On the bipanpositionable bipanconnectedness of hypercubes

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\textbf{A B S T R A C T}

A bipartite graph $G$ is bipanconnected if, for any two distinct vertices $x$ and $y$ of $G$, it contains an $[x, y]$-path of length $l$ for each integer $l$ satisfying $d_G(x, y) \leq l \leq |V(G)| - 1$ and $2|l - d_G(x, y)|$, where $d_G(x, y)$ denotes the distance between vertices $x$ and $y$ in $G$ and $V(G)$ denotes the vertex set of $G$. We say a bipartite graph $G$ is bipanpositionably bipanconnected if, for any two distinct vertices $x$ and $y$ of $G$ and for any vertex $z \in V(G) - \{x, y\}$, it contains a path $P_{l,k}$ of length $l$ such that $x$ is the beginning vertex of $P_{l,k}$, $z$ is the $(k + 1)$-th vertex of $P_{l,k}$, and $y$ is the ending vertex of $P_{l,k}$ for each integer $l$ satisfying $d_G(x, z) + d_G(y, z) \leq l \leq |V(G)| - 1$ and $2|l - d_G(x, z) - d_G(y, z)|$ and for each integer $k$ satisfying $d_G(x, z) \leq k \leq l - d_G(y, z)$ and $2|k - d_G(x, z)|$. In this paper, we prove that an $n$-cube is bipanpositionably bipanconnected if $n \geq 4$. As a consequence, many properties of hypercubes, such as bipancyclicity, bipanconnectedness, bipanpositionable Hamiltonicity, etc., follow directly from our result.

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1. Introduction

In many parallel and distributed computer systems, processors are connected on the basis of interconnection networks. Thus, the interconnection network has been a critical factor affecting the system performance and is widely addressed in the researches \cite{4,5,9,12,18,20}. In this paper, the topological structure of an interconnection network is modeled as a loopless undirected graph in the aspect of network analysis. For the graph definitions and notations, we follow the ones given by Bondy and Murty \cite{3}. A graph $G$ consists of a vertex set $V(G)$ and an edge set $E(G)$ that is a subset of $\{(u, v) \mid (u, v) \text{ is an unordered pair of } V(G)\}$. Two vertices $u$ and $v$ of $G$ are adjacent if $(u, v) \in E(G)$. A graph $G$ is bipartite if its vertex set can be partitioned into two disjoint partite sets $V_0(G)$ and $V_1(G)$ such that every edge joins a vertex of $V_0(G)$ and a vertex of $V_1(G)$.

A graph $H$ is a subgraph of a graph $G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A path $P$ of length $k$ from vertex $x$ to vertex $y$ in graph $G$ is a sequence of distinct vertices $(v_1, v_2, \ldots, v_k)$ such that $v_1 = x$, $v_k = y$, and $(v_i, v_{i+1}) \in E(G)$ for every $1 \leq i \leq k$ if $k \geq 1$. Moreover, a path of length zero from vertex $x$ is denoted by $(x)$. For convenience, we write $P$ as $\langle v_1, v_2, \ldots, v_k \rangle$, where $Q = \langle v_1, \ldots, v_i \rangle$. The $i$-th vertex of $P$ is denoted by $P(i)$; i.e., $P(i) = v_i$. To emphasize the beginning and ending vertices of $P$, we call $P$ an $[x, y]$-path. We use $\ell(P)$ to denote the length of $P$. The distance between two distinct vertices $u$ and $v$ of $G$, denoted by $d_G(u, v)$, is the length of the shortest path between $u$ and $v$. A cycle is a path with at least three vertices such that the last vertex is adjacent to the first one. For clarity, a cycle of length $k$ is represented by $\langle v_1, v_2, \ldots, v_k, v_1 \rangle$. A path (or cycle) in a graph $G$ is a Hamiltonian path (or Hamiltonian cycle) of $G$ if it spans $G$. A bipartite graph is Hamiltonian laceable \cite{16} if there is a Hamiltonian path between any two vertices that are in different partite sets. A Hamiltonian laceable graph $G$ is hyper-Hamiltonian laceable \cite{10} if, for $i \in \{0, 1\}$ and for any vertex $v \in V_i(G)$, there is a Hamiltonian path of $G - \{v\}$ between any two vertices of $V_{1-i}(G)$.

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A graph $G$ is pancyclic [2] if it contains a cycle of length $l$ for each integer $l$ from 3 to $|V(G)|$ inclusive. Since there is no odd cycle in any bipartite graph, Mitchem and Schmeichel [14] defined the bipancyclicity for bipartite graphs. A bipartite graph $G$ is bipancyclic if it contains cycles of all even lengths from 4 to $|V(G)|$ inclusive. On the other hand, a graph $G$ is said to be $\pi$-connected [1] if, for any two distinct vertices $x$ and $y$, it has an $[x, y]$-path of length $l$ for each $d_G(x, y) \leq l \leq |V(G)| - 1$. Obviously, every $\pi$-connected graph is pancyclic. Moreover, it is easy to see that any bipartite graph with at least three vertices is not $\pi$-connected. Therefore, the concept of bipanconnected graphs was proposed. A bipartite graph $G$ is bipanconnected if, for any two distinct vertices $x$ and $y$ of $G$, it contains an $[x, y]$-path of length $l$ for each integer $l$ satisfying both $d_G(x, y) \leq l \leq |V(G)| - 1$ and $2(l - d_G(x, y))$.

A graph $G$ is panpositionally Hamiltonian [7] if, for any two distinct vertices $x$ and $y$ of $G$, it contains a Hamiltonian cycle $C$ such that $d_G(x, y) = k$ for any integer $k$ satisfying $d_G(x, y) \leq k \leq |V(G)|/2$. Recently, Teng et al. [17] studied the panpositionally Hamiltonian of the arrangement graphs. In contrast, a bipartite graph $G$ is bipanpositionally Hamiltonian [7] if, for any two distinct vertices $x$ and $y$ of $G$, it has a Hamiltonian cycle $C$ such that $d_G(x, y) = k$ for any integer $k$ satisfying both $d_G(x, y) \leq k \leq |V(G)|/2$ and $2(l - d_G(x, y))$. In this paper, we further define a property for bipartite graphs. We say a bipartite graph $G$ is relay-bipanpositionable between two distinct vertices $x$ and $y$ if, for any vertex $z \in V(G) - \{x, y\}$, it contains an $[x, y]$-path $P_{i,k}$ of length $l$ such that $P_{i,k}(1) = x$, $P_{i,k}(k + 1) = y$, and $P_{i,k}(l + 1) = y$ for each integer $l$ satisfying both $d_G(x, z) + d_G(y, z) \leq l \leq |V(G)| - 1$ and $2(l - d_G(x, z) - d_G(y, z))$ for each integer $k$ satisfying both $d_G(x, z) \leq k \leq l - d_G(y, z)$ and $2(l - d_G(x, z))$. Then a bipartite graph $G$ is said to be bipanpositionally bipanconnected if it is relay-bipanpositionable between every pair of distinct vertices.

The hypercube is an attracting underlying network topology for parallel systems [9,19]. For clarity, we use boldface letters to denote $n$-bit binary strings. Let $u = b_{n-1} \ldots b_1 \ldots b_0$ be an $n$-bit binary string. For any $0 \leq i \leq n - 1$, we use $(u)^i$ to denote the binary string $b_{n-1} \ldots b_i \ldots b_0$. Moreover, we use $(u)$ to denote the bit $b_i$ of $u$. The Hamming weight of $u$, denoted by $w_H(u)$, is defined as $|\{0 \leq j \leq n - 1 : (u)^j = 1\}|$. The $n$-dimensional hypercube (or $n$-cube for short) $Q_n$ consists of $2^n$ vertices and $2^{n-1}$ edges. Each vertex corresponds to an $n$-bit binary string. Two vertices $u$ and $v$ are adjacent if and only if $v = (u)^i$ for some $i$ and we call the edge $(u, (u)^i)$ an i-dimensional edge. The Hamming distance between $u$ and $v$, denoted by $h(u, v)$, is defined to be $|\{0 \leq j \leq n - 1 : (u)^j \neq (v)^j\}|$. Hence, two vertices $u$ and $v$ are adjacent if and only if $h(u, v) = 1$.

Clearly, $d_{Q_n}(u, v) = h(u, v)$, and $Q_n$ is a bipartite graph with partite sets $V(Q_n) = \{v \in V(Q_n) : h(u, v) \text{ is even}\}$ and $V(Q_n) = \{v \in V(Q_n) : h(u, v) \text{ is odd}\}$. Moreover, $Q_n$ is vertex-transitive and edge-transitive [9]. It was proved that $Q_n$, $n \geq 2$, is panpositionally Hamiltonian if $n \geq 4$. Recently, there are several interesting studies on hypercubes [6,12,13]. In this paper, we are going to prove that $Q_n$ is bipanpositionally bipanconnected if $n \geq 4$. As an immediate consequence, many other properties of hypercubes, such as bipancyclicity, bipanconnectedness, bipanpositionally Hamiltonian, etc., follow from our result.

