Inner functions of numerical contractions

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\textbf{ABSTRACT}

We prove that, for a function \(f\) in \(H^\infty\) of the unit disc with \(\|f\|_\infty \leq 1\), the existence of an operator \(T\) on a complex Hilbert space \(H\) with its numerical radius at most one and with \(\|f(T)x\| = 2\) for some unit vector \(x\) in \(H\) is equivalent to that \(f\) be an inner function with \(f(0) = 0\). This confirms a conjecture of Drury [S.W. Drury, Symbolic calculus of operators with unit numerical radius, Linear Algebra Appl. 428 (2008) 2061–2069]. Moreover, we also show that any operator \(T\) satisfying the above conditions has a direct summand similar to the compression of the shift \(S(\phi)\), where \(\phi(z) = zf(z)\) for \(|z| < 1\). This generalizes the result of Williams and Crimmins [J.P. Williams, T. Crimmins, On the numerical radius of a linear operator, Amer. Math. Monthly 74 (1967) 832–833] for \(f(z) = z\) and of Crabb [M.J. Crabb, The powers of an operator of numerical radius one, Michigan Math. J. 18 (1971) 253–256] for \(f(z) = z^n\) \((n \geq 2)\).

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An operator $A$ is a numerical contraction (resp., contraction) if $\|A\| \leq 1$ (resp., $\|A\| \leq 1$). In 1967, Sz.-Nagy and Foiaş [16] proved that every numerical contraction is similar to a contraction. Some years later, Okubo and Ando [13] gave another proof basing it on a factorization of the numerical contraction by Ando [1], which has the advantage of a sharp control on the invertible operator implementing the similarity. As a consequence, an estimate on the norm of a function of a numerical contraction can easily be obtained.

**Theorem 1.** (a) An operator $A$ is a numerical contraction if and only if $A = 2(I - B^*B)^{1/2}B$ for some contraction $B$.

(b) If $A$ is a numerical contraction, then $A = XCX^{-1}$ for some invertible operator $X$ with $\|X\|, \|X^{-1}\| \leq \sqrt{2}$ and some contraction $C$.

(c) If $A$ is a numerical contraction and $f : \mathbb{T} \to \mathbb{C}$ is a function analytic on $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and continuous on $\mathbb{T}$, then $\|f(A)\| \leq 2\|f\|_{\infty}$, where $\|f\|_{\infty} = \sup\{|f(z)| : z \in \mathbb{T}\}$.

For our later use, we briefly sketch a proof of Theorem 1(b) based on (a), which is slightly different from the one in [13, Theorem 2]. Let $A$ be factored as in (a). If

$$g(t) = \begin{cases} \sqrt{2(1 - t)} & \text{if } 0 \leq t \leq 1/2, \\ 1/\sqrt{2t} & \text{if } 1/2 \leq t \leq 1, \end{cases}$$

then both $g$ and $1/g$ are continuous functions on $[0, 1]$ with $\|g\|_{\infty} = 1/g(1) = \sqrt{2}$, where $\| \cdot \|_{\infty}$ denotes the supremum of a function over $[0, 1]$. It is easily seen that $X = g(B^*B)$ is invertible, $\|X\|, \|X^{-1}\| \leq \sqrt{2}$ and

$$\|X^{-1}AX\| \leq 2\|g(B^*B)^{-1}(I - B^*B)^{1/2}\| \cdot \|(B^*B)^{1/2}g(B^*B)\| \\ \leq 2 \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = 1.$$

More recently, Drury [7] in studying the norm and numerical radius of $f(A)$ proposed a conjecture on the sharpness of the inequality in Theorem 1(c). The purpose of this paper is to confirm this conjecture with a more detailed information on the structure of $A$.

In the following, we will consider a more general functional calculus than the one in Theorem 1(c) for numerical contractions. Indeed, if $A$ is a numerical contraction on $H$, then the Berger dilation theorem [3] says that there is a unitary operator $U$ on a space $K$ containing $H$ such that $A^n = 2P_HU^n|H$ for all $n \geq 1$, where $P_H$ denotes the (orthogonal) projection from $K$ onto $H$. Such a unitary 2-dilation $U$ of $A$ can be taken to be minimal in the sense that $K = \sqrt{U^nH} : n = 0, \pm 1, \pm 2, \ldots$. In this case, $U$ is uniquely determined up to isomorphism, and, moreover, if $A$ is completely nonunitary, that is, if $A$ has no unitary direct summand, then $U$ is absolutely continuous (cf. [8, Theorem 1] and [14, Proposition 2]).

(We thank G. Cassier and L. Kérchy for providing us the relevant references on this subject.) Hence if $A' = U' \oplus A$ on $L \oplus H$ is a numerical contraction, where $U'$ is absolutely continuous unitary and $A$ is completely nonunitary, then $f(A') = f(U') \oplus (2P_Hf(U)|H - f(0)I)$ for $f$ in $H^{\infty}$ is well-defined, where $U$ is the minimal unitary 2-dilation of $A$. Note that Theorem 1(c) is obviously true for $A$ a numerical contraction with no singular unitary part and $f$ in $H^{\infty}$.

