Vertex-bipancyclicity of the generalized honeycomb tori

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Abstract

Assume that m, n and s are integers with \( m \geq 2, n \geq 4, 0 \leq s < n \) and s is of the same parity of m. The generalized honeycomb tori \( \text{GHT}(m, n, s) \) have been recognized as an attractive architecture to existing torus interconnection networks in parallel and distributed applications. A bipartite graph \( G \) is bipancyclic if it contains a cycle of every even length from 4 to \( |V(G)| \) inclusive. \( G \) is vertex-bipancyclic if for any vertex \( v \in V(G) \), there exists a cycle of every even length from 4 to \( |V(G)| \) that passes \( v \). A bipartite graph \( G \) is called k-vertex-bipancyclic if every vertex lies on a cycle of every even length from \( k \) to \( |V(G)| \).

In this article, we prove that \( \text{GHT}(m, n, s) \) is 6-bipancyclic, and is bipancyclic for some special cases. Since \( \text{GHT}(m, n, s) \) is vertex-transitive, the result implies that any vertex of \( \text{GHT}(m, n, s) \) lies on a cycle of length \( l \), where \( l \geq 6 \) and is even. Besides, \( \text{GHT}(m, n, s) \) is vertex-bipancyclic in some special cases. The result is optimal in the sense that the absence of cycles of certain lengths on some \( \text{GHT}(m, n, s) \)'s is inevitable due to their hexagonal structure.

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1. Introduction

Network topology is a crucial factor for an interconnection network since it determines the performance of the network [1]. Stojmenovic [2] proposed variations of honeycomb tori and those honeycomb tori have been recognized as an attractive architecture to existing torus interconnection networks in parallel and distributed applications [2–6]. Cho and Hsu [7] proved that all these honeycomb torus networks can be characterized in a unified way, called the generalized honeycomb torus. Recently, there have been many studies about honeycomb networks [8–13].

The cycle-embedding problem is one of the most popular research problems [14]. From the applicational point of view, efficient algorithms and execution methods are required for communication patterns in networks. The study of certain topological structures on network designs provides a systematic and logical analysis for the desired performance. Since cycles in networks are useful in embedding linear arrays and rings, the existence of cycles with various lengths on networks has been largely investigated. (See [15–18] and their references.) A graph \( G \) is pancyclic if it contains cycles of all lengths from 3 to \( |V(G)| \). A graph \( G \) is k-pancyclic if it contains cycles of all lengths from \( k \) to \( |V(G)| \). A graph \( G \) is called vertex-pancyclic (resp. edge-pancyclic) if every vertex (resp. edge) lies on a cycle of every length from 3 to \( |V(G)| \). Moreover, \( G \) is called k-vertex-pancyclic if every vertex lies on a cycle of every even length from \( k \) to \( |V(G)| \). These concepts are defined for bipartite graphs similarly. Let \( H = (B \cup W, E) \) be a bipartite graph, where \( B \cup W = V(H) \) and \( E \subseteq \{ (u, v) : u \in B \text{ and } v \in W \} \). Obviously, \( H \) has no odd cycles. We say that \( H \) is bipancyclic if it has cycles of all even lengths from 4 to \( |V(G)| \). \( H \) is k-bipancyclic if it contains cycles of all even lengths from \( k \) to \( |V(G)| \). \( H \) is called vertex-bipancyclic (resp. edge-bipancyclic) if every vertex

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Fig. 2 shows that crossing edges while the edges ((1, 0), (2, 0)) and ((5, 0), (6, 0)) are the horizontal edges while the edges ((10, 0), (7, 4)) and ((0, 8), (7, 2)) are crossing edges.

(resp. edge) lies on a cycle of every even length from 4 to $|V(G)|$. Also, $H$ is called $k$-vertex-bipancyclic if every vertex lies on a cycle of every even length from $k$ to $|V(G)|$. There are numerous studies about the pancyclicity of hypercubes and their variants [19,20], products of graphs [21–23], and some classes of graphs [24,25]. Vertex-pancyclicity and edge-pancyclicity were discussed in [26–29], and many related studies were published recently [30–36].

In this article, we prove that the generalized honeycomb tori are vertex-bipancyclic in some special cases and contain cycles of length $l$, where $l \geq 6$ is an even integer, in most cases. More specifically, let $GHT(m, n, s)$ be a generalized honeycomb torus, where $m \geq 3$ is an integer, $n \geq 6$ is an even integer, and $s \geq 0$ is an integer with $m + s$ even. We study the existence of cycles of different lengths in $GHT(m, n, s)$ with various combinations of $m$, $n$, and $s$. The result is optimal in the sense that the absence of cycles of certain lengths on some $GHT(m, n, s)$'s is inevitable due to their hexagonal structure.