2. Preliminaries

Obviously, $Q_2$ is not only bipanconnected but also bipanpositionally bipanconnected. It is easy to see that $Q_3$ has no $[000, 011]$-paths $P$ of length six such that $P(4) = 001$. Hence, $Q_3$ is not relay-bipanpositionable between vertices 000 and 011. It is noticed that vertices 000 and 011 are in the same partite set of $Q_3$. However, we can show that $Q_3$ is relay-bipanpositionable between every two vertices in different partite sets.

**Lemma 1.** The 3-cube $Q_3$ is relay-bipanpositionable between every two vertices in different partite sets.

**Proof.** Let $x \in V_0(Q_3)$ and $y \in V_1(Q_3)$. Without loss of generality, we suppose that $z \in V_0(Q_3) - \{x\}$. Since $Q_3$ is vertex-transitive, we can assume that $x = 000$. Hence, we have $z \in \{011, 101, 110\}$ and $y \in \{001, 010, 100, 111\}$. Since $Q_3$ is edge-transitive, we only consider the case that $y \in \{001, 111\}$. We list all the required $[x, y]$-paths obtained by brute force in Table 1.

The following lemma shows that the relay-bipanpositionability between every two vertices in different partite sets of $Q_n$ implies the bipanconnectedness of $Q_n$.

**Lemma 2.** Suppose that the $n$-cube $Q_n$, $n \geq 2$, is relay-bipanpositionable between every two vertices in different partite sets. Then $Q_n$ is bipanconnected.

**Proof.** Let $e = 0^n$ and $v \in V(Q_n) - \{e\}$. Since $Q_n$ is vertex-transitive, we only concern the paths between $e$ and $v$.

**Case 1:** Suppose that $v \in V_1(Q_n)$. Let $i$ be an integer of $\{0, 1, \ldots, n - 1\}$ with $(e)^i \neq (v)$ and let $j$ be an integer of $\{0, 1, \ldots, n - 1\} - \{i\}$. We set $w$ to be $(v)^i$ if $e \neq (v)^i$ and set $w$ to be $(v)^i$ if $e = (v)^i$. Hence we have $d_{Q_n}(w, v) = 1$. Since $Q_n$ is relay-bipanpositionable between any two vertices in different partite sets, it has an $[e, v]$-path $P$ of length $l$ such that $P(1) = e$, $P(l) = v$, and $P(l + 1) = (v)^i$ for any odd integer $l$ from $d_{Q_n}(e, w) + 1$ to $2^n - 1$ inclusive. If $e \neq (v)^i$, then we have $d_{Q_n}(e, w) + 1 = d_{Q_n}(e, v)$. Otherwise, we have $d_{Q_n}(e, w) + 1 = 3$. Thus, $Q_n$ has an $[e, v]$-path of any odd length from $d_{Q_n}(e, v)$ to $2^n - 1$.

**Case 2:** Suppose that $v \in V_0(Q_n)$. Let $k$ be any integer of $\{0, 1, \ldots, n - 1\}$. Obviously $Q_n$ is relay-bipanpositionable between $e$ and $(v)^k$ because $e$ and $(v)^k$ belong to different partite sets of $Q_n$. Hence, $Q_n$ has an $[e, (v)^k]$-path $P$ of length $l$ such that $P(1) = e$, $P(l) = v$, and $P(l + 1) = (v)^k$ for any odd integer $l$ from $d_{Q_n}(e, v) + 1$ to $2^n - 1$ inclusive. For convenience, path $P$ can be written as $(e, P', (v)^k)$, where $P'$ is an $[e, v]$-path of length $l - 1$. Clearly $Q_n$ has an $[e, v]$-path of any even length in the range from $d_{Q_n}(e, v)$ to $2^n - 2$.

In summary, $Q_n$ is bipanconnected.
Two paths $P_1$ and $P_2$ are vertex-disjoint if $V(P_1) \cap V(P_2) = \emptyset$. Let $Q_n$, $i \in \{0, 1\}$, be the subgraph of $Q_n$ induced by $\{u \in V(Q_n) \mid (u)_{n-i} = i\}$. Obviously, $Q_n^i$ is isomorphic to $Q_{n-1}$. Then we can prove the following lemma.

**Lemma 3.** Suppose that an $n$-cube $Q_n$ is bipanconnected if $n \geq 2$. Let $(w_1, b_1)$ and $(w_2, b_2)$ be two vertex-disjoint edges of $Q_n$ such that $\{w_2, b_2\} \cap \{(w_1)^i, (w_1)^i, (w_1)^i, (w_1)^i, (b_1)^i, (b_1)^i\} \neq \emptyset$ if $n = 3$. For each even integer $l$ from 2 to $2^n - 2$ and for each odd integer $k$ from 1 to $l - 1$, $Q_n$ has two vertex-disjoint paths $S_{l,k}^{(1)}$ and $S_{l,k}^{(2)}$ such that $S_{l,k}^{(1)}$ is a $(w_1, b_1)$-path of length $k$ and $S_{l,k}^{(2)}$ is a $(w_2, b_2)$-path of length $l - k$.

**Proof.** We claim that $Q_n$ can be partitioned along some dimension in such a way that $(w_1, b_1)$ and $(w_2, b_2)$ are located on different subcubes. Without loss of generality, we assume that $w_1, w_2 \in V_0(Q_n)$ and $b_1, b_2 \in V_1(Q_n)$. Moreover, we assume that $h(w_1, w_2) = \max(h(w_1, w_2), h(b_1, b_2))$. Let $(w_1, b_1)$ be an $i_1$-dimensional edge and $(w_2, b_2)$ be an $i_2$-dimensional edge. If $|\{0, 1, \ldots, n-1\} \setminus \{i_1, i_2\}| + h(w_1, w_2) > n + 1$, then there exists an integer $i \in [0 \leq i \leq n - 1 \mid (w_i) = (w_2) = 2)$. Moreover, we assume that $|\{0, 1, \ldots, n-1\} \setminus \{i_1, i_2\}| + h(w_1, w_2) = n$. Thus, we have $i_1 \neq i_2$ and $(w_i, w_2) = 2$. For convenience, let $(j_1, j_2) = \{0 \leq j \leq n - 1 \mid (w_j) = (w_1)\}$. Clearly we have $(j_1, j_2) - (i_1, i_2) \neq \emptyset$. Let $i$ be any integer of $(j_1, j_2) - (i_1, i_2)$. Then we partition $Q_n$ along dimension $i$ so that $(w_1, b_1)$ and $(w_2, b_2)$ are located on different subcubes.

Since $Q_n$ is edge-transitive, we assume that $i = n - 1$. Without loss of generality, we assume that $(w_1, b_1) \in E(Q_n^0)$ and $(w_2, b_2) \in E(Q_n^1)$.

This lemma can be proved by induction on $n$. First of all, the result is trivial for $n = 2$. When $n = 3$, let $x = (w_1)^2$, $y = (b_1)^2$, and $\{u, v, x, y\} = V(Q_3)$. See Fig. 1(a) for illustration. Since $(w_2, b_2) \cap \{(w_1)^1, (w_1)^1, (w_1)^1, (w_1)^1, (b_1)^1, (b_1)^1\} \neq \emptyset$, we have $(w_2, b_2) \in \{(u, x), (x, y), (y, v)\}$. Then it is easy to see that there are two vertex-disjoint paths $S_{l,k}^{(1)}$ and $S_{l,k}^{(2)}$ in $Q_3$ such that $S_{l,k}^{(1)}$ is a $(w_1, b_1)$-path of length $k$ and $S_{l,k}^{(2)}$ is a $(w_2, b_2)$-path of length $l - k$ for any $l \in \{2, 4, 6\}$ and for any odd integer $k$ from 1 to $l - 1$.

As the inductive hypothesis, we suppose that the result is true for $Q_{n-1}$, $n \geq 4$. Obviously, at least one of $k$ and $l - k$ is less than $2^{n-1}$. By symmetry, we only consider the case that $1 \leq k \leq 2^{n-1} - 1$. Then we distinguish the following two cases.