For an inner function $\phi$ ($\phi$ bounded analytic on $\mathbb{D}$ with $|\phi| = 1$ almost everywhere on $\partial \mathbb{D}$), the compression of the shift $S(\phi)$ is defined on $H(\phi) = H^2 \ominus \phi H^2$ by

$$S(\phi)f = P_{H(\phi)}(zf(z))|H(\phi) \quad \text{for } f \in H(\phi).$$

Such operators have been studied extensively since the 1960s starting with the work of Sarason [15]. A nice account of their properties together with those of the more general $C_0$ contractions can be found in [2]. Sz.-Nagy and Foiaş [17] is the classical treatise on further developments of this subject. In particular, if $\phi$ is a Blaschke product with $n$ zeros (counting multiplicity), then $H(\phi)$ is $n$-dimensional.

Its closure $\overline{W(A)}$ contains the spectrum $\sigma(A)$ of $A$. For other properties of the numerical range and numerical radius, the reader may consult [11, Chapter 22] or [10].
Our main result is the following:

**Theorem 2.** Let \( f \) be a function in \( H^\infty \) with \( \|f\|_\infty \leq 1 \). Then there exists a numerical contraction \( T \) with no unitary part such that \( \|f(T)x\| = 2 \) for some unit vector \( x \) if and only if \( f \) is inner and \( f(0) = 0 \). Moreover, any operator \( T \) satisfying the above conditions has a direct summand similar to \( S(\phi) \), where \( \phi(z) = zf(z) \) for \( |z| < 1 \).

A finite-dimensional version of this confirms Drury’s Conjecture 6 in [7].

**Corollary 3.** Let \( f : \mathbb{D} \to \mathbb{C} \) be analytic on \( \mathbb{D} \) and continuous on \( \overline{\mathbb{D}} \) with \( \|f\|_\infty \leq 1 \). Then there exists a numerical contraction \( T \) with \( \|f(T)x\| = 2 \) for some unit vector \( x \) if and only if \( f \) is a finite Blaschke product and \( f(0) = 0 \). In this case, \( f \) has \( n \) zeros (counting multiplicity), then any such \( T \) is unitarily equivalent to an operator of the form \( A \oplus A' \), where \( A \) can be represented by the \((n + 1)\times(n + 1)\) upper-triangular matrix \( [a_{ij}]_{i,j=1}^{n+1} \) with \( a_i = a_j \) satisfying \( a_1 = a_{n+1} = 0 \) and \( |a_i| < 1 \) for all \( i \), and

\[
a_{ij} = \begin{cases} 
\sqrt{2}b_{ij} & \text{if } 1 \leq i \leq j \leq n \text{ or } 2 \leq i < j = n + 1, \\
2b_{ij} & \text{if } i = 1 \text{ and } j = n + 1, \\
b_{ij} & \text{if } 2 \leq i < j \leq n, \\
0 & \text{if } i > j,
\end{cases}
\]

where

\[
b_{ij} = (-1)^{j-i}a_{i+1} \cdots a_{j-1}(1 - |a_i|^2)(1 - |a_j|^2)^{1/2} \quad \text{for } i < j.
\]

The matrix form of \( A \) here is a consequence of Theorem 8(b) below and the matrix representation of the finite-dimensional compression of the shift \( S(\phi) \) (cf. [9, Corollary 1.3]).

A special case of this yields a result of Crabb [5, Theorem 2].

**Corollary 4.** If \( T \) is a numerical contraction and \( \|T^n\| = 2 \) for some \( n \geq 1 \) and some unit vector \( x \), then \( T \) is unitarily equivalent to an operator of the form \( A \oplus A' \), where \( A \) is the \((n + 1)\times(n + 1)\) matrix

\[
\begin{bmatrix}
0 & \sqrt{2} \\
0 & 1 \\
\end{bmatrix}
\quad \text{or} \quad
\begin{bmatrix}
0 & \sqrt{2} \\
0 & 1 \\
\end{bmatrix}
\]

depending on whether \( n = 1 \) or \( n \geq 2 \).

The case \( n = 1 \) was obtained earlier by Williams and Crimmins [18]. It will be invoked in the proof of Theorem 8(b).

We start by proving the sufficiency part of Theorem 2.

**Theorem 5.** Let \( f \) be an inner function with \( f(0) = 0 \) and let \( \phi(z) = zf(z) \) for \( |z| < 1 \). Let \( X = \sqrt{2} \oplus I \oplus (1/\sqrt{2}) \) on \( H(\phi) = H_1 \oplus H_2 \oplus H_3 \), where \( H_1 = \ker S(\phi) \), \( H_3 = \ker S(\phi)^* \), and \( H_2 = H(\phi) \ominus (H_1 \oplus H_3) \), and let \( A = XS(\phi)X^{-1} \). Then \( A \) is a cyclic irreducible operator with no unitary part such that \( W(A) = \overline{\mathbb{D}} \) and \( \|f(A)x\| = 2 \) for some unit vector \( x \).

The next corollary is a special case (cf. [6, Theorem 3.1]).