2. Preliminaries

An interconnection network is represented by a graph with vertices and edges symbolizing the processors and communication links between processors, respectively. In this paper, a network is represented as an undirected graph. For the graph definition and notation we follow [37]. $G = (V, E)$ is a graph if $V$ is a finite set and $E$ is a subset of $\{(u, v) \mid (u, v)\}$ is an unordered pair of $V$. We say that $V$ is the vertex set and $E$ is the edge set of $G$. Two vertices $u$ and $v$ are adjacent if $(u, v) \in E$. A path is represented by $(v_0, v_1, v_2, \ldots, v_k)$. We also write the path $P_1 = (v_0, v_1, v_2, \ldots, v_k)$, where $P_1$ is the path $(v_0, v_1, \ldots, v_{i-1}, v_i)$ and $P_2$ is the path $(v_j, v_{j+1}, \ldots, v_k)$. Hence, it is possible to write a path $(v_0, v_1, v_2, v_3, \ldots, v_k)$ if the length of $P$ is zero. If a path $Q = (v_0, v_1, v_2, \ldots, v_k)$, then $Q^{-1}$ denotes the path $(v_k, v_{k-1}, \ldots, v_1, v_0)$. A cycle is a path of at least three vertices such that the first vertex and the last vertex are identical. Let $C$ be a cycle and $P$ a path. We use $|C|$ to denote the total number of distinct vertices/edges on $C$ and $|P|$ the total number of distinct edges of $P$.

Throughout this paper, we use the following notations. For any two positive integers $r$ and $d$, $[r]_d$ denotes $r \mod d$. Let $m$, $n$, and $s$ be positive integers with $m \geq 2$, $n \geq 4$, and $m + s$ are even. The generalized honeycomb torus $GHT(m, n, s)$ is the graph with the vertex set $\{(i, j) \mid 0 \leq i < m, 0 \leq j < n\}$ such that $(i, j)$ and $(k, l)$ with $i \leq k$ are adjacent if they satisfy one of the following conditions:

1. $(k, l) = (i, j) \pm 1_n$;
2. $0 \leq i \leq m - 2, i + j$ is odd and $(k, l) = (i + 1, j)$;
3. $i = 0, j$ is even, and $(k, l) = (m - 1, j + s)n$.

Any edge satisfying the second condition is called a horizontal edge, and any edge satisfying the third condition is called a crossing edge. Fig. 1 gives an illustration of the graph $GHT(8, 10, 4)$. For example, the edges $((1, 0), (2, 0))$ and $((5, 0), (6, 0))$ are the horizontal edges while the edges $((0, 0), (7, 4))$ and $((0, 8), (7, 2))$ are crossing edges. Fig. 2 shows that crossing edges can become horizontal edges in different layouts for the same graph. Obviously, any generalized honeycomb torus is a 3-regular bipartite graph. Moreover, any generalized honeycomb torus is vertex-transitive [7].

3. Bipancyclicity of $GHT(m, n, s)$

Let $C_i$ denote a cycle in $GHT(m, n, s)$ with $|C_i| = i$, where $i \in \{4, 6 + 2t \mid 0 \leq t \leq \frac{mn}{2} - 3\}$. Since $GHT(m, n, s)$ consists of hexagons, the existence of $C_4$ is missed in all cases except for $n = 4$. In addition, $GHT(m, n, s)$ contains $mn$ vertices, so
we should construct cycles of even lengths from 6 to \( mn \). Yang et al. [13] proved that every generalized honeycomb torus is Hamiltonian. Therefore, we only need to construct cycles of even lengths from 6 to \( mn - 2 \). That is, we should construct \( C_i \) in \( \text{GHT}(m, n, s) \), where \( i = 6 + 2t \) for \( 0 \leq t \leq \frac{mn}{4} - 4 \). Obviously, \( \{2t : 0 \leq t \leq \frac{mn}{4} - 4\} = \{4t : 0 \leq t \leq \frac{mn}{4} - 2\} \cup \{4t + 2 : 0 \leq t \leq \frac{mn}{4} - 3\} \). Thus, it suffices to construct \( C_{4+4t} \) for \( 0 \leq t \leq \frac{mn}{4} - 2 \) and \( C_{6+4t} \) for \( 0 \leq t \leq \frac{mn}{4} - 3 \) for the 6-bipancyclic property of \( \text{GHT}(m, n, s) \). In the following, some path patterns in generalized honeycomb tori are defined in order to construct \( C_i \) with various \( i \).

\[
\begin{align*}
I_t(i, j) &= \langle (i, j), (i, [j + 1]_n), (i, [j + 2]_n), \ldots, (i, [j + t - 1]_n), (i, [j + t]_n) \rangle, \quad t \in \mathbb{Z}; \\
Q_0(i, j) &= \langle (i, j), I_t(i, j), (i, [j + t]_n), (i + 1, [j + t]_n), I^{-1}(i + 1, [j + t]_n), (i + 1, j) \rangle, \quad t \in \mathbb{Z}; \\
Q_2(i, j) &= \langle (i, j), (i, [j - 1]_n), (i - 1, [j - 1]_n), (i - 1, j), (i - 2, j) \rangle; \\
P_0(i, j) &= \langle (i, j), (i, [j - 1]_n), (i - 1, [j - 1]_n) \rangle; \\
P_1(i, j) &= \langle (i, j), Q_0(i, j), (i + 1, j), Q_2(i + 1, j), (i + 1, j) \rangle, \quad t \in \mathbb{N}; \\
P_2(i, j) &= \langle (i, j), Q_0(i, j), (i + 1, j), (i + 2, j) \rangle, \quad t \in \mathbb{N}; \\
Q_6(i, j) &= \langle (i, j), (i, [j + 1]_n), (i, [j + 2]_n), \ldots, (i, [j + t - 1]_n), (i, [j + t]_n) \rangle.
\end{align*}
\]