**Case 1:** Suppose that $k \leq 2^{n-1} - 3$. Since $Q_{n,m}, m \leq n - 1$, is supposed to be bipanconnected, $Q_{n,m}^0$ has a $(w_2, b_2)$-path $R_k$ of length $r$ for each odd integer $r$ from $1$ to $2^{n-1} - 1$. Let $\tilde{r} = 2^{n-1} - 1$ and let $A = \{(R_i, (i + 1)| 1 \leq i \leq \tilde{r} \text{ and } i \equiv 1 \pmod{2}\}$. Since $|A| = 2^{n-2} > 3$ for $n = 4$, there exists an odd integer $\tilde{i} \leq \tilde{r}$, such that $\{(R_{\tilde{i}}(i))^{n-1}, (R_{\tilde{i}}(i + 1))^{n-1}\} \neq \emptyset$ and $(R_{\tilde{i}}(i))^{n-1}, (R_{\tilde{i}}(i + 1))^{n-1} \cap \{(w_1)^1, (w_1)^1, (w_1)^1, (b_1)^1, (b_1)^1\} \neq \emptyset$ if $n = 4$. Since $|A| = 2^{n-2} > 7$ for $n \geq 5$, there exists an odd integer $\tilde{i} \leq \tilde{r}$, such that $\{(R_{\tilde{i}}(i))^{n-1}, (R_{\tilde{i}}(i + 1))^{n-1} \cap \{w_1, b_1\} = \emptyset$ if $n \geq 5$. For convenience, let $u = R_{\tilde{i}}(i)$ and $v = R_{\tilde{i}}(i + 1)$. Thus, $R_{\tilde{i}}$ can be written as $(w_2, R_{\tilde{i}}, u, v, R_{\tilde{i}}, b_2)$. By the inductive hypothesis, $Q_{n-1}^0$ has two vertex-disjoint paths $p_{\tilde{i},q}$ and $\tilde{p}_{\tilde{i},q}$ such that $p_{\tilde{i},q}$ is a $(w_1, b_1)$-path of length $q$ and $\tilde{p}_{\tilde{i},q}$ is a $(u)^{n-1}, (v)^ {n-1}$-path of length $p - q$ for any even integer $p$ satisfying $2 \leq p \leq 2^{n-1} - 2$ and for any odd integer $q$ satisfying $1 \leq q \leq p - 1$. Then we set $S_{l,k}^{(1)} = p_{\tilde{i},q}$ and $S_{l,k}^{(2)} = R_{\tilde{i}-k}$ if $l - k \leq 2^{n-1} - 1$ (See Fig. 1(b)), we set $S_{l,k}^{(1)} = \tilde{p}_{\tilde{i},q}$ and $S_{l,k}^{(2)} = (w_2, R_{\tilde{i}-k}, u, (u)^{n-1}, (b_1)^{n-1}, (v)^{n-1}, (v, R_{\tilde{i}}, b_2)$ if $l - k \geq 2^{n-1} - 1$ (See Fig. 1(c)).

**Case 2:** Suppose that $k = 2^{n-1} - 1$. Since $Q_{n-1}$ is bipanconnected, $Q_{n-1}$ has a $(w_2, b_2)$-path $R_k$ of length $r$ for each odd integer $r$ from 1 to $2^{n-1} - 1$. Similarly, $Q_{n-1}^0$ has a $(w_1, b_1)$-path $H$ of length $2^{n-1} - 1$. Then we set $S_{l,k}^{(1)} = H$ and $S_{l,k}^{(2)} = R_{l-k}$. See Fig. 1(d).

Therefore, the proof is completed. □
Lemma 5. \[ \square \]

Since \( d \leq 2^{n-1} - 1 \) and \( 1 \leq l - k \leq 2^{n-1} - 1 \), the result is trivial for \( k \)

\[
\text{A bipanconnected graph } G \text{ is hyper-bipanconnected if, for any vertex } w \in V_i(G) \text{ (} i \in \{0, 1\} \text{) and for any two distinct vertices } u \text{ and } v \text{ of } V_i(G), G - \{w\} \text{ has a } [u, v]\text{-path of length } l \text{ for any even integer } l \text{ ranging from } d_G(u, v) \text{ to } |V(G)| - 2 \text{ inclusive.}
\]

**Lemma 4.** Suppose that the \( n \)-cube \( Q_n, n \geq 2 \), is relay-bipanpositionable between every two vertices in different partite sets. Then \( Q_n \) is hyper-bipanconnected.

**Proof.** The result is trivial for \( n = 2 \). In what follows, we consider the case that \( n \geq 3 \). Since \( Q_n \) is relay-bipanpositionable between every two vertices in different partite sets, **Lemma 2** ensures that \( Q_n \) is bipanconnected. Hence, we only concern the paths between every pair of distinct vertices in the same partite set. Let \( u \) and \( v \) be any two distinct vertices of \( V_i(Q_n) \) and let \( w \) be any vertex of \( V_{i-1}(Q_n) \) for some \( i \in \{0, 1\} \). Then we have to show that \( Q_n - \{w\} \) has \([u, v]\)-paths of all possible lengths. Since \( Q_n \) is relay-bipanpositionable between every two vertices in different partite sets, it has a \([u, w]\)-path \( P_k \) such that \( P_k(1) = u, P_k(k + 1) = v \), and \( d_{Q_n}(w, v) = d_{Q_n}(u, v) \) for each even integer \( k \) satisfying \( d_{Q_n}(u, v) \leq k \leq 2^n - d_{Q_n}(w, v) \). Thus, for clarity, we can write \( P_k \) as \( (u, l_k, v, j_k, w) \), where \( l_k \) is a \([u, v]\)-path of length \( k \) and \( j_k \) is some shortest path between \( v \) and \( w \). That is, \( Q_n - \{w\} \) has \([u, v]\)-paths of all lengths in the range from \( d_{Q_n}(u, v) \) to \( 2^n - d_{Q_n}(w, v) \). See **Fig. 2(a).**

The \([u, v]\)-paths of lengths greater than \( 2^n - d_{Q_n}(w, v) - 1 \) can be constructed as follows. Since \( Q_n \) is edge-transitive, we can assume that \((u)_{n-1} \neq (v)_{n-1} \). Thus, we can partition \( Q_n \) into \( Q_n^u \) and \( Q_n^v \) in such a way that \( u \) and \( v \) are located on different subcubes. Without loss of generality, we assume that \( u, w \in V(Q_n^u) \) and \( v \in V(Q_n^v) \). Let \( j \) be an integer of \( \{0, 1, \ldots, n - 2\} \) with \((w)_j \neq u \). Since \( Q_{n-1} \) is relay-bipanpositionable between every two vertices in different partite sets, \( Q_n^u \) has a \([u, w]\)-path \( T_k \) such that \( T_k(1) = u, T_k(k + 1) = (w)_j \), and \( d_{Q_n}(w, (w)_j) = 1 \) for each even integer \( k \) satisfying \( d_{Q_{n-1}}(u, (w)_j) \leq k \leq 2^{n-1} - 2 \). Thus, we can write \( T_k \) as \( (u, R_k, (w)_j, w) \), where \( R_k \) is a \([u, (w)_j]\)-path of length \( k \). By **Lemma 2**, \( Q_{n-1} \) is bipanconnected. Accordingly, \( Q_n^u \) has a \([[w]_j]^{n-1}, v]\)-path \( H \) of length \( 2^{n-1} - 1 \). Then \( (u, R_k, (w)_j, [[w]_j]^{n-1}, v, H, v) \) turns out to be a \([u, v]\)-path of length \( 2^{n-1} + k \). See **Fig. 2(b).** Obviously, we have \( 2^{n-1} + d_{Q_n}(u, (w)_j) \leq 2^{n-1} + k \leq 2^n - 2 \). Since \( 2^{n-1} + d_{Q_n}(u, (w)_j) \leq 2^{n-1} + n - 1 \) and \( (2^n - d_{Q_n}(w, v) - 1) + 2 \geq (2^n - n - 1) + 2 = 2^n - n + 1 \geq 2^{n-1} + n - 1 \), we have\( 2^{n-1} + d_{Q_n}(u, (w)_j) \leq 2^n - d_{Q_n}(w, v) + 1 \) if \( n \geq 3 \). Hence, all possible lengths have been concerned and the proof is completed. \( \square \)

**Lemma 5.** Suppose that \( x \) is any vertex of \( Q_3 \) and \((w, b)\) is any edge of \( Q_3 - \{x\} \). Then \( Q_3 - \{x\} \) has a \([w, b]\)-path of length \( l \) for each \( l \in \{1, 3, 5\} \).

**Proof.** Since \( Q_3 \) is vertex-transitive, we assume that \( x = 000 \). It is easy to see that any edge \((w, b) \in E(Q_3 - \{000\}) \) lies on a cycle of length four. Therefore, \( Q_3 - \{000\} \) has \([w, b]\)-paths of lengths one and three. The \([w, b]\)-path of length five is listed in **Table 2.** \( \square \)
Table 2

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<th>(w, b)</th>
<th>(w, b)-path of length five</th>
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<tr>
<td>(011,010)</td>
<td>(011, 111, 101, 110, 001, 111)</td>
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<tr>
<td>(101,001)</td>
<td>(101, 100, 111, 011, 001, 011)</td>
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<tr>
<td>(011,101)</td>
<td>(011, 001, 101, 110, 110, 111)</td>
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</table>

3. Bipanpositionable bipanconnectedness

Applying Theorem 1, we are able to prove the following theorem.

Theorem 1. The n-cube Q_n is relay-bipanpositionable between every two vertices in different partite sets if n >= 2.

Proof. The result is trivial for n = 2. We prove this theorem by induction for n >= 3. The induction basis follows from Lemma 1. As the inductive hypothesis, we assume that Q_{n-1}, n >= 4, is relay-bipanpositionable between every two vertices in different partite sets. Let x and y be any two vertices in different partite sets of Q_n. We have to show that for any vertex z in V(Q_n) - {x, y}, Q_n contains an [x, y]-path P_{k} of length l such that P_{k}(1) = x, P_{k}(k + 1) = z, and P_{k}(l + 1) = y for each odd integer l from d_{Q_n}(x, z) + d_{Q_n}(y, z) to 2^{n-1} - 1 and for each integer k satisfying both d_{Q_n}(x, z) <= k <= l - d_{Q_n}(y, z) and 2(k - d_{Q_n}(x, z)). For convenience, we write P_{k} = (x, P_{1}, z, P_{2}, y) with \ell(P_{1}) = k and \ell(P_{2}) = l - k. Since Q_n is vertex-transitive, we can assume that x, z \in V_0(Q_n) and y \in V_1(Q_n). Since x and z are in the same partite set of Q_n, we have d_{Q_n}(x, z) >= 2. Obviously, there exists an integer a of {0, 1, ..., n - 1} such that (x)_a \neq (z)_a and (z)_a \neq y. By symmetry, we assume that a = n - 1. Thus, Q_n can be partitioned into Q'_a and Q'_{n-1} so that x and z are on different subcubes. Without loss of generality, we assume that x is on Q'_a.