**Corollary 6.** If \( f \) is a Blaschke product with \( n \) zeros (counting multiplicity), then there is an \((n + 1)\times(n + 1)\) matrix \( A \) with \( W(A) = \overline{\mathbb{D}} \) and \( \|f(A)\| = 2 \).
An operator $A$ on $H$ is cyclic with cyclic vector $x$ if $H = \sqrt{\{A^nx : n \geq 0\}}$. It is easily seen that for a cyclic $A$ the dimension of $\ker A^*$ is at most one.

An operator is irreducible if it is not unitarily equivalent to the direct sum of two other operators. To prove the irreducibility of the operator $A$ in Theorem 5, we need the following lemma.

**Lemma 7.** If $A$ is cyclic with a cyclic vector in $\ker A^*$, then $A$ is irreducible.

**Proof.** Assume that $A = A_1 \oplus A_2$ on $H = H_1 \oplus H_2$. Let $x = x_1 \oplus x_2$, where $x_j \in H_j$, $j = 1, 2$, be a cyclic vector of $A$ in $\ker A^*$. Then $A^*x = (A_1 \oplus A_2^*)(x_1 \oplus x_2) = 0$ implies that $(A_1 \oplus A_2^*)(x_1 \oplus 0) = (A_1 \oplus A_2^*)(0 \oplus x_2) = 0$. On the other hand, since $H_1 \oplus H_2 = \sqrt{\{A_1^n x_1 \oplus A_2^nx_2 : n \geq 0\}}$, we infer that $x_j \neq 0$ for $j = 1, 2$. Thus $x_1 \oplus 0$ and $0 \oplus x_2$ are linearly independent, and, therefore, $\dim \ker A^* = \dim \ker (A_1 \oplus A_2^*) \geq 2$, a contradiction. This proves our assertion. \(\square\)

**Proof of Theorem 5.** Since $\phi(0) = 0$, the function $g \equiv 1$ is in $H(\phi)$. It is a unit cyclic vector for $S(\phi)$ and generates the one-dimensional subspace $H_1$. The other hand, from the facts that $f$ is inner and $\phi(z) = zf(z)$ on $\mathbb{D}$ we can easily check that $f = P_H(\phi)f = f(S(\phi))g$ and $\phi(f) = 0$. Thus $f$ is a unit vector which generates the one-dimensional $H_1$. That $f$ and $g$ are orthogonal follows from a simple computation using $f(0)$. Note also that

$$f(A)g = Xf(S(\phi))X^{-1}g = \sqrt{2}Xf(S(\phi))g = \sqrt{2}Xf = 2f,$$

which shows that $\|f(A)g\| = 2$. Since $g \in \ker S(\phi)^*$ is a cyclic vector for $S(\phi)$, $Xg = g/\sqrt{2} \in \ker A^*$ is cyclic for $A = XS(\phi)X^{-1}$. The irreducibility of $A$ then follows from Lemma 7. Moreover, since $S(\phi)^n$ converges to 0 in the strong operator topology (SOT), the same is true for $A^n$. Hence $A$ has no unitary part.

To prove that $\overline{W(A)} \subseteq \mathbb{D}$, let $B = S(\phi)X^{-1}/\sqrt{2}$. Since $\rank(I - S(\phi)^*S(\phi)) = 1$ and $S(\phi)^*S(\phi)f = 0$, we have $S(\phi)^*S(\phi) = 0 \oplus I \oplus 1$ and hence $B^*B = 0 \oplus (1/2)I \oplus 1$ on $H(\phi) = H_1 \oplus H_2 \oplus H_3$. Therefore, $B$ is a contraction and

$$2(I - B^*B)^{1/2}B = 2 \left(1 \oplus \frac{1}{\sqrt{2}}I \oplus 0\right) \frac{1}{\sqrt{2}}S(\phi)X^{-1}$$

$$= XS(\phi)X^{-1} = A.$$ 

Theorem 1(a) then implies that $\overline{W(A)} \subseteq \mathbb{D}$.

To prove the converse, let $\lambda$ be any point in $\mathbb{D}$. Then the operator $I - \lambda S(\phi)$ is invertible and $u = (I - \lambda S(\phi))^{-1}g - g = \sum_{n=1}^{\infty} \lambda^n S(\phi)^n g$ in norm. Let $v = u - \langle u, f \rangle f$. Note that

$$\langle v, g \rangle = \sum_{n=1}^{\infty} \lambda^n \langle S(\phi)^n g, g \rangle - \langle u, f \rangle \langle f, g \rangle$$

$$= 0 - \langle u, f \rangle \cdot 0 = 0$$

and

$$\langle v, f \rangle = \langle u, f \rangle - \langle u, f \rangle \langle f, f \rangle$$

$$= \langle u, f \rangle - \langle u, f \rangle = 0.$$ 

Hence $v$ is in $H_2$. Finally, letting $y = \langle u, f \rangle f \oplus \sqrt{2}v \oplus g$ in $H(\phi) = H_1 \oplus H_2 \oplus H_3$, we show that $\tilde{\lambda}By = (I - B^*B)^{1/2}y$. Indeed, on the one hand, we have

$$\tilde{\lambda}By = \tilde{\lambda}S(\phi) \left(\frac{1}{2} \oplus \frac{1}{\sqrt{2}}I \oplus 1\right) \langle u, f \rangle f \oplus \sqrt{2}v \oplus g$$

$$= \tilde{\lambda} \left(\frac{1}{2} \langle u, f \rangle S(\phi)f + S(\phi)v + S(\phi)g\right)$$

$$= \tilde{\lambda} \langle S(\phi)(I - \tilde{\lambda} S(\phi))^{-1}g - S(\phi)g \rangle + \tilde{\lambda} S(\phi)g$$