3.1. \( m \) is even

**Lemma 1.** \( \text{GHT}(4, 4, s) \) is bipancyclic.

**Proof.** By brute force, we construct cycles with different lengths in \( \text{GHT}(4, 4, s) \) below.

\[
\begin{align*}
C_4 &= \langle (0, 1), I^4(0, 1), (0, 1) \rangle; \\
C_6 &= \langle (0, 1), (0, 2), P_0(0, 2), (0, 1) \rangle; \\
C_8 &= \langle (0, 0), P_1(0, 0), (2, 0), Q_0(2, 0), (3, 0), (0, 0) \rangle, \quad \text{if } s = 0; \\
or \quad C_8 &= \langle (0, 1), (0, 2), (3, 0), (3, 1), (2, 1), Q_1(2, 1), (0, 1) \rangle, \quad \text{if } s = 2; \\
C_{10} &= \langle (0, 1), (0, 2), P_2(0, 2), (2, 1), Q_1(2, 1), (0, 1) \rangle; \\
C_{12} &= \langle (0, 0), Q_0(0, 0), (1, 0), (Q_1(1, 0))^{-1}, (3, 0), (0, 0) \rangle, \quad \text{if } s = 0; \\
or \quad C_{12} &= \langle (0, 0), P_1(0, 0), (2, 0), (2, 1), (3, 1), (3, 2), (0, 0) \rangle, \quad \text{if } s = 2; \\
C_{14} &= \langle (0, 1), (0, 2), P_1(0, 2), (2, 2), P_0(2, 2), (2, 1), Q_1(2, 1), (0, 1) \rangle.
\end{align*}
\]

The lemma is proved. \( \square \)

**Lemma 2.** \( \text{GHT}(m, 4, s) \) contains \( C_l \) for \( l \in \{4, 2m + 4t : 0 \leq t \leq \frac{m}{2} - 1\} \cup \{6 + 4t : 0 \leq t \leq m - 2\} \), if \( m \geq 6 \).

**Proof.** It is obvious that \( \text{GHT}(m, n, s) \) consists of many hexagons and any two adjacent hexagons have two vertices in common. This structure implies the usage of a crossing edge in \( C_{4+4t} \) of \( \text{GHT}(m, 4, s) \). The smallest size of \( C_l \) with \( |l|_4 \equiv 0 \) is \( 2m \). By brute force, we construct cycles with various lengths in \( \text{GHT}(m, 4, s) \) as follows.

Fig. 2. The graph \( \text{GHT}(3, 8, 3) \). Notice that crossing edges in (a) can become horizontal edges in (b).
\[ C_4 = \langle (0, 0), I^4(0,0), (0,0) \rangle; \]
\[ C_{2m+4r} = \left\langle (0,0),(P_{1.1}(0,0))^t,(2t,0),(P_{1.0}(2t,0))^{m-1-t},(m-2,0),Q_0,1(m-2,0),(m-1,0), (0,0) \right\rangle \]
\[ \text{for } s = 0; \]
\[ C_{2m+4r} = \left\langle (0,0),(P_{1.1}(0,0))^t,(2t,0),(P_{1.0}(2t,0))^{m-1-t},(m-2,0),(m-2,1),(m-1,1),(m-1,2), (0,0) \right\rangle \]
\[ \text{for } s = 2; \]
\[ C_{6+4r} = \left\langle (0,1),(0,2),(P_{1.0}(0,2))^{\left\lfloor \frac{t}{2} \right\rfloor},(2\left\lfloor \frac{t}{2} \right\rfloor,2),P_{2\left\lfloor \frac{t}{2} \right\rfloor,0}\left(2\left\lfloor \frac{t}{2} \right\rfloor,2\right),\left(2\left\lfloor \frac{t}{2} \right\rfloor +2\left\lfloor t \right\rfloor,1\right),Q_1\left(2\left\lfloor \frac{t}{2} \right\rfloor +2\left\lfloor t \right\rfloor,1\right)\right\rangle. \]

This proves the lemma. \[ \square \]

**Lemma 3.** GHT(m, n, s) is 6-bipancyclic for \( m \geq 4, n \in \{6, 8\} \).

**Proof.** By brute force, we construct cycles with various lengths in GHT(m, n, s) for \( n \in \{6, 8\} \) as follows.

**Case 1.** For GHT(m, 6, s).

**Case 1.1.** \( C_{6+4r} \) for \( 0 \leq t \leq \frac{3}{2}m-2 \):

\[
\begin{align*}
C_{6+4r} &= \langle (0,1),(0,2),(P_{1.1}(0,2))^{\left\lfloor \frac{t}{2} \right\rfloor},(2\left\lfloor \frac{t}{2} \right\rfloor,2),\Delta, (2\left\lfloor \frac{t}{2} \right\rfloor,1),(Q_1(2\left\lfloor \frac{t}{2} \right\rfloor,1)),(0,1) \rangle, \text{ where } \Delta = P_{0,\left\lfloor t \right\rfloor}(2\left\lfloor \frac{t}{2} \right\rfloor,2) \text{ if } [t+1]_3 \neq 0 \text{ and is empty otherwise.}
\end{align*}
\]