Case 1: Suppose that y is on Q'_{n-1}. Based on the inductive hypothesis, Lemma 2 ensures that Q'_0 and Q'_{n-1} are bipanconnected. Let j be an integer of {0, 1, ..., n - 2} such that (x)_j \neq (z)_j. Then we consider the following subcases.

Subcase 1.1: Suppose that d_{Q_n}(x, z) = 2. Therefore, we have ((x)_j)^{n-1} = z. First, we consider the case that \ell(P_{1}) = k <= 2^{n-1} - 2. Since Q'_0 is bipanconnected, it has a [x, y]-path R_{k}, of length r for every odd integer r satisfying d_{Q_n}(y, z) <= r <= 2^{n-1} - 1. Let \ell = 2^{n-1} - 1 and A = \{(R_{i}(i), R_{i}(i + 1)) | 1 <= i <= \ell and i \equiv 0 (mod 2)\}. Since |A| = 2^{n-2} > 3 for n = 4, there exists an odd integer i, 1 <= i <= \ell, such that \{(R_{i}(i))^{n-1}, (R_{i}(i + 1))^{n-1}\} \cap \{(x, (x)_j)\} = \emptyset and \{(R_{i}(i))^{n-1}, (R_{i}(i + 1))^{n-1}\} \cap \{(x, (y)_a)\} \neq \emptyset if n = 4. Since |A| = 2^{n-2} > 7 for n >= 5, there exists an odd integer i, 1 <= i <= \ell, such that \{(R_{i}(i))^{n-1}, (R_{i}(i + 1))^{n-1}\} \cap \{(x, (x)_j)\} = \emptyset if n >= 5. For convenience, let w = R_{i}(i) and b = R_{i}(i + 1). Hence, path R_{k} can be written as (z, R', w, b, R', y). For each even integer p from 2 to 2^{n-1} - 2 and for each odd integer q from 1 to p - 1, Lemma 3 ensures that Q'_0 has two vertex-disjoint paths S_{p,1}^{(1)} and S_{p,2}^{(1)} such that S_{p,1}^{(1)} is an [x, y]-path of length q and S_{p,2}^{(1)} is a [(w)_j]-path of length p - q. Therefore, we set P_{1} = (\langle x, S_{1-2\ldots 1-2\ldots k-1}, (x)_j^{(n-1)} = z \rangle) and P_{2} = R_{i-k} if \ell(P_{1}) = k <= 2^{n-1} - 2 and \ell(P_{2}) = l - k <= 2^{n-1} - 1 (See Fig. 3(a)); we set P_{1} = \langle x, S_{1-2\ldots 1-1\ldots k-1}, (x)_j^{(n-1)} = z \rangle and P_{2} = (z, R', w, (w)_j^{n-1}, S_{1-2\ldots 1-1\ldots k-1}^{(2)}, (b)_j^{n-1}, b, R', y) if \ell(P_{1}) = k <= 2^{n-1} - 2 and \ell(P_{2}) = l - k <= 2^{n-1} + 1 (See Fig. 3(b)). Since Q'_0 is bipanconnected, it has an [x, y]-path H of length 2^{n-1} - 1. Thus, we set P_{1} = (x, H, (x)_j^{(n-1)} = z) and P_{2} = R_{i-k} if \ell(P_{1}) = k = 2^{n-1} - 1 (See Fig. 3(c)). As a result, P_{1} is indeed an [x, y]-path of length k and P_{2} is indeed an [z, y]-path of length l - k.

Next, we consider the case that \ell(P_{1}) = k > 2^{n-1} + 2. Let w be a vertex of Q'_0 with d_{Q_n}(w, z) = 2. Since Q'_0 is bipanconnected, it has an [x, (w)_j]-path H of length 2^{n-1} - 1. By the inductive hypothesis, Q'_1 is relay-bipanpositionable between every two vertices in different partite sets; thus, Q'_1 has a [w, y]-path J of length l = 2^{n-1} such that J(1) = w, J(k - 2^{n-1} + 1) = z, and J(l - 2^{n-1} + 1) = y. For clarity, path J can be written as j = (w, J_{k-2^{n-1}}, z, J_{l-k}, y), where j_{k-2^{n-1}} is a [w, z]-path of length k - 2^{n-1} and j_{l-k} is a [z, y]-path of length l - k. Then we set P_{1} = (x, H, (w)_j^{n-1}, w, J_{l-2\ldots 1-1\ldots k-1}, (b)_j^{n-1}, b, R', y) and P_{2} = J_{l-k}' (See Fig. 3(d)). As a consequence, P_{1} is indeed an [x, z]-path of length k and P_{2} is indeed an [z, y]-path of length l - k. Obviously, \ell(P_{2}) = l - k can be any odd integer from d_{Q_n}(y, z) to 2^{n-1} - \ell(P_{1}) - 1.

Subcase 1.2: Suppose that d_{Q_n}(x, z) > 2. By the inductive hypothesis, Q'_1 is relay-bipanpositionable between two arbitrary vertices in different partite sets. Hence, Q'_1 has an [((x)_j)^{n-1}, y]-path H_{c}, of length s such that H_{c}(1) = ((x)_j)^{n-1}, H_{c}(t + 1) = z, and H_{c}(s + 1) = y for any odd integer s from d_{Q_n}(((x)_j)^{n-1}, z) + d_{Q_n}(y, z) = d_{Q_n}(x, z) + d_{Q_n}(y, z) - 2 to 2^{n-1} - 1 and for any even integer t from d_{Q_n}(((x)_j)^{n-1}, z) = d_{Q_n}(x, z) - 2 to 2^{n-1} - d_{Q_n}(y, z) - 1. For clarity, path H_{c} can be written as (((x)_j)^{n-1}, H_{c}(1), z, H_{c}(2), y), where H_{c}(1) is an [((x)_j)^{n-1}, z]-path of length t and H_{c}(2) is a [z, y]-path of length s - t.
First, we consider the case that \( \ell(P_2) = l-k \leq 2^{n-1} - d_{Q_0}(x, z) + 1 \). Since \( Q_0^0 \) is biconnected, it has an \([x, (x)]\)-path \( R_t \) of length \( r \) for every odd integer \( r \) satisfying \( 1 \leq r \leq 2^{n-1} - 1 \). Then we set \( P_1 \) to be the path \((x, R_t, (x^j)^{(i-1)}, H_{t+1-k, t}^{(i)}, z)\) with \( r + t = k - 1 \). We set \( P_2 = P_1 \cdot P_1 \cdot P_1 \). See Fig. 3(e).

Next, we consider the case that \( \ell(P_2) = l-k \geq 2^{n-1} - d_{Q_0}(x, z) + 3 \). For convenience, let \( m_1 = d_{Q_0}(x, z) - 2 \) and \( m_2 = 2^{n-1} - d_{Q_0}(x, z) + 1 \). Since \( m_2 \geq 2^{n-1} - n + 1 \geq 5 \) for \( n \geq 4 \), path \( H_{m_1+m_2, m_1}^{(i)} \) can be written as \( (z, b, w, j, y) \), where \( b \) is some vertex adjacent to \( z \), \( w \) is some vertex adjacent to \( b \), and \( j \) is a \([w, y]\)-path. Since \( d_{Q_0}(x, z) > 2 \), we have \([x, (x^j)] \cap ([z]^{n-1}, (b)^{(n-1)}, (w)^{(n-1)}) = \emptyset \) and \( m_1 = k - m_1 - 1 \) and \( m_2 = l - k - m_2 - 1 \).

I. When \( n \geq 5 \), Lemma 3 ensures that \( Q_0^0 \) has two vertex-disjoint \( S_{m_1+m_2, m_1}^{(i)}, m_1 \) and \( S_{m_1+m_2, m_1}^{(i)}, m_2 \) such that \( S_{m_1+m_2, m_1}^{(i)} \) is an \([x, (x^j)]\)-path of length \( m_1 \) and \( S_{m_1+m_2, m_1}^{(i)} \) is a \([z]^{n-1}, (b)^{(n-1)}, (w)^{(n-1)}) \)-path of length \( m_2 \). Then we set \( P_1 = ([x, (x^j)] \cap ([z]^{n-1}, (b)^{(n-1)}, (w)^{(n-1)}) = \emptyset \) and \( m_1 = k - m_1 - 1 \) and \( m_2 = l - k - m_2 - 1 \).