On the other hand,
\[(I - B^*B)^{1/2}y = \left(1 \oplus \frac{1}{\sqrt{2}} I \oplus 0\right)(u,f) + \sqrt{2}v + g\]

\[= (u,f) + v\]

\[= u\]

\[= (I - \bar{\lambda}S(\phi))^{-1}g - g\]

\[= \bar{\lambda}S(\phi)(I - \bar{\lambda}S(\phi))^{-1}g.\]

Thus \(\bar{\lambda}By = (I - B^*B)^{1/2}y\) holds. Hence

\[|\lambda|^2\|By\|^2 = \|(I - B^*B)^{1/2}y\|^2 = \|y\|^2 - \|By\|^2,\]

which implies that \(\|By\|^2 = \|y\|^2/(1 + |\lambda|^2).\) Therefore,

\[\langle Ay, y \rangle = \langle (I - B^*B)^{1/2}By, y \rangle\]

\[= 2\langle By, (I - B^*B)^{1/2}y \rangle = 2\langle By, \bar{\lambda}By \rangle\]

\[= 2\lambda\|By\|^2 = \frac{2\lambda}{1 + |\lambda|^2}\|y\|^2.\]

This shows that \(2\lambda/(1 + |\lambda|^2)\) is in \(W(A)\) for any \(\lambda\) in \(\mathbb{D}\). Hence \(\mathbb{D} \subset W(A)\) and thus \(\overline{W(A)} = \overline{\mathbb{D}}\) as asserted. This completes the proof. \(\square\)

We now proceed to prove the necessity part of Theorem 2.

**Theorem 8.** Let \(f\) be a function in \(H^\infty\) with \(\|f\|_\infty \leq 1\). If \(T\) is a numerical contraction with no singular unitary part such that \(\|f(T)x\| = 2\) for some unit vector \(x\), then

(a) \(f\) is inner with \(f(0) = 0\), and

(b) \(T\) is unitarily equivalent to an operator of the form \(XS(\phi)X^{-1} \oplus A'\), where \(\phi(z) = zf(z)\) for \(|z| < 1\) and \(X = \sqrt{2} \oplus I \oplus (1/\sqrt{2})\) on \(H(\phi) = H_1 \oplus H_2 \oplus H_3\) (\(H_1 = \ker S(\phi)\) and \(H_3 = \ker S(\phi)^*\)).

For the proof of its part (b), we need the following lemma.

**Lemma 9.** Let \(A\) be a \(C_0\) contraction on \(H\) with minimal function \(\phi\). Then there is an operator \(\tilde{A}\) on \(\tilde{H} \supseteq H\) of class \(C_0\) such that (a) \(\tilde{A}H \subseteq H\), (b) \(A = \tilde{A}|H\), and (c) \(\tilde{A}\) is unitarily equivalent to \(\sum_{n=1}^{d} \oplus S(\phi)\), where \(d = \text{rank}(I - A^*A)^{1/2} \leq \infty\).

This appeared in [12, Lemma 4] (with \(T\) there replaced by \(A^*\)) and is dependent on the Sz.-Nagy–Foiaş contraction theory.

**Proof of Theorem 8.** (a) That \(f(0) = 0\) follows from Drury [7, Theorem 4]. Indeed, since the latter is also valid for functions \(f\) in \(H^\infty\) with \(\|f\|_\infty \leq 1\), we have \(\|f(T)\| \leq \nu(f(0))\), where

\[\nu(t) = (2 - 3t^2 + 2t^4 + 2(1 - t^2)(1 - t^2 + t^4)^{1/2})^{1/2} \leq 0 \text{ for } 0 \leq t \leq 1.\]

Our assumption yields that

\[2 = \|f(T)x\| \leq \|f(T)\| \leq \nu(f(0)) \leq 2\]

or \(\nu(f(0)) = 2\). This is equivalent to \(f(0) = 0\).