**Case 1.2.** \( C_{8+4r} \) for \( 0 \leq t \leq \frac{3}{2}m-3 \):

\[
\begin{align*}
t &= 0 & \langle (0,5),I^4(0,5),(0,3),(1,3),(1,4),(1,5),(0,5) \rangle \\
t &= \{1,2,3\} & \langle (0,5),I^4(0,5),(0,3),(1,3),(1,2),(2,2),\Delta, (2,1),(Q_2(2,1))^2,(0,5) \rangle, \text{ where } \Delta \text{ is empty if } t = 1 \text{ and is } P_{0,\left\lfloor t \right\rfloor}(2,2) \text{ otherwise.}
\end{align*}
\]

**Case 2.** For GHT(m, 8, s).

**Case 2.1.** \( C_{6+4r} \) for \( 0 \leq t \leq 2m-2 \):

\[
\begin{align*}
C_{6+4r} &= \langle (0,1),(0,2),(P_{1.2}(0,2))^{\left\lfloor \frac{t}{2} \right\rfloor},(2\left\lfloor \frac{t}{2} \right\rfloor,2),\Delta, (2\left\lfloor \frac{t}{2} \right\rfloor,1),(Q_1(2\left\lfloor \frac{t}{2} \right\rfloor,1)),(0,1) \rangle, \text{ where } \Delta = P_{0,\left\lfloor t \right\rfloor}(2\left\lfloor \frac{t}{2} \right\rfloor,2) \text{ if } [t+1]_4 \neq 0 \text{ and is empty otherwise.}
\end{align*}
\]

**Case 2.2.** \( C_{8+4r} \) for \( 0 \leq t \leq 2m-3 \):

\[
\begin{align*}
t &= 0 & \langle (0,1),I^3(0,1) \rangle \\
t &= 1 & \langle (0,7),I^6(0,7),(0,5),(1,5),(1,2),(2,2),(2,1),(Q_2(2,1))^2,(0,7) \rangle \\
t &= \{2,3,4,5\} & \langle (0,7),I^6(0,7),(0,5),(1,5),I^{-3}(1,5),(1,2),(2,2),\Delta, (2,1),(Q_2(2,1))^2,(0,7) \rangle, \text{ where } \Delta \text{ is empty if } t = 2 \text{ and is } P_{0,\left\lfloor t \right\rfloor}(2,2) \text{ otherwise.}
\end{align*}
\]

\[
\begin{align*}
6 \leq t \leq 2m-3 & \langle (0,7),I^6(0,7),(0,5),(1,5),I^{-3}(1,5),(1,2),(2,2),(P_{1.2}(2,2))^{\left\lfloor \frac{t}{2} \right\rfloor},(2+2\left\lfloor \frac{t}{2} \right\rfloor,2),\Delta, (2+2\left\lfloor \frac{t}{2} \right\rfloor,1),(Q_1(2+2\left\lfloor \frac{t}{2} \right\rfloor,1)),(2,1),(Q_2(2,1))^2,(0,7) \rangle, \text{ where } \Delta = P_{0,\left\lfloor t \right\rfloor-3}(2+2\left\lfloor \frac{t}{2} \right\rfloor,2) \text{ if } [t-2]_4 \neq 0 \text{ and is empty otherwise.}
\end{align*}
\]

\[ \square \]

**Theorem 1.** Let \( n \geq 10 \) and \( s \geq 0 \) be even integers. GHT(4, n, s) is 6-bipancyclic if \( s \in \{0, 2, 4\} \). And GHT(4, n, s) contains any cycle with length \( l \) for \( l \in \{6, 10 + 2t|0 \leq t \leq 2n - 6| \} \) if \( s \geq 6 \). Moreover, there exists no 8-cycle in GHT(4, n, s) for \( s \geq 6 \).

**Proof.** The corresponding cycles are constructed below.

**Case 1.** \( s \in \{0, 2, 4\} \).

**Case 1.1.** \( C_{6+4r} \) for \( 0 \leq t \leq n-2 \):

...
By brute force, we construct cycles with different lengths in $GHT$.

**Lemma 4.**

**Proof.**

By brute force, we construct cycles with different lengths in $GHT(3,4,s)$ below.

- $C_4 = \langle (0,1), l^2(0,1), (0,1) \rangle$;
- $C_5 = \langle (0,1), (0,2), P_{0,0}(0,2), (0,1) \rangle$;
- $C_6 = \langle (0,0), l^3(0,0), (0,3), (1,3), (1,2), (2,2), (2,1), (0,0) \rangle$;
- $C_{10} = \langle (0,1), (0,2), P_{2,0}(0,2), (2,1), Q_2(2,1), (0,1) \rangle$.
Fig. 3. The smallest cycle with a crossing edge in $GHT(4, 12, 6)$ for $[l]_4 = 0$ is $C_{12}$.

Fig. 4. $C_{mn-4}$ of $GHT(m, n, s)$ maps to $C_{mn}$ of $GHT(m + 2, n, s)$ in Definition 1 for $m$ is even. Example: (a) $C_{44}$ in $GHT(4, 12, s)$; (b) $C_{48}$ in $GHT(6, 12, s)$. Note that the crossing edges are omitted in this figure, and the edges in $C_{44}$ ($C_{48}$, resp.) are plotted by thick lines.