II. When \( n = 4 \), let \( A = ([x]^{0}, (x)^{(1)}, (x)^{(0)}), ([x]^{0}, (x)^{(1)}, (x)^{(0)})) \) and \( ([x]^{0}, (x)^{(1)}, (x)^{(0)})) \). Then we have \([z]^{n-1}, (b)^{(n-1)}, (w)^{(n-1)}) \cap A = \emptyset \) if \( ([z]^{n-1}, (b)^{(n-1)}, (w)^{(n-1)}) \cap A = \emptyset \), the desired path can be obtained as discussed for the case that \( n \geq 5 \). Otherwise, Lemma 3 ensures that \( Q_0^0 \) has two vertex-disjoint \( T_{m_1+m_2, m_1}^{(i)}, m_1 \) and \( T_{m_1+m_2, m_1}^{(i), m_2} \) such that \( T_{m_1+m_2, m_1}^{(i), m_1} \) is an \([x, (x^j)]\)-path of length \( m_1 \) and \( T_{m_1+m_2, m_1}^{(i), m_2} \) is a \([n]^{n-1}, (b)^{(n-1)}, (w)^{(n-1)}) \)-path of length \( m_2 \). Then the desired path can be formed similarly.

**Case 2**: Suppose that \( y \) is on \( Q_0^0 \). Recall that \( ([z]^{n-1}, (b)^{(n-1)}, (w)^{(n-1)}) \) and \( ([x, (x^j)] \cap ([z]^{n-1}, (b)^{(n-1)}, (w)^{(n-1)}) = \emptyset \) and \( m_1 = k - m_1 - 1 \) and \( m_2 = l - k - m_2 - 1 \).

First, we consider the case that \( \ell(P_2) = l-k \leq 2^{n-1} - d_{Q_0}(x, z) + 1 \). For convenience, let \( m_1 = d_{Q_0}(x, z) - 1 \) and \( m_2 = 2^{n-1} - d_{Q_0}(x, z) + 1 \). Since \( m_2 \geq 2^{n-1} - n \) for \( n \geq 4 \), we have \([x, (x^j)] \cap ([z]^{n-1}, (b)^{(n-1)}, (w)^{(n-1)}) = \emptyset \) and \( m_1 = k - m_1 - 1 \) and \( m_2 = l - k - m_2 - 1 \). For clarity, path \( H_{i+1-k, t}^{(i)} \) can be written as \( ([x, (x^j)] \cap ([z]^{n-1}, (b)^{(n-1)}, (w)^{(n-1)}) = \emptyset \) and \( m_1 = k - m_1 - 1 \) and \( m_2 = l - k - m_2 - 1 \). See Fig. 4(a).

Next, we consider the case that \( \ell(P_2) = l-k \geq 2^{n-1} - d_{Q_0}(x, z) + 3 \). For convenience, let \( m_1 = d_{Q_0}(x, z) - 1 \) and \( m_2 = 2^{n-1} - d_{Q_0}(x, z) + 1 \), and \( A = ([H_{m_1+m_2, m_1}^{(i)}, (i + 1)] \mid 2 \leq i \leq m_2 \) and \( i \equiv 0 \pmod{2} \)). Moreover, we write \( H_{m_1+m_2, m_1}^{(i), m_2} \) as \( ([H_{m_1+m_2, m_1}^{(i), m_2}(i)] \mid 2 \leq i \leq m_2 \) and \( i \equiv 0 \pmod{2} \)). Clearly, we have \((H_{m_1+m_2, m_1}^{(i), m_2}(i))^\sim \) and \( ([H_{m_1+m_2, m_1}^{(i), m_2}(i + 1))] \) for any \( 2 \leq i \leq m_2 \). Since \( m_2 \geq 2^{n-1} - n \), we have \([A] = [m_2/2] \geq 2^{n-1} - n/2 \) for \( n = 4 \). Hence, there exists an even integer \( i \), \( 2 \leq i \leq m_2 \), such that \((H_{m_1+m_2, m_1}^{(i), m_2}(i))^\sim \) and \( ([H_{m_1+m_2, m_1}^{(i), m_2}(i + 1))] \) for any \( 2 \leq i \leq m_2 \). For convenience, let \( u = H_{m_1+m_2, m_1}^{(i), m_2}(i) \) and \( v = H_{m_1+m_2, m_1}^{(i), m_2}(i) \). Accordingly, \( H_{m_1+m_2, m_1}^{(i), m_2}(i) \) can be represented as \(([z]^{n-1}, l', u, v, l', y) \). Let \( m_1 = k - m_1 \) and \( m_2 = l - k - m_2 - 1 \).
As a result, has a assumption that $P \equiv \langle y \rangle$ is bipan connected, it has an $\ell(7) = \langle z \rangle$-path of length $m_1'$ and $S^{(2)}_{m_1' + m_2'}$ is a $\langle (w)^n \rangle$-path of length $m_2'$. Then we set $P_1 = \langle x, j, w, (w)^{n-1}, S^{(1)}_{m_1' + m_2'}, z \rangle$ and $P_2 = \langle x, (z)^{n-1}, l', u, (u)^{n-1}, S^{(2)}_{m_1' + m_2'}, v \rangle$. See Fig. 4(b).

In summary, $P_1$ is indeed an $[x, z]$-path of length $k$ and $P_2$ is indeed an $[z, y]$-path of length $l - k$. Hence, the proof is completed. □

**Lemma 6.** The 4-cube $Q_4$ is relay-bipanpositionable between every pair of distinct vertices in the same partite set.

**Proof.** Let $x$ and $y$ be any two distinct vertices in the same partite set of $Q_4$ and let $z$ be any vertex of $V(Q_4) - \{x, y\}$. We have to construct an $[x, y]$-path $P_{x,y}$ of length $l$ such that $P_{x,y}(1) = x$, $P_{x,y}(k+1) = y$, and $P_{x,y}(l+1) = y$ for any even integer $l$ from $d_{Q_4}(x, z) + d_{Q_4}(y, z)$ to 14 and for any integer $k$ satisfying $d_{Q_4}(x, z) \leq k \leq 14 - d_{Q_4}(y, z)$ and $2(k - d_{Q_4}(z, z))$. For convenience, we write $P_{x,k} = \langle x, P_{1,k}, z, P_{2,k}, y \rangle$ with $\ell(P_1) = k$ and $\ell(P_2) = l - k$. Without loss of generality, we assume that $x, y \in V_0(Q_4)$. Then we distinguish the following two cases.

**Case 1:** Suppose that $z \in V_0(Q_4)$. Since $Q_4$ is edge-transitive, we assume that $(x)_3 \neq (y)_3$. Without loss of generality, we assume that $x, y \in V(Q_4^0)$ and $y, z \in V(Q_4')$.

First, we consider the case that $\ell(P_1) = k \geq 8$ and $\ell(P_2) = l - k \leq 6$. By **Lemma 2** and **Theorem 1**, $Q_4$ is bipanconnected. Therefore, $Q_4^0$ has an $[x, (z)_3]$-path $S_p$ of length $p$ for each odd integer $p$ from $d_{Q_4}(x, (z)_3) = d_{Q_4}(x, z) - 1$ to 7. Similarly, $Q_4^0$ has a $[y, (z)_3]$-path $R_l$ of length $l$ for each even integer $l$ from $d_{Q_4}(y, z)$ to 6. Then we set $P_1 = \langle x, S_{k-1}, (z)_3, z, P_{2,l} \rangle$. As a result, $P_{x,k} = \langle x, P_{1,k}, z, P_{2,l}, y \rangle$ is indeed an $[x, y]$-path of length $l$ such that $P_{x,k}(k+1) = z$. See Fig. 5(a). Next we consider the case that $\ell(P_1) = k \leq 10$. Let $b$ be a vertex of $Q_4$ such that $d_{Q_4}(b, z) = 1$ and $d_{Q_4}(b, b) \neq x$. Since $Q_4^0$ is bipanconnected, it has an $[x, (b)_3]$-path $H$ of length six. By **Theorem 1**, $Q_4$ is relay-bipanpositionable between any two vertices in different partite sets. Therefore, $Q_4^0$ has a $[b, y]$-path $l_{1-7,k-1}$ of length $l - 7$ such that $l_{1-7,k-1}(1) = b$, $l_{1-7,k-1}(k-6) = z$, and $l_{1-7,k-1}(l-6) = y$. For convenience, we write $l_{1-7,k-1} = \langle b, l_{1,k-1}', z, l_{1,k-1}', y \rangle$, where $l_{1,k-1}'$ is a $[y, z]$-path of length $k - 7$ and $l_{1,k-1}'$ is a $[z, y]$-path of length $l - k$. Then we set $P_1 = \langle x, H, (b)_3, b, l_{1,k-1}', z, P_{2,l} \rangle$. See Fig. 5(b). Consequently, $P_1$ is indeed an $[x, z]$-path of length $k$ and $P_2$ is indeed an $[z, y]$-path of length $l - k$.