Let \(M = \sqrt{(T^*T : n \geq 0)}\) and \(A = T|M\). Then \(w(A) \leq 1\) and \(\|f(A)x\| = \|f(T)x\| = 2\). By Theorem 1(a), \(A = 2(I - B^*B)^{1/2}B\) for some contraction \(B\). Let \(g\) be as in (1) and \(X = g(B^*B)\). Then, as indicated before, \(X\) is positive definite and invertible with \(\|X\|, \|X^{-1}\| \leq \sqrt{2}\) and \(C = X^{-1}AX\) is a contraction. It is easily seen that \(C\), being similar to the operator \(A\) with no singular unitary part, is itself without singular unitary part. Thus \(f(C)\) is well-defined. The chain of inequalities
2 = \|f(A)x\| = \|Xf(C)X^{-1}x\|
\leq \|X\|\|f(C)X^{-1}x\| \leq \|X\|\|f(C)\|\|X^{-1}x\|
\leq \|X\|\|f(C)\|\|X^{-1}\| \leq \sqrt{2}\|f\|\infty \sqrt{2} \leq 2,

where \|f(C)\| \leq \|f\|\infty is by the von Neumann inequality, yields equalities throughout. In particular, we have

\|X\| = \|X^{-1}\| = \|f(C)X^{-1}x\| = \|X^{-1}x\| = \sqrt{2}

and \|f(C)\| = \|f\|\infty = 1. Note that for a positive semidefinite operator \(Y\) and vector \(u\), the equalities \(\|Yu\| = \|Y\|\|u\|\) and \(Yu = Y\|u\) are equivalent. Thus from \(\|X^{-1}x\| = \sqrt{2} = \|X^{-1}\|\|X\|\), we infer that \(X^{-1}x = \sqrt{2}x\) or \(Xx = (1/\sqrt{2})x\). Similarly, for \(y = f(C)x\), we have

\[\|y\| = \|f(C)x\| = \frac{1}{\sqrt{2}}\|f(C)X^{-1}x\| = 1\]

and

\[\|Xy\| = \|Xf(C)x\| = \frac{1}{\sqrt{2}}\|Xf(C)X^{-1}x\| = \frac{1}{\sqrt{2}}\|f(A)x\| = \sqrt{2} = \|X\|\|y\|\].

As above, this yields \(Xy = \sqrt{2}y\). Thus \(x\) and \(y\) are eigenvectors associated with the eigenvalues \(1/\sqrt{2}\) and \(\sqrt{2}\) of the positive definite \(X\), respectively. Hence they are orthogonal to each other. Since \(X = g(B^*B)\) with \(g\) defined in (1), we infer that 1 and 0 are eigenvalues of \(B^*B\) with corresponding eigenvectors \(x\) and \(y\), respectively. We also have

\[f(A)x = Xf(C)X^{-1}x = \sqrt{2}Xf(C)x = \sqrt{2}Xy = 2y.\] (2)

From \(B^*By = 0\), we obtain \(By = 0\). Thus

\[A^nf(A)x = A^n(2A^ny) = 2(l - B^*B)^{1/2}By = 0\]

and, consequently,

\[Af(A)A^nx = A^n(AF(A)x) = 0\]

for all \(n \geq 0\). Since \(M\) is generated by \(A^n\), \(n \geq 0\), this yields \(Af(A) = 0\). Hence \(Ce(C) = X^{-1}Af(A)X = 0\), which shows that \(C\) is a \(C_0\) contraction. Let \(\psi\) be its minimal (inner) function, and let \(\phi(z) = zf(z)\). Then \(\psi\) divides \(\phi\). We necessarily have \(\psi(0) = 0\) for otherwise \(\psi\) would divide \(f\), which would imply \(f(C) = 0\), contradicting \(\|f(C)\| = 1\). Hence \(\psi(z) = z\psi(z)\) for some inner function \(\eta\) and \(f(z) = \xi(z)\eta(z)\) for some \(\xi\) in \(H^\infty\) with \(\|\xi\|\infty = 1\). Let \(\xi(z) = \xi(0) + z\xi(z)\) for \(\xi\) in \(H^\infty\). We have \(f(z) = \xi(0)\eta(z) + \zeta(z)\psi(z)\) and thus \(f(C) = \xi(0)\eta(C)\). From

\[1 = \|f(C)\| = \|\xi(0)\|\|\eta(C)\| \leq \|\eta(C)\| \leq 1,\]

we obtain \(\|\xi(0)\| = 1\). Therefore, \(\xi(z) = \xi(0)\xi\) is constant and \(f = \xi(0)\eta\) is inner.

(b) We first show that \(C\) is unitarily equivalent to \(S(\phi)\), where \(\phi(0) = zf(z)\). Note that, from the proof of (a), \(\phi\) is the minimal function of \(C\). By Lemma 9, \(C\) can be extended to an operator unitarily equivalent to \(\sum_{n=1}^\infty \oplus S(\phi)\). Hence \(f(C)\) extends to \(\sum_{n=1}^\infty \oplus f(S(\phi))\). Let \(x = \sum_{n=1}^\infty \oplus g_n\) with \(g_n\) in \(H(\phi)\) for all \(n\). We infer from

\[1 = \|y\|^2 = \|f(C)x\|^2 = \sum_{n=1}^\infty \|f(S(\phi))g_n\|^2 \leq \sum_{n=1}^\infty \|g_n\|^2 = \|x\|^2 = 1\]

that \(\|f(S(\phi))g_n\| = \|g_n\|\) for all \(n\). Since \(f(S(\phi))\) is a contraction, we have \(f(S(\phi))^*f(S(\phi))g_n = g_n\). Thus \(g_n\) is in ran \((f(S(\phi))^*\), a one-dimensional space generated by the function \(g = 1\). Hence, for each \(n \geq 1\), \(g_n = a_ng\) for some scalar \(a_n\). Define the operator \(V : M \rightarrow H(\phi)\) by

