beta_0 \) and \( \sqrt{(w^2 + K_\xi \xi^2 + p) / K_\xi} = \beta_0 \). The values of \( K_\gamma, K_\xi, w \) and \( \xi \) are positive while \( p \) is negative. The value of \( w^2 + K_\xi \xi^2 + p \) could therefore be greater or smaller than zero. We let \( \sqrt{(w^2 + K_\xi \xi^2 + p) / K_\xi} = \beta_0 \)
if $w^2 + k_y z^2 + p > 0$ and $\sqrt{(w^2 + k_y z^2 + p)/k_z} = \beta_0$ if $w^2 + k_y z^2 + p < 0$. Thus, both $\beta_0$ and $\beta_n$ must be positive for any value of $w^2 + k_y z^2 + p$.

References