Finally, we consider the case that $\ell(P_1) = l - k \geq 8$. Since $x, y, z \in V_0(Q_4)$, we have $[x, y] \neq (y, z)$. By **Theorem 1**, $Q_4^0$ has an $[(x)_3, y]$-path $l_{1-7,k-1}$ of length $l - 7$ such that $l_{1-7,k-1}(1) = (x)_3$, $l_{1-7,k-1}(k) = z$, and $l_{1-7,k-1}(l-6) = y$. For convenience, we write $l_{1-7,k-1} = \langle (x)_3, l_{1,k-1}', z, l_{1,k-1}', y \rangle$, where $l_{1,k-1}'$ is an $[(x)_3, z]$-path of length $k - 1$ and $l_{1,k-1}'$ is a $[y, z]$-path of length $l - k - 6$. Moreover, we write $l_{1,k-1}' = \langle (z, v, l_{1,k-1}', y) \rangle$, where $v$ is some vertex adjacent to $z$ and $l_{1,k-1}'$ is a $[v, y]$-path of length $l - k - 7$. By **Lemma 5**, $Q_4^0 - [x]$ has a $[z]_3$-path $R_l$ of length $r \in \{1, 3, 5\}$. Then we set $P_1 = \langle x, (x)_3, l_{1,k-1}', z, P_{2,l} \rangle$ and $P_2 = \langle z, (z)_3, R_l, (v)_3, v, l_{1,k-1}', y \rangle$. See Fig. 5(c).

**Case 2:** Suppose that $z \in V_1(Q_4)$. Since $x$ and $z$ are in different partite sets of $Q_4$, we have $d_{Q_4}(x, z) \in \{1, 3\}$.

**Subcase 2.1:** Suppose that $d_{Q_4}(x, z) = 3$. Since $x, y \in V_0(Q_4)$, we have $d_{Q_4}(x, y) \geq 2$. Hence, we can find an integer $i \in \{0, 1, 2, 3\}$ such that $(x)_i \neq (y)_i$ and $(x)_i \neq (z)_i$. Since $Q_4$ is edge-transitive, we assume that $i = 3$. Without loss of generality, we assume that $x$ is on $Q_4^0$ and both $y$ and $z$ are on $Q_4'$.

First we consider the case that $\ell(P_1) = k \leq 7$ and $\ell(P_2) = l - k \leq 7$. Since $Q_4^0$ is bipanconnected, it has an $[x, (z)_3]$-path $S_p$ of length $p$ for any even integer $p$ from $d_{Q_4}(x, z) - 1 = 2$ to 6. Similarly, $Q_4^0$ has a $[z, y]$-path $R_l$ of length $q$ for any odd integer $q$ from $d_{Q_4}(y, z)$ to 7. Then we set $P_1 = \langle x, S_{k-1}, (z)_3, z, P_{2,l} \rangle$ and $P_2 = \langle x, P_{1,k}, z, y \rangle$. As a result, $P_{x,k} = \langle x, P_{1,k}, z, P_{2,l}, y \rangle$ is indeed an $[x, y]$-path of length $l$ such that $P_{x,k}(k+1) = z$. The illustration is similar to Fig. 5(a).

Next, we consider the case that $\ell(P_1) = k \geq 9$. Let $b \in V_1(Q_4^0)$ such that $d_{Q_4}(b, z) = 2$ and $d_{Q_4}(b, b) \neq x$. Since $Q_4^0$ is bipanconnected, it has an $[x, (b)_3]$-path $H$ of length six. By **Theorem 1**, $Q_4^0$ has a $[b, y]$-path $l_{1-7,k-7}$ of length $l - 7$ such that...
Finally, we consider the case that $\ell(P_2) = l - k \geq 9$. Since $d_{Q_4}(x, z) = 3$, we have $(x)^3 \neq z$. Because $Q_4$ is relay-bipanpositionable between $(x)^3$ and $y$, it has an $(x)^3, y,l$-path $j_{l-k-1}$ of length $l - 7$ such that $j_{l-k-1}(1) = (x)^3$, $j_{l-k-1}(k) = z$, and $j_{l-k-1}(l - 6) = y$. For convenience, we write $j_{l-k-1} = (j_{l-k-1}, z, j_{l-k-6}, y)$, where $j_{l-k-6}$ is an $(x)^3, y,l$-path of length $l - 1$ and $j_{l-k-6}$ is a $(y, z, 1)$-path of length $l - 6$. Furthermore, we can write $j_{l-k-6}$ as $\langle v, j_{l-k-7}, y \rangle$, where $v$ is some vertex adjacent to $z$ and $j_{l-k-7}$ is a $(v, y,l)$-path of length $l - k - 7$. By Lemma $5$, $Q_4^0 - \{x\}$ has an $(z)^3, y,l$-path $R_r$ of length $r \in \{1, 3, 5\}$. Then we set $P_1 = \langle x, (x)^3, j_{l-k-1}, z \rangle$ and $P_2 = \langle z, (z)^3, R_r, (v)^3, v, j_{l-k-7}, y \rangle$. The illustration is similar to Fig. 5(c).

Subcase 2.2: Suppose that $d_{Q_4}(x, z) = 1$. Since $x$ and $y$ are in the same partite set of $Q_4$, we have $d_{Q_4}(x, y) \in \{2, 4\}$. Thus, there exists an integer $i$ of $\{0, 1, 2, 3\}$ such that $x \neq (y)^i$, and $(x) = (z)$. Since $Q_4$ is edge-transitive, we assume that $i = 3$. Without loss of generality, we assume that $x, z \in V(Q_4^0)$ and $y \notin V(Q_4^0)$. Since $y$ and $z$ are in different partite sets of $Q_4$, we have $d_{Q_4}(x, z) \in \{1, 3\}$. Therefore, we distinguish the following subcases.

Subcase 2.2.1: Suppose that $d_{Q_4}(x, y) = 3$. On the one hand, we concern the case that $\ell(P_1) = k \leq 7$ and $\ell(P_2) = l - k \leq 7$. Since $Q_4^{0}$ is bipanconnected, it has an $(x, z, y,l)$-path $S_p$ of length $p \in \{1, 3, 5, 7\}$. Since $d_{Q_4}(y, z) = 3$, we have $(x)^3 \neq y$. Similarly, $Q_4^{0}$ has an $(z)^3, y,l$-path $K_r$ of length $r \in \{2, 4, 6\}$. Then we set $P_1 = S_k$ and $P_2 = (z, (z)^3, R_{l-k-1}, y)$. On the other hand, we concern the case that $\ell(P_1) = k \geq 9$ or $\ell(P_2) = l - k \geq 9$. Without loss of generality, we assume that $x = 0000$ and $z = 0001$. Since $y \notin V(Q_4^0)$ and $d_{Q_4}(y, z) = 3$, we have $y \in \{1010, 1100, 1111\}$. Then the required paths obtained by brute force are listed in Table 3. Then we list the required paths obtained by brute force in Table 4.

We apply Lemma 6 to prove the following theorem.

**Theorem 2.** The $n$-cube $Q_n$ is relay-bipanpositionable between every pair of distinct vertices in the same partite set if $n \geq 4$.

**Proof.** We prove this theorem by induction on $n$. The induction basis follows from Lemma 6. As the inductive hypothesis, we assume that $Q_{n-1}$ is relay-bipanpositionable between every pair of distinct vertices in the same partite set for $n \geq 5$. Let $x$ and $y$ be any two distinct vertices in the same partite set of $Q_n$ and let $z$ be any vertex of $V(Q_n) - \{x, y\}$. We have to construct an $(x, y,l)$-path $P_{n-k}$ of length $l$ such that $P_{n-k}(1) = x$, $P_{n-k}(k + 1) = z$, and $P_{n-k}(k + 1) = y$ for any even integer $l$ from $d_{Q_n}(x, z) + d_{Q_n}(y, z) \geq 2^{n-2}$ and for any integer $k$ satisfying $d_{Q_n}(x, z) \leq k \leq 2^n - d_{Q_n}(y, z) - 2$ and $2(k - d_{Q_n}(x, z))$. For convenience, we write $P_{n-k} = (x, P_1, z, P_2, y)$ with $\ell(P_1) = k$ and $\ell(P_2) = l - k$. Without loss of generality, we assume that $l \leq k$ and $x = 0000, y = 1001, z = 1011$. We will consider the cases that $k = 2, 4, 6, 8$ and $l = 4, 6, 8, 10$. In each case, we will use the following notations:

1. $b_l(x, z)$: a $(x, z,l)$-path of length $l$ with $b_l(x, z) = \langle x, P_1, z \rangle$.
2. $c_l(x, y, z)$: a $(y, z,l)$-path of length $l$ with $c_l(x, y, z) = \langle y, P_2, z \rangle$.
3. $d_l(x, y, z)$: a $(x, y,l)$-path of length $l$ with $d_l(x, y, z) = \langle x, P_1, y \rangle$.
4. $e_l(x, z)$: a $(x, z,l)$-path of length $l$ with $e_l(x, z) = \langle x, P_1 \rangle$.