\[V(p(C)x) = p(S(\phi))g\]
for any polynomial \( p \). Since \( p(C)x = \sum_{n=1}^{\infty} p(S(\phi))g_n \), we have
\[
\| p(C)x \| = \left( \sum_{n=1}^{\infty} \| p(S(\phi))g_n \|^2 \right)^{1/2} = \| p(S(\phi))g \| \left( \sum_{n=1}^{\infty} |d_n|^2 \right)^{1/2} = \| p(S(\phi))g \| \| x \| = \| p(S(\phi))g \|.
\]

Note that \( M \) being generated by \( A^\dagger x, \; n \geq 0 \), is also generated by \( C^nX^{-1}x = \sqrt{2}C^n x, \; n \geq 0 \). Thus the set of vectors \( p(C)x, \; p \) polynomial, is dense in \( M \). From above, we obtain that \( V \) is an isometry with \( VC = S(\phi)V \). Since \( \phi \) is the minimal function of \( C \) and \( S(\phi) \) follows.

Let \( H_1 \) and \( H_2 \) be the one-dimensional subspaces of \( M \) which are generated by \( y \) and \( x \), respectively, and let \( H_2 = M \oplus (H_1 \oplus H_3) \). On \( M = H_1 \oplus H_2 \oplus H_3 \), the operators \( X \) and \( B^*B \) can be decomposed as \( X = \sqrt{\overline{\mathcal{E}}} \oplus X_1 \oplus (1/\sqrt{\overline{\mathcal{E}}}) \) and \( B^*B = 0 \oplus D \oplus 1 \). Let \( B = [B_{ij}]_{i,j=1}^{3} \) on \( M = H_1 \oplus H_2 \oplus H_3 \). From \( B^*B = 0 \oplus D \oplus 1 \), we obtain \( B_{11}^*B_{11} + B_{21}^*B_{21} + B_{31}^*B_{31} = 0 \), which implies that \( B_{11}, B_{21} \) and \( B_{31} \) are all zero operators. Hence
\[
A = 2(I - B^*B)^{1/2}B
\]
\[
= 2\begin{bmatrix}
1 & (I - D)^{1/2} \\
(I - D)^{1/2} & 0 \\
0 & 0 & B_{12} & B_{13} \\
0 & 0 & B_{22} & B_{23} \\
0 & 0 & B_{32} & B_{33}
\end{bmatrix}
\]
\[
= \begin{bmatrix}
0 & 2B_{12} & 2B_{13} \\
0 & 2(I - D)^{1/2}B_{22} & 2(I - D)^{1/2}B_{23} \\
0 & 0 & 0
\end{bmatrix}
(3)
\]

We now show that \( X_1 = I \). This is done by proving \( DB_{22} = B_{22}/2 \) and \( DB_{23} = B_{23}/2 \). Note that
\[
C = X^{-1}AX
\]
\[
= \begin{bmatrix}
1/\sqrt{2} \\
X_1^{-1} \sqrt{2}X_1 \sqrt{2}X_1^{-1} \sqrt{2}X_1^{-1} \\
0 & 0 & 2B_{12} & 2B_{13} & 2(I - D)^{1/2}B_{22} & 2(I - D)^{1/2}B_{23}
\end{bmatrix}
\]
\[
= \begin{bmatrix}
0 & \sqrt{2}B_{12}X_1 \\
0 & 2X_1^{-1}(I - D)^{1/2}B_{22}X_1 & \sqrt{2}X_1^{-1}(I - D)^{1/2}B_{23}
\end{bmatrix}
= \begin{bmatrix}
0 & C_{12} & C_{13} \\
0 & C_{22} & C_{23}
\end{bmatrix}
\]

Since
\[
I - C^*C = \begin{bmatrix}
1 & 0 & 0 \\
0 & I - C_{12}C_{12} - C_{22}^*C_{22} & 0 \\
0 & 0 & 1 - |C_{13}|^2 - C_{23}^*C_{23}
\end{bmatrix}
\]

has rank one, we have
\[
C_{12}C_{12} + C_{22}^*C_{22} = I
(4)
\]

and
\[
|C_{13}|^2 + C_{23}^*C_{23} = 1.
(5)
\]

From (4), we obtain
\[
I = 2X_1^*B_{12}^*X_1 + 4X_1^*B_{22}^*(I - D)^{1/2}X_1^{*^{-1}}X_1^{-1}(I - D)^{1/2}X_{22}X_1
= 2X_1(B_{12}^*B_{12} + 2B_{22}^*X_1^{-2}(I - D)B_{22})X_1.
(6)
\]

Note that
yields $B_{12}^*B_{12} + B_{22}^*B_{22} + B_{32}^*B_{32} = D$. We derive from (6) that $(1/2)X_1^{-2} = D - B_{22}^*B_{22} - B_{32}^*B_{32} + 2B_{22}^*X_1^{-2}(I - D)B_{22}$ or

$$B_{32}^*B_{32} + B_{22}^*(I - 2X_1^{-2}(I - D))B_{22} = D - \frac{1}{2}X_1^{-2}. \tag{7}$$