Fig. 5. $C_{mn-2}$ of $GHT(m, n, s)$ maps to $C_{mn+2}$ of $GHT(m + 2, n, s)$ in Definition 1 for $m$ is even. Example: (a) $C_{46}$ in $GHT(4, 12, s)$; (b) $C_{50}$ in $GHT(6, 12, s)$. Note that the crossing edges are omitted in this figure, and the edges in $C_{46}$ ($C_{50}$, resp.) are plotted by thick lines.

**Lemma 5.** $GHT(m, 4, s)$ contains $C_l$ for $l \in \{4, 2m + 2 + 4t | 0 \leq t \leq \frac{m-3}{2}\} \cup \{6 + 4t | 0 \leq t \leq m - 2\}$, if $m \geq 5$. 
Fig. 6. $C_{mn}^{4}$ of GHT$(m, n, s)$ maps to $C_{mn}^{4}$ of GHT$(m + 2, n, s)$ in Definition 2 for $m$ is even. Example: (a) $C_{44}$ in GHT$(4, 12, s)$; (b) $C_{52}$ in GHT$(6, 12, s)$. Note that the crossing edges are omitted in this figure, and the edges in $C_{44}$ ($C_{52}$, resp.) are plotted by thick lines.

Fig. 7. $C_{mn}^{8}$ of GHT$(m, n, s)$ maps to $C_{mn}^{8}$ of GHT$(m + 2, n, s)$ in Definition 2 for $m$ is even. Example: (a) $C_{44}$ in GHT$(4, 12, s)$; (b) $C_{56}$ in GHT$(6, 12, s)$. Note that the crossing edges are omitted in this figure, and the edges in $C_{44}$ ($C_{56}$, resp.) are plotted by thick lines.

Fig. 8. $C_{mn}^{2}$ of GHT$(m, n, s)$ maps to $C_{mn}^{2}$ of GHT$(m + 2, n, s)$ in Definition 2 for $m$ is even. Example: (a) $C_{46}$ in GHT$(4, 12, s)$; (b) $C_{54}$ in GHT$(6, 12, s)$. Note that the crossing edges are omitted in this figure, and the edges in $C_{46}$ ($C_{54}$, resp.) are plotted by thick lines.

Proof. It is obvious that GHT$(m, n, s)$ consists of many hexagons and any two adjacent hexagons have two vertices in common. This structure implies the usage of a crossing edge in $C_{4+4\ell}$ of GHT$(m, 4, s)$. The smallest size of $C_{\ell}$ with $[\ell]_{4} \equiv 0$ is
By brute force, we construct cycles with various lengths in GHT(m, 4, s) as follows.

\[ C_4 = \langle (0, 0), l^4(0, 0), (0, 0) \rangle; \]
\[ C_{2m+2+4t} = \left\langle (0, 0), \left( P_{1,1}(0, 0) \right)^t, (2t, 0), \left( P_{1,0}(2t, 0) \right)^{m-1-t}, (m-1, 0), l^{-3}(m-1, 0), (m-1, 1), (0, 0) \right\rangle; \]
\[ C_{6+4t} = \left\langle (0, 1), (0, 2), \left( P_{1,0}(0, 2) \right)^{\left\lfloor \frac{t}{2} \right\rfloor}, \left( 2 \left\lfloor \frac{t}{2} \right\rfloor + 2 \right), P_{2\lfloor t \rfloor, 0}\left(2 \left\lfloor \frac{t}{2} \right\rfloor, 2\right), \left(2 \left\lfloor \frac{t}{2} \right\rfloor + 2 \left\lfloor \frac{t}{2} \right\rfloor + 1 \right) \right\rangle. \]

The lemma is proved. \[ \square \]

**Lemma 6.** GHT(m, n, s) is 6-bipancyclic for \( m \geq 3, n \in \{6, 8\} \).

**Proof.** By brute force, we construct cycles with various lengths in GHT(m, n, s) for \( n \in \{6, 8\} \) as follows.

**Case 1.** For GHT(m, 6, s).

**Case 1.1.** \( C_{6+4t} \) for \( 0 \leq t \leq \frac{3}{2}m - \frac{5}{2} \):

\[ C_{6+4t} = \langle (0, 1), (0, 2), \left( P_{1,1}(0, 2) \right)^{\left\lfloor \frac{t}{2} \right\rfloor} \left( 2 \left\lfloor \frac{t}{2} \right\rfloor + 2 \right), (2 \left\lfloor \frac{t}{2} \right\rfloor + 2, 3), \left( Q_{0, -2[t+1]_3 + 1}(2 \left\lfloor \frac{t}{2} \right\rfloor + 2, 3) \right)^{-1} \rangle \text{ if } [t+1]_3 \neq 0 \text{ and is empty otherwise.} \]

**Case 1.2.** \( C_{6+4t} \) for \( 0 \leq t \leq \frac{3}{2}m - \frac{5}{2} \):

**Case 2.** For GHT(m, 8, s).

**Case 2.1.** \( C_{6+4t} \) for \( 0 \leq t \leq 2m - 2 \):

**Fig. 9.** \( C_{mn-2} \) of GHT(m, n, s) maps to \( C_{mn+10} \) of GHT(m + 2, n, s) in Definition 2 for m is even. Example: (a) \( C_{46} \) in GHT(4, 12, s); (b) \( C_{58} \) in GHT(6, 12, s). Note that the crossing edges are omitted in this figure, and the edges in \( C_{46} \) (\( C_{58} \), resp.) are plotted by thick lines.