We will consider the cases that $k = 2, 4, 6, 8$ and $l = 4, 6, 8, 10$. In each case, we will use the following notations:

1. $b_l(x, z)$: a $(x, z,l)$-path of length $l$ with $b_l(x, z) = \langle x, P_1, z \rangle$.
2. $c_l(x, y, z)$: a $(y, z,l)$-path of length $l$ with $c_l(x, y, z) = \langle y, P_2, z \rangle$.
3. $d_l(x, y, z)$: a $(x, y,l)$-path of length $l$ with $d_l(x, y, z) = \langle x, P_1, y \rangle$.
4. $e_l(x, z)$: a $(x, z,l)$-path of length $l$ with $e_l(x, z) = \langle x, P_1 \rangle$.
Table 4
The required paths for Subcase 2.2.2 of Lemma 6.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$l - k$</th>
<th>Paths between $x = 0000$ and $y = 1001$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>$x = 0000, 0001 = z, 1001 = y$</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>$x = 0000, 0001 = z, 0011, 1001, 1001 = y$</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>$x = 0000, 0001 = z, 0011, 1111, 1100, 1001, 1001 = y$</td>
</tr>
<tr>
<td>7</td>
<td>6</td>
<td>$x = 0000, 0001 = z, 0011, 1111, 1110, 1000, 1000, 1001, 1001 = y$</td>
</tr>
<tr>
<td>11</td>
<td>10</td>
<td>$x = 0000, 0001 = z, 0101, 0111, 0011, 1111, 1110, 1000, 1000, 1001, 1001 = y$</td>
</tr>
<tr>
<td>13</td>
<td>12</td>
<td>$x = 0000, 0001 = z, 0101, 0100, 0110, 0111, 1111, 1110, 1000, 1000, 1001, 1001 = y$</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>$x = 0000, 0100, 0101, 0001 = z, 1001 = y$</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>$x = 0000, 0100, 0101, 0001 = z, 0011, 1111, 1100, 1001, 1001 = y$</td>
</tr>
<tr>
<td>7</td>
<td>0</td>
<td>$x = 0000, 0100, 0101, 0001 = z, 0111, 1111, 1110, 1000, 1001, 1001 = y$</td>
</tr>
<tr>
<td>9</td>
<td>1</td>
<td>$x = 0000, 0100, 0101, 0001 = z, 0111, 1111, 1110, 1000, 1000, 1001, 1001 = y$</td>
</tr>
<tr>
<td>11</td>
<td>3</td>
<td>$x = 0000, 0100, 0101, 0001 = z, 0111, 1111, 1110, 1000, 1001, 1001 = y$</td>
</tr>
<tr>
<td>13</td>
<td>5</td>
<td>$x = 0000, 0100, 0101, 0001 = z, 0111, 1111, 1110, 1000, 1000, 1001, 1001 = y$</td>
</tr>
</tbody>
</table>

$x, y \in V(Q_n)$. Since $x$ and $y$ are in the same partite set of $Q_n$, we have $d_{Q_n}(x, y) \geq 2$. Therefore, there exists an integer $i$ of $\{0, 1, \ldots, n - 1\}$ such that $(x_i) \neq (y_i)$. Since $Q_n$ is edge-transitive, we assume that $i = n - 1$. Thus, $Q_n$ can be partitioned into $Q^0_n$ and $Q^1_n$ so that $x$ and $y$ are on different subgraphs. Without loss of generality, we assume that $x \in V(Q^0_n)$ and $y, z \in V(Q^1_n)$. By Theorem 1 and Lemma 2, both $Q^0_n$ and $Q^1_n$ are bipanconnected. Then we distinguish the following cases.

**Case 1:** Suppose that $d_{Q_n}(x, z) = 1$; i.e., $z = (x)^{n-1}$. We concern the following subcases.

**Subcase 1.1:** Suppose that $\ell(P_1) = k = 1$. Obviously, $P_1 = (x, z)$ is the desired path. On the one hand, we consider the case that $\ell(P_2) = l - k \leq 2^{n-1} - 1$. Since $Q^1_n$ is bipanconnected, it has a $[z, y]$-path $T_l$ of length $t$ for any odd integer $t$ from $d_{Q_n}(y, z)$ to $2^{n-1} - 1$. Then we set $P_2 = T_{l-1}$. See Fig. 6(a). On the other hand, we consider the case that $\ell(P_2) = l - k \geq 2^{n-1} + 1$. Since $2^{n-1} - 1 \geq 15$ for $n \geq 5$, we can write $T_{2^{n-1} - 1}$ as $T_1, T'_2$, $b, y$, where $b$ is some vertex adjacent to $y$ and $w$ is some vertex adjacent to $b$. By Lemma 4, $Q_{n-1}$ is hyper-bipanconnected. Thus, $Q^0_{n-1} - \{x\}$ has a $([w]^{n-1}, (y)^{n-1})$-path $R_l$ of length $r$ for any even integer $r$ from 2 to $2^{n-1} - 2$. Then we can set $P_2 = (z, T', w, (w)^{n-1}, R_l, k-2^{n-1} - 1, (y)^{n-1}, y, R_l)$. See Fig. 6(b).

**Subcase 1.2:** Suppose that $\ell(P_1) = k \geq 3$. Let $\{i, j, \ldots, n - 2\}$ be an integer of $\{0, 1, \ldots, n - 2\}$ such that $(x)^{n-1} \neq y$. Since $x$ and $z$ are adjacent, $(x^{n-1})$ and $z$ are also adjacent. By the inductive hypothesis, $Q^1_n$ has an $([x], (y)^{n-1}, y]$-path $l_{p,q}$ of length $p$ such that $l_{p,q}(1) = ((x)^{n-1}, l_{p,q}(q+1) = z$, and $l_{p,q}(p+1) = y$ for any even integer $p$ from $d_{Q_n}((x)^{n-1}, z) + d_{Q_n}(y, z) = 1 + d_{Q_n}(y, z)$ to $2^{n-2} - 2$ and for any odd integer $q$ from 1 to $2^{n-1} - d_{Q_n}(y, z) - 2$. For convenience, we write $l_{p,q}$ as $((x)^{n-1}, l_{p,q}(1),$ $l_{p,q}(q+1), (x)^{n-1}, z)]$-path of length $q$ and $l_{p,q}^{(2)}$ is a $[z, y]$-path of length $p - q$.

First we consider the case that $\ell(P_2) = l - k \leq 2^{n-1} - 3$. Since $Q^0_n$ is bipanconnected, it has an $([x], (y)^{n-1}, z]$-path $R_0$ of any odd length $r$ in the range from 1 to $2^{n-1} - 1$. Then we set $P_1 = (x, R_0, (y), (x)^{n-1}, l_{p,q}^{(2)}_{q+r-1-k,q}, z)$ with $r + q = k - 1$ and $P_2 = l_{p,q}^{(2)}_{q+r-1-k,q}$. See Fig. 6(c).

Now we consider the case that $\ell(P_2) = l - k \geq 2^{n-1} - 1$. Let $m = 2^{n-1} - 3$ and $A = \{(l_{i+1,m,1}(i), l_{i+1,m,1}(i + 1)) \mid 1 \leq i \leq m, m \equiv i (\text{mod} 2)\}$. Obviously, we have $|A| = [m/2] = 2^{n-2} - 1$. Since $|A| \geq 7$ for $n \geq 5$, there exists an odd integer $l$, $1 \leq l \leq m$, such that $(l_{i+1,m,1}(i) - 1, l_{i+1,m,1}(i + 1) - 1) \subseteq (x, y) = \emptyset$. For convenience, let $b = l_{i+1,m,1}(i)$ and $w = l_{i+1,m,1}(i + 1)$. Accordingly, $l_{i+1,m,1}^{(2)}$ can be written as $(z, l', b, w, l', y)$. Let $m' = l - k - m = l - k - 2^{n-1} + 2$. By Lemma 3, $Q^0_n$ has two vertex-disjoint paths $S^{(1)}_{k-2,m,k-2}$ and $S^{(2)}_{k-2,m,k-2}$ such that $S^{(1)}_{k-2,m,k-2}$ is an $[x, (y)]$-path of length $k - 2$ and $S^{(2)}_{k-2,m,k-2}$ is a $([b]^{n-1}, (w)^{n-1})$-path of length $m'$. Then we can set $P_1 = (x, S^{(1)}_{k-2,m,k-2}, (y), (x)^{n-1}, z)$ and $P_2 = (z, l', b, (b)^{n-1}, S^{(2)}_{k-2,m,k-2}, (w)^{n-1}, w, l', y)$. See Fig. 6(d).
Case 2: Suppose that \( d_{Q_0}(x, z) = 2 \). Clearly there exists an integer \( j \) of \([0, 1, \ldots, n-2]\) such that \((x)_j \neq (z)_j\). Therefore, we have \((x)^{n-1}_j = z\). Since \( Q_n \) is bipartitioned, it has a \([z, y]\)-path \( T_1 \) of any even length \( t \) from \( d_{Q_0}(y, z) \) to \( 2^{n-1} - 2 \).