Since $X_1 = g(D)$, a simple computation involving the expression of $g$ in (1) yields that $I - 2X_1^{-2}(I - D) \succeq 0$. Hence (7) gives $D \geqslant X_1^{-2}/2 = g(D)^{-2}/2$. Again, from the expression of $g$ in (1), we derive that $D \geqslant 1/2$ and thus

$$X_1 = g(D) = \frac{1}{\sqrt{2}}D^{-1/2}. \tag{8}$$

It follows from (7) that $B_{22}^*(I - 4D(I - D))B_{22} = 0$, which is the same as

$$0 = B_{22}^*(I - 4D + 4D^2)B_{22} = B_{22}^*(I - 2D)^2B_{22}. \tag{9}$$

We thus obtain $(I - 2D)B_{22} = 0$ or $DB_{22} = B_{22}/2$ as asserted.

To prove $DB_{23} = B_{23}/2$, we use (5) to derive that

$$1 = |B_{13}|^2 + 2B_{23}^*(I - D)^{1/2}X_1^{-2}(I - D)^{1/2}B_{23} = |B_{13}|^2 + 2B_{23}^*X_1^{-2}(I - D)B_{23}. \tag{10}$$

Since $B$ is a contraction, we have $|B_{13}|^2 + B_{23}^*B_{23} \leqslant 1$. These two together yield $1 \leqslant 1 - B_{23}^*B_{23} + 2B_{23}^*X_1^{-2}(I - D)B_{23}$ or $B_{23}^*(I - 2X_1^{-2}(I - D))B_{23} \leqslant 0$. Since $I - 2X_1^{-2}(I - D) \succeq 0$ as was noted before, we obtain $B_{23}^*(I - 2X_1^{-2}(I - D))B_{23} = 0$ and thus

$$0 = B_{23}^*(I - 4D(I - D))B_{23} = B_{23}^*(I - 2D)^2B_{23} \tag{11}$$

by (8). Therefore, $(I - 2D)B_{23} = 0$ or $DB_{23} = B_{23}/2$ as required.

From $DB_{22} = B_{22}/2$ and $DB_{23} = B_{23}/2$, we have $(I - D)B_{22} = B_{22}/2$ and $(I - D)B_{23} = B_{23}/2$ and thus $(I - D)^{1/2}B_{22} = B_{22}/\sqrt{2}$ and $(I - D)^{1/2}B_{23} = B_{23}/\sqrt{2}$. It follows from (3) that

$$A = \begin{bmatrix} 0 & 2B_{12} & 2B_{13} \\ 0 & \sqrt{2}B_{22} & \sqrt{2}B_{23} \\ 0 & 0 & 0 \end{bmatrix} \text{ on } M = H_1 \oplus H_2 \oplus H_3. \tag{12}$$

On the other hand, since $M = \sqrt{(A^nx : n \geqslant 0)}$ and $H_2 = M \ominus (\sqrt{\langle x, y \rangle})$, we have $H_2 = \sqrt{(P_2A^nx : n \geqslant 1)}$, where $P_2$ denotes the (orthogonal) projection from $M$ onto $H_2$. A simple computation with (9) shows that $P_2A^nx = (\sqrt{2}B_{22})^{-n-1}(\sqrt{2}B_{23})x$ for all $n \geqslant 1$. Therefore,

$$D(P_2A^nx) = D(\sqrt{2}B_{22})^{-n-1}(\sqrt{2}B_{23})x = \frac{1}{2}(\sqrt{2}B_{22})^{-n-1}(\sqrt{2}B_{23})x = \frac{1}{2}P_2A^nx$$

if $n \geqslant 2$, and

$$D(P_2Ax) = D(\sqrt{2}B_{23})x = \frac{1}{2}(\sqrt{2}B_{23})x = \frac{1}{2}P_2Ax. \tag{13}$$

These show that $D = I/2$ and hence $X_1 = D^{-1/2}/\sqrt{2} = I$ by (8) or $X = \sqrt{2} \oplus I \oplus (1/\sqrt{2})$.

Finally, we prove that $M$ is a reducing subspace of $T$. Since $f$ is inner with $f(0) = 0$, we have $w(f(T)) \leqslant 1$ (cf. [4, Theorem 4]). This, together with $\|f(T)x\| = 2$, yields that the subspace $K = H_1 \oplus H_3$ reduces