2. By brute force, we construct cycles with various lengths in GHT(m, 4, s) as follows.

\[ C_4 = \langle (0, 0), l^4(0, 0), (0, 0) \rangle; \]
\[ C_{2m+2+4t} = \left\langle (0, 0), \left( P_{1,1}(0, 0) \right)^t, (2t, 0), \left( P_{1,0}(2t, 0) \right)^{m-1-t}, (m-1, 0), l^{-3}(m-1, 0), (m-1, 1), (0, 0) \right\rangle; \]
\[ C_{6+4t} = \left\langle (0, 1), (0, 2), \left( P_{1,0}(0, 2) \right)^{\left\lfloor \frac{t}{2} \right\rfloor}, \left( 2 \left\lfloor \frac{t}{2} \right\rfloor + 2 \right), P_{2\lfloor t \rfloor, 0}\left(2 \left\lfloor \frac{t}{2} \right\rfloor, 2\right), \left(2 \left\lfloor \frac{t}{2} \right\rfloor + 2 \left\lfloor \frac{t}{2} \right\rfloor + 1 \right) \right\rangle. \]
Case 2.2. $C_{8+4t}$ for $0 \leq t \leq 2m - 3$:

<table>
<thead>
<tr>
<th>$t$</th>
<th>$C_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t = 0$</td>
<td>$\langle (0,1), I^3(0,1), (0,1) \rangle$</td>
</tr>
<tr>
<td>$t = {1,2}$</td>
<td>$\langle (0,1), I^3(0,1), (0,4), P_{2,(t-1)}(0,4), (2,3), (Q_2,2,3)^2, (0,1) \rangle$</td>
</tr>
<tr>
<td>$t = 3$</td>
<td>$\langle (0,1), (0,0), (0,7), (1,7), (1,6), (2,6), (2,5), Q_1(2,5), (0,5), (0,4), (0,3), (1,3), (1,2), (2,2), (2,1), Q_1(2,1), (0,1) \rangle$</td>
</tr>
<tr>
<td>$4 \leq t \leq 2m - 3$</td>
<td>$\langle (0,1), (0,0), (0,7), (Q_4(0,7))^{-1}, (2[\frac{1}{4}],7), (1+2[\frac{1}{4}],7), (1+2[\frac{1}{4}],6), \Delta, (1+2[\frac{1}{4}],5), (P_{1,-3}(1+2[\frac{1}{4}],5))^{-1}(\frac{1}{2}), (3,5), (2,5), Q_1(2,5), (0,5), (0,4), (0,3), (1,3), (1,2), (2,2), (2,1), Q_1(2,1), (0,1) \rangle$, where $\Delta$ is $(2+2[\frac{1}{4}],6), (2+2[\frac{1}{4}],5), (Q_0,-2[t]_{4+1}(2+2[\frac{1}{4}],5))^{-1}$ if $[t]_4 \neq 0$ and is empty otherwise.</td>
</tr>
</tbody>
</table>

\[ \square \]

**Theorem 3.** Let $n \geq 10$ be an even integer and $s \geq 1$ be an odd integer. GHT($3, n, s$) is 6-bipancyclic if $s \in \{1, 3, 5\}$. And GHT($3, n, s$) contains any cycle with length $l$ for $l \in \{6, 10 + 2t | 0 \leq t \leq \frac{n}{2} - 6\}$ if $s \geq 7$. Moreover, there exists no 8-cycle in GHT($3, n, s$) for $s \geq 7$.

**Proof.** The corresponding cycles are constructed below. $C_{l_1}$ of GHT($3, n, s$) for $l_1 \in \{6, 10 + 2t | 0 \leq t \leq \frac{n}{2} - 5\}$ is the same as in GHT($4, n, s$). And $C_{l_2}$ of GHT($3, n, s$) for $l_2 \in \{n + 2 + 2t | 0 \leq t \leq n - 2\}$ is constructed as follows.

Case 1. $s \in \{1, 3, 5\}$.

Case 1.1. $C_8$:

$C_8 = \langle (0,0), I^3(0,0), (0,3), (1,3), (1,2), (2,2), (2,1), (0,0) \rangle$, if $s = 1$;

$C_8 = \langle (0,0), I^3(0,0), (0,3), (Q_{n-1}(0,3))^{-1}, (2,3), (0,0) \rangle$, if $s = \{3, 5\}$.