Subcase 2.1: Suppose that \( \ell(P_1) = k \leq 2^{n-1} \). Since \( Q_0^n \) is bipartitioned, it has an \([x, (x')]\)-path \( T \), of any odd length \( r \) from 1 to \( 2^{n-1} - 1 \). For the case that \( \ell(P_2) \leq 2^{n-1} - 2 \), we can set \( P_2 = (x, R_{1-1}, (x'), ((x'))^{n-1} = z) \) and \( P_2 = T_{1-k} \). See Fig. 6(e). In what follows, we consider the case that \( \ell(P_2) \geq 2^{n-1} \). Let \( m = 2^{n-1} - 2 \) and \( A = \{(T_m(i), T_m(i+1)) \mid 2 \leq i \leq m \text{ and } i \equiv 0 \pmod{2}\} \). Obviously, we have \(|A| = \lceil m/2 \rceil = 2^{n-2} - 1 \). Since \(|A| \geq 7\) for \( n \geq 5 \), there exists an even integer \( \hat{i} \), \( 2 \leq \hat{i} \leq m \), such that \( \{(T_m(\hat{i}))(n) = (T_m(\hat{i} + 1))^{n-1} \} \cap \{(x, (x'))\} = \emptyset\). For convenience, let \( b = T_m(\hat{i}) \) and \( w = T_m(\hat{i} + 1) \). Accordingly, path \( T_m \) can be written as \((z, T', b, w, T'', y)\). Let \( m' = l - k = m - 1 = l - k - 2^{n-1} + 1 \). By Lemma 3, \( Q_0^n \) has two vertex-disjoint paths \( S_{k-1+m', k-1}^{(1)} \) and \( S_{k-1+m', k-1}^{(2)} \) such that \((x, (x'))\)-path of length \( k - 1 \) and \((y, (y'))\)-path of length \( m' \). Then we can set \( P_1 = \{(x, S_{k-1+m', k-1}^{(1)}), (x'), ((x'))^{n-1} = z) \) and \( P_2 = (z, T', b, (b')^{n-1}, S_{k+1+m', k-1}^{(2)})^{n-1}, (w, T'', y)\). See Fig. 6(f).

Subcase 2.2: Suppose that \( \ell(P_1) = k \geq 2^{n-1} + 2 \). Let \( w \) be a vertex of \( Q_0^n \) such that \( d_{Q_0}(w, z) = 2 \). Obviously, \((w)^{n-1} \) and \( x \) are in the different partite sets of \( Q_0^n \). Since \( Q_0^n \) is bipartitioned, it has an \([x, (w)]\)-path \( H \) of length \( 2^{n-1} - 1 \). By the inductive hypothesis, \( Q_1^n \) has a \([w, y]\)-path \( l_{p, q}(n-1, k-2^{n-1}) \) path of length \( l - 2^{n-1} \) such that \( l_{p, q}(n-1, k-2^{n-1}) = w \) and \( l_{p, q}(n-1, k-2^{n-1}) = y \). For convenience, we write \( l_{p, q}(n-1, k-2^{n-1}) = (w, l', (z, l'', z') \) in Fig. 6(g).

Case 3: Suppose that \( d_{Q_0}(x, z) > 2 \). Hence there exists an integer \( j \) of \([0, 1, \ldots, n-2]\) such that \((x)_j \neq (z)_j\) and \((x)^{n-1} \neq y\). By the inductive hypothesis, \( Q_1^n \) has an \([x, (y)]\)-path \( l_{p, q} \) of length \( p \) such that \( l_{p, q}(1) = (x)^{n-1} \), \( l_{p, q}(q - 1) = z \), and \( l_{p, q}(p + 1) = y \) for any even integer \( p \) from \( d_{Q_0^n}(x, z) \geq 2^{n-1} - 2 \) and for any integer \( q \) satisfying both \( d_{Q_0^n}(x, z) \geq 2 \) \( q < 2^{n-1} - d_{Q_0^n}(x, z) - 2 \) \( q < 2^{n-1} - 2 \) \( q \) \( q \). For convenience, we write \( l_{p, q} = ((x)^{n-1}, l_{p, q}^{(1)}, l_{p, q}^{(2)}, (z, l'', z') \) in Fig. 6(h).

First we consider the case that \( \ell(P_2) = l - k \leq 2^{n-1} - d_{Q_0}(x, z) \). Because \( Q_0^n \) is bipartitioned, it has an \([x, (x')]\)-path \( R_r \) of odd length \( r \) from 1 to \( 2^{n-1} - 1 \). Then we set \( P_1 = \{(x, R_r, (x'), (((x'))^{n-1}, l_{p, q}^{(1)}, l_{p, q}^{(2)}, k, q) \) with \( r + q = k - 1 \) and set \( P_2 = l_{p, q}^{(2)}(k-1) \). See Fig. 6(h).

Now we consider the case that \( \ell(P_2) = l - k \geq 2^{n-1} - d_{Q_0}(x, z) \). Let \( m_1 = d_{Q_0}(x, z) \), \( m_2 = 2^{n-1} - d_{Q_0}(x, z) \), and \( A = \{(l_{m_2-1+m_2-1}(l), l_{m_2-1+m_2-1}(l+1)) \mid 1 \leq l \leq m_2 \text{ and } i \equiv 1 \pmod{2}\} \). Obviously, we have \(|A| = \lceil m_2/2 \rceil \geq \lceil (2^{n-1} - n)/2 \rceil \). Since \(|A| \geq 6\) for \( n \geq 5 \), there exists an odd integer \( l \), \( 1 \leq l \leq m_2 \), such that \( \{(l_{m_2-1+m_2-1}(l), l_{m_2-1+m_2-1}(l+1))^{n-1} \} \cap \{(x, (x'))\} = \emptyset\). For convenience, let \( u = l_{m_2-1+m_2-1}(l) \) and \( v = l_{m_2-1+m_2-1}(l+1) \). Accordingly, \( l_{m_2-1+m_2-1}^{(2)}(k-1) \) can be written as \((z, l', u, v, l'', y)\). For simplicity, let \( m'_1 = k - m_1 - 1 \) and \( m'_2 = l - k - m_2 - 1 \).
By Lemma 3, $Q_n^0$ has two vertex-disjoint paths $S_1^{(1)}_{m_1, m_2, m_1}$ and $S_1^{(2)}_{m_1, m_2, m_1}$ such that $S_1^{(1)}_{m_1, m_2, m_1}$ is an $[x, (x)^l]$-path of length $m_1$ and $S_1^{(2)}_{m_1, m_2, m_1}$ is a $(u^{n-1}, v^{n-1})$-path of length $m_2$. Then we can set $P_1 = (x, S_1^{(1)}_{m_1, m_2, m_1}, (x)^l, (y)^{n-1}, l^{(1)}_{m_1, m_2, m_1}, z)$ and $P_2 = (z, l, u, (u^{n-1}, v^{n-1}, v, l', y)$. See Fig. 6(i).

In summary, $P_1$ is indeed an $[x, z]$-path of length $k$ and $P_2$ is indeed an $[z, y]$-path of length $l - k$. Consequently, the proof is completed. □

According to Theorems 1 and 2, we have the following result.

**Theorem 3.** The $n$-cube $Q_n$ is bipanpositionally bipanconnected if $n \geq 4$.

By Theorem 1, we have shown that $Q_n$ is not only bipanconnected but also hyper-bipanconnected if $n \geq 2$. Moreover, it is also easy to prove that $Q_n$ is bipancyclic and bipanpositionally Hamiltonian.

**Corollary 1 ([15]).** The $n$-cube $Q_n$ is bipancyclic if $n \geq 2$.

**Corollary 2 ([17]).** The $n$-cube $Q_n$ is bipanpositionally Hamiltonian if $n \geq 2$.

### 4. Conclusion

In this paper, we defined the bipanpositionable bipanconnectedness for bipartite graphs and discussed such property on hypercubes. That is, for any two vertices $x$ and $y$ of $Q_n$ and for any vertex $z \in V(Q_n) - \{x, y\}$, the $Q_n$ contains an $[x, y]$-path $P_{z,k}$ of length $l$ such that $P_{z,k}(1) = x$, $P_{z,k}(l+1) = y$ for any integer $l$ satisfying $h(x, z) + h(y, z) \leq l \leq 2^n - 1$ and $2(l - h(x, z) - h(y, z))$ for any integer $k$ satisfying both $h(x, z) + h(y, z) \leq l \leq 2^n - 1$. In particular, path $P_{z,k}$ turns out to be a Hamiltonian path of $Q_n$ while $l = 2^n - 1$. Recently, Lee et al. [8] presented a method to construct a Hamiltonian path in $Q_n$ with a required vertex in a fixed position. It is noticed that their result is just a special case included in our addressed bipanpositionable bipanconnectedness. Therefore, our study can be thought of a generalization of the previous result. Based on the bipanpositionable bipanconnectedness of hypercubes, many other properties of hypercubes, such as bipancyclic, bipanpositioned Hamiltonicity, etc., can be easily derived. In other words, our study unifies the related researches in a general sense.

It is straightforward to define the panpositionable panconnectedness for the non-bipartite graphs. That is, a graph $G$ is said to be panpositionably panconnected if, for any two distinct vertices $x$ and $y$ of $G$ and for any vertex $z \in V(G) - \{x, y\}$, it contains a path $P_{z,k}$ of length $l$ such that $P_{z,k}(1) = x$, $P_{z,k}(l+1) = y$ for each integer $l$ from $d_G(x, z) + d_G(y, z)$ to $|V(G)| - 1$ and for each integer $k$ from $d_G(x, z)$ to $l - d_G(y, z)$. Then it is also intriguing to address such issue on various kinds of non-bipartite network topologies.

### References