\begin{align*}
B^*B &= \begin{bmatrix} 0 & 0 & 0 & B_{12} & B_{13} \\ B_{12}^* & B_{22}^* & B_{23}^* & 0 & 0 \\ B_{13}^* & B_{23}^* & B_{33}^* & 0 & 0 \\ 0 & 0 & 0 & B_{32} & B_{33} \end{bmatrix} \begin{bmatrix} 0 & B_{12} & B_{13} \\ 0 & B_{22} & B_{23} \\ 0 & B_{32} & B_{33} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \end{align*}
\( f(T) \) and \( f(T)|K \) has the matrix representation \[
\begin{bmatrix}
0 & 2 \\
0 & 0
\end{bmatrix}
\] relative to the orthonormal basis \((y, x)\) of \(K\) (cf. Corollary 4 or [18]). In particular, this gives \( f(T)^*x = 0 \) and \( f(T)^*y = 2x \). Now we repeat these with \( T \) and \( f \) replaced by \( T^* \) and \( \tilde{f} \), where \( \tilde{f} \) is the inner function \( \tilde{f}(z) = f(\overline{z}) \) for \( |z| < 1 \). Since \( \tilde{f}(T^*) = f(T)^* \), we have \( \|f(T)^*y\| = 2 \). Letting \( \tilde{M} = \sqrt{(T^*n)y : n \geq 0} \), we infer from what were proved before for \( T \) and \( f \) that \( \tilde{A} = T^*|\tilde{M} = \tilde{X}\tilde{C}\tilde{X}^{-1} \) for some operator

\[
\tilde{C} = \begin{bmatrix}
0 & \tilde{c}_{12} & \tilde{c}_{13} \\
0 & \tilde{c}_{22} & \tilde{c}_{23} \\
0 & 0 & 0
\end{bmatrix}
\quad \text{on } \tilde{M} = \tilde{H}_1 \oplus \tilde{H}_2 \oplus \tilde{H}_3
\]

(\( \tilde{H}_1 = \sqrt{\{\tilde{f}(\tilde{C})y\} \) and \( \tilde{H}_3 = \sqrt{\{|y\}} \) which is unitarily equivalent to \( S(\tilde{\phi}) (\tilde{\phi}(z) = \tilde{f}(z) \) on \( \mathbb{D} \)), and \( \tilde{X} = \sqrt{2} \oplus I \oplus (1/\sqrt{2}) \) on \( \tilde{M} = \tilde{H}_1 \oplus \tilde{H}_2 \oplus \tilde{H}_3 \). We check that \( \tilde{A} \) is unitarily equivalent to \( A^* \). Indeed, since \( C^* \) is unitarily equivalent to \( S(\tilde{\phi}) \) and the latter is in turn unitarily equivalent to \( \tilde{C} \), there is a unitary operator \( U \) mapping \( M \) onto \( \tilde{M} \) such that \( UC^* = \tilde{C}U \). In particular, we have \( U(\ker C^*) = \ker \tilde{C} \) and \( U(\ker C) = \ker \tilde{C}^* \). Note that

\[
\tilde{f}(\tilde{C})y = \frac{1}{2} \tilde{f}(\tilde{A})y = \frac{1}{2} \tilde{f}(T^*)y = \frac{1}{2} f(T)^*y = x
\]

by the analogue of (2). Hence \( \ker C^* = \ker \tilde{C} = \sqrt{\{|x\}} \) and also \( \ker C = \ker \tilde{C}^* = \sqrt{\{|y\}} \). Therefore, \( Ux = \lambda_1 x \) and \( Uy = \lambda_2 y \) for some scalars \( \lambda_1 \) and \( \lambda_2 \) of modulus one. Thus \( U \) is of the form

\[
U = \begin{bmatrix}
\lambda_2 & U_1 \\
\lambda_1 & U_2
\end{bmatrix}
\]

from \( M = \tilde{H}_1 \oplus \tilde{H}_2 \oplus \tilde{H}_3 \) to \( \tilde{M} = \tilde{H}_1 \oplus \tilde{H}_2 \oplus \tilde{H}_3 \) and hence

\[
U^*\tilde{A}U = U^*\tilde{X}\tilde{C}\tilde{X}^{-1}U
= \begin{bmatrix}
\lambda_1 & U_1 \\
\lambda_2 & U_2
\end{bmatrix}
\begin{bmatrix}
\sqrt{2} & I & \tilde{C} & \lambda_1 \\
1/\sqrt{2} & I & \sqrt{2} & \lambda_2 \\
\lambda_2 & U_1 \\
\lambda_1 & \sqrt{2} & U_2 \\
\end{bmatrix}
\begin{bmatrix}
\frac{1}{\sqrt{2}} & I & \sqrt{2} & \lambda_2 \\
\lambda_1 & \sqrt{2} & U_1 \\
\end{bmatrix}
= \begin{bmatrix}
X^{-1}U^*\tilde{C}UX & X^{-1}C^*X = A^*
\end{bmatrix}
\]

Finally, we check that \( \tilde{M} \) is contained in \( M \). This is because, for any \( n \geq 0 \), the equalities

\[
\|T^*y\| = \|\tilde{A}^ny\| = \|UA^nU^*y\|
= \|UA^n(\lambda_2y)\| = \|A^n\| = \|(T|M)^ny\|
\]

hold, which yields that \( T^*y \) belongs to \( M \). Similarly, we can show that \( M \subseteq \tilde{M} \). Hence \( M = \tilde{M} \) and \( T^*M = T^*\tilde{M} \subseteq \tilde{M} = M \). Thus \( M \) reduces \( T \). This completes the proof. \( \square \)

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**References**