Case 1.2. $C_{n+2+4t}$ for $0 \leq t \leq \frac{n}{2} - 1$:

<table>
<thead>
<tr>
<th>$t$</th>
<th>$C_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t = 0$</td>
<td>$\langle (0,0), (0, n - 1), (1, n - 1), (1, n - 2), (1, n - 3), (0, n - 3), I^{-(n-3)}(0, n - 3), (0,0) \rangle$</td>
</tr>
<tr>
<td>$1 \leq t \leq \frac{n}{2} - 1$</td>
<td>$\langle (0,0), (0, n - 1), (1, n - 1), (1, n - 2), (2, n - 2), (2, n - 3), (Q_0,-2[t]_{4+1}(2, n - 3))^{-1}, (1, n - 3), (0, n - 3), I^{-(n-3)}(0, n - 3), (0,0) \rangle$</td>
</tr>
</tbody>
</table>

Case 1.3. $C_{n+4+4t}$ for $0 \leq t \leq \frac{n}{2} - 2$:

<table>
<thead>
<tr>
<th>$t$</th>
<th>$C_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t = 0$</td>
<td>$\langle (0,0), (0, n - 1), (1, n - 1), (1, n - 2), (1, n - 3), (0, n - 3), (0, n - 4), (0, n - 5), (1, n - 5), (1, n - 6), (1, n - 7), (0, n - 7), I^{-(n-7)}(0, n - 7), (0,0) \rangle$</td>
</tr>
<tr>
<td>$1 \leq t \leq \frac{n}{2} - 2$</td>
<td>$\langle (0,0), (0, n - 1), (1, n - 1), (1, n - 2), (2, n - 2), (2, n - 3), Q_1(2, n - 3), (0, n - 3), (0, n - 4), (0, n - 5), (1, n - 5), (1, n - 6), \Delta, (1, n - 7), (0, n - 7), I^{-(n-7)}(0, n - 7), (0,0) \rangle$, where $\Delta$ is $(2, n - 6), (2, n - 7), (Q_0,-2[t]_{4+1}(2, n - 7))^{-1}$ if $t \neq 1$ and is empty otherwise.</td>
</tr>
</tbody>
</table>

Notice that the construction of all $C_i$’s except for $C_8$ in Case 1 contains no crossing edge of GHT($3, n, s$) for $s \in \{1, 3, 5\}$.

Case 2. $s \geq 7$.

It is obvious that GHT($m, n, s$) consists of many hexagons and any two adjacent hexagons have two vertices in common. Therefore, construction of cycles without crossing edges result in cycles with length $6 + 4t$, $0 \leq t \leq \frac{n}{2} - 2$, only. This implies that the usage of a crossing edge in $C_8$ is necessary. However, when $s \geq 7$, the smallest size of $C_{i}$ with $[t]_4 \equiv 0$ is $s + 3 + [s + 3]_2 \geq 12$, as shown in Fig. 10. Thus it is impossible to have $C_8$ in GHT($3, n, s$) for $s \geq 7$. On the other hand, we can construct $C_i$ for $l \in \{6, 10 + 2t | 0 \leq t \leq \frac{n}{2} - 6\}$ the same as in Case 1. $\square$

**Definition 3.** Let $f_0$ be a function that maps $((m - 1, n - 2), (m - 1, n - 3))$ in $C_{m-4}$ of GHT($m, n, s$) to $\langle (m - 1, n - 2), Q_0(1-m-1, n - 2), (m - 1, n - 3), (m - 1, n - 3) \rangle$ in $C_{m-4}$ of GHT($m + 2, n, s$) and maps $((m - 1, n - 2), (m - 1, n - 3))$ in $C_{m-2}$ of GHT($m, n, s$) to $\langle (m - 1, n - 2), Q_0(1-m-1, n - 2), (m - 1, n - 2), (m - 1, n - 3), (m - 1, n - 3) \rangle$ in $C_{m+2}$ of GHT($m + 2, n, s$). Examples are given in Figs. 11 and 12.
Fig. 10. The smallest cycle with a crossing edge in GHT(3, 14, 7) for $|l_4| = 0$ is $C_{12}$.

Fig. 11. $C_{mn-4}$ of GHT$(m, n, s)$ maps to $C_{mn}$ of GHT$(m+2, n, s)$ in Definition 3 for $m$ is odd. Example: (a) $C_{32}$ in GHT$(3, 12, s)$; (b) $C_{36}$ in GHT$(5, 12, s)$. Note that the crossing edges are omitted in this figure, and the edges in $C_{32}$ ($C_{36}$, resp.) are plotted by thick lines.

Fig. 12. $C_{mn-2}$ of GHT$(m, n, s)$ maps to $C_{mn+2}$ of GHT$(m+2, n, s)$ in Definition 3 for $m$ is odd. Example: (a) $C_{34}$ in GHT$(3, 12, s)$; (b) $C_{38}$ in GHT$(5, 12, s)$. Note that the crossing edges are omitted in this figure, and the edges in $C_{34}$ ($C_{38}$, resp.) are plotted by thick lines.

Definition 4. Let $g_0(k)$ be a function that maps ($(m - 1, n - 2), (m - 1, n - 3)$) in $C_{mn-4}$ of GHT$(m, n, s)$ to $((m - 1, n - 2), Q_{0,1}(m - 1, n - 2), (m, n - 2), (m + 1, n - 2), (m + 1, n - 3), (Q_{0,1}^{-1}(m + 1, n - 3))^{-1}, (m, n - 3), (m - 1, n - 3))$ in $C_{mn+4+4k}$ of GHT$(m+2, n, s)$ and maps ($(m - 1, n - 2), (m - 1, n - 3)$) in $C_{mn-2}$ of GHT$(m, n, s)$ to $((m - 1, n - 2), Q_{0,1}(m-1, n-2), (m, n-2), (m+1, n-2), (m+1, n-3), (Q_{0,1}^{-1}(m+1, n-3))^{-1}, (m, n-3), (m-1, n-3))$ in $C_{mn+4+4k}$ of GHT$(m+2, n, s)$.
Theorem 3. Let $m$ be odd integers and $n \geq 10$ be an even integer. $GHT(m, n, s)$ contains $6$-cycle and all cycles with lengths $l$ where $l = 10 + 2t$ for $0 \leq t \leq \frac{mn}{2} - 6$. Moreover, there exists no 8-cycle in $GHT(m, n, s)$.

Proof. We prove the theorem by the mathematical induction. For $GHT(5, n, s)$ we can construct $C_l$ for $l \in \{6, 10+2t|0 \leq t \leq \frac{5n}{2}-6\}$ the same as in Case 1 of Theorem 3 because there involves no crossing edge in those cycles. Then with Definitions 3 and 4, we construct $C_k$ of $GHT(5, n, s)$ for $k \in \{10+2t|\frac{5n}{2}-5 \leq t \leq \frac{5n}{2}-6\}$ by using $f_{50}$ and $g_{60}$. Using the induction hypothesis, we assume that $GHT(m, n, s)$ contains any cycle with length $l$ for $l \in \{6, 10+2t|0 \leq t \leq \frac{mn}{2} - 6\}$. Obviously, $GHT(m+2, n, s)$ contains the same $C_0, C_{10}, C_{12}, \ldots, C_{mn-4}, C_{mn-2}$ as in $GHT(m, n, s)$ since $C_l \in \{6, 10+2t|0 \leq t \leq \frac{mn}{2} - 6\}$, contains no crossing edge in $GHT(m, n, s)$. Then with Definitions 3 and 4, we construct $C_{mn+2t}$ of $GHT(m+2, n, s)$ for $0 \leq t \leq n-1$ by using $f_{50}, g_{60}$.

By induction, we know that $GHT(m, n, s)$ contains 6-cycle and all cycles with lengths $l$ for $l \in \{10+2t|0 \leq t \leq \frac{mn}{2} - 6\}$. Moreover, for the same reason as in Case 2 of Theorem 3, there exists no 8-cycle in $GHT(m, n, s)$ for odd $m \geq 5$, even $n \geq 10$ and odd $s \geq 1$. \hfill \Box

4. Conclusion

In this article, we study the vertex-bipancyclic of the generalized honeycomb tori. In Section 3, we prove that $GHT(m, n, s)$ is 6-bipancyclic, and is bipancyclic for some special cases. Moreover, some $GHT(m, n, s)$ contains cycles with length $l$ for any even integer $l \geq 6$ except 8 due to its hexagonal structure. Since $GHT(m, n, s)$ is vertex-transitive, our
theorems in Section 3 imply that given any vertex \( v \) of \( \text{GHT}(m, n, s) \), there exists a cycle with the required lengths that contains \( v \). The results are summarized in the following tables and are shown to be optimal in the sense that the absence of cycles of certain lengths on some \( \text{GHT}(m, n, s) \)'s is inevitable due to their hexagonal structure. Let \( G \) be \( \text{GHT}(m, n, s) \).

When \( m \) is even:

<table>
<thead>
<tr>
<th>( n ) ( m )</th>
<th>( m = 4 )</th>
<th>( m \geq 6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n = 4 )</td>
<td>Lemma 1. ( G ) is vertex-bipancyclic.</td>
<td>Lemma 2. ( G ) contains ( C_l ) for ( l \in {4, 2m + 4t, 6 + 4t</td>
</tr>
<tr>
<td>( n = 6, 8 )</td>
<td>Lemma 3. ( G ) is 6-vertex-bipancyclic.</td>
<td>Lemma 3. ( G ) is 6-vertex-bipancyclic.</td>
</tr>
<tr>
<td>( n \geq 10 )</td>
<td>Theorem 1. ( G ) is 6-vertex-bipancyclic for ( s \in {0, 2, 4}; G ) contains a 6-cycle and is 10-vertex-bipancyclic for ( s \geq 6 ).</td>
<td>Theorem 2. ( G ) contains a 6-cycle and is 10-vertex-bipancyclic.</td>
</tr>
</tbody>
</table>

When \( m \) is odd:

<table>
<thead>
<tr>
<th>( n ) ( m )</th>
<th>( m = 3 )</th>
<th>( m \geq 5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n = 4 )</td>
<td>Lemma 4. ( G ) is vertex-bipancyclic.</td>
<td>Lemma 5. ( G ) contains ( C_l ) for ( l \in {4, 2m + 2 + 4t, 6 + 4t</td>
</tr>
<tr>
<td>( n = 6, 8 )</td>
<td>Lemma 6. ( G ) is 6-vertex-bipancyclic.</td>
<td>Lemma 6. ( G ) is 6-vertex-bipancyclic.</td>
</tr>
<tr>
<td>( n \geq 10 )</td>
<td>Theorem 3. ( G ) is 6-vertex-bipancyclic for ( s \in {1, 3, 5}; G ) contains a 6-cycle and is 10-vertex-bipancyclic for ( s \geq 7 ).</td>
<td>Theorem 4. ( G ) contains a 6-cycle and is 10-vertex-bipancyclic.</td>
</tr>
</tbody>
</table>
References