Conditional fault hamiltonian connectivity of the complete graph

Tung-Yang Ho a,*, Yuan-Kang Shih b, Jimmy J.M. Tan b, Lih-Hsing Hsu c

a Department of Information Management, Ta Hwa Institute of Technology, Hsinchu, Taiwan 30740, ROC
b Department of Computer Science, National Chiao Tung University, Hsinchu, Taiwan 30010, ROC
c Department of Computer Science and Information Engineering, Providence University, Taichung, Taiwan 43301, ROC

1. Introduction

For the graph definitions and notations, we follow [1]. Let G = (V, E) be a graph if V is a finite set and E is a subset of \{ (u, v) | (u, v) is an unordered pair of V \}. We say that V is the vertex set and E is the edge set. Two vertices u and v are adjacent if (u, v) ∈ E. The complete graph Kn is the graph with n vertices such that any two distinct vertices are adjacent. The degree of a vertex u in G, denoted by deg_G(u), is the number of vertices adjacent to u. We use δ(G) to denote min\{deg_G(u) | u ∈ V(G)\}. A path of length m – 1, \{v_0, v_1, ..., v_{m-1}\}, is an ordered list of distinct vertices such that v_i and v_{i+1} are adjacent for 0 ≤ i ≤ m – 2. We also write the path \{v_0, v_1, P, v_1, ..., v_{m-1}\} for P = \{v_k, ..., v_l\}. A cycle is a path with at least three vertices such that the first vertex is the same as the last one. A hamiltonian cycle of G is a cycle that traverses every vertex of G exactly once. A graph is hamiltonian if it has a hamiltonian cycle. A hamiltonian path is a path of length V(G) – 1.

A Hamiltonian graph G is k edge-fault tolerant Hamiltonian if G – F remains Hamiltonian for every F ∈ E(G) with |F| ≤ k. The edge-fault tolerant Hamiltonicity, \mathcal{H}_k(G), is defined as the maximum integer k such that G is k edge-fault tolerant Hamiltonian connected if G is Hamiltonian connected and is undefined otherwise. Assume that G is a Hamiltonian graph, and x is a vertex such that deg_G(x) = δ(G). We arbitrarily choose deg_G(x) – 1 edges from those edges incident to x to form an edge faulty set F. Obviously, deg_{G-F}(x) = 1; hence, G – F is not Hamiltonian. Therefore, \mathcal{H}_k(G) ≤ δ(G) – 2 if \mathcal{H}_k(G) is defined. Assume that n is an integer with n ≥ 3. It is proved by Ore [9] that any n-vertex graph with at least C(n, 2) – (n – 3) edges is Hamiltonian. Moreover, there exists a non-Hamiltonian n-vertex graph with C(n, 2) – (n – 2) edges. In other words, \mathcal{H}_k(K_n) = n – 3 for n ≥ 3. In [5], it is proved that \mathcal{H}_k(Q_n) = n – 2 for n ≥ 2 where Q_n is the n-dimensional hypercube. In [6], it is proved that \mathcal{H}_k(S_n) = n – 3 for n ≥ 3 where S_n is the n-dimensional star graph.

Chan and Lee [2] began the study of the existence of Hamiltonian cycle in a graph such that each vertex is incident to at least two fault-free edges. A graph G is conditional k edge-fault tolerant Hamiltonian if G – F is Hamilto-
nian for every $F \subseteq E(G)$ with $|F| \leq k$ and $\delta(G - F) \geq 2$. The conditional edge-fault tolerant hamiltonicity, $\mathcal{H}_k^e(G)$, is defined as the maximum integer $k$ such that $G$ is conditional $k$ edge-fault tolerant hamiltonian if $G$ is hamiltonian and is undefined otherwise. Chan and Lee [2] proved that $\mathcal{H}_2^e(Q_n) = 2n - 5$ for $n \geq 3$. Recently, Fu [3] studies the conditional edge-fault tolerant hamiltonicity of the complete graph.

Fault tolerant hamiltonian connectivity is another important parameter for graphs [4]. A graph $G$ is hamiltonian connected if there exists a hamiltonian path between any two distinct vertices of $G$. It is easy to see that a hamiltonian connected graph with at least three vertices is hamiltonian. It is proved by Moon [7] that the degree of any vertex in a hamiltonian connected graph with at least four vertices is at least $3$. A graph $G$ is $k$ edge-fault tolerant hamiltonian connected if $G - F$ remains hamiltonian connected for any $F \subseteq E(G)$ with $|F| \leq k$. The edge-fault tolerant hamiltonian connectivity of a graph $G$, $\mathcal{H}_C(G)$, is defined as the maximum integer $k$ such that $G$ is $k$ edge-fault tolerant hamiltonian connected if $G$ is hamiltonian connected and is undefined otherwise. Assume that $G$ is a hamiltonian connected graph with at least four vertices and $x$ is a vertex such that $\deg_G(x) = \delta(G)$. We arbitrarily choose $\deg_G(x) - 2$ edges from those edges incident to $x$ to form an edge faulty set $F$. Obviously, $\deg_{G - F}(x) = 2$; hence, $G - F$ is not hamiltonian connected. Therefore, $\mathcal{H}_C(G) \leq \delta(G) - 3$ if $\mathcal{H}_C(G)$ is defined. Again, Ore [8] proved that $\mathcal{H}_C(G)(K_n) = n - 4$ for $n \geq 4$.

In this paper, we study the concept of conditional edge-fault tolerant hamiltonian connectivity. Since the degree of any vertex in a hamiltonian connected graph with at least four vertices is at least 3, it is natural to assume that each vertex is incident to at least three fault-free edges. A graph $G$ is conditional $k$ edge-fault tolerant hamiltonian connected if $G - F$ is hamiltonian connected for every $F \subseteq E(G)$ with $|F| \leq k$ and $\delta(G - F) \geq 3$. The conditional edge-fault tolerant hamiltonian connectivity, $\mathcal{H}_C^3(G)$, is defined to be the maximum integer $k$ such that $G$ is conditional $k$ edge-fault tolerant hamiltonian connected if $G$ is hamiltonian connected and is undefined otherwise.

Assume that $n$ is an integer with $n \geq 4$. In this paper, we prove that $\mathcal{H}_C^3(K_n) = 2n - 10$ for $n \notin \{4, 5, 8, 10\}$, $\mathcal{H}_C^3(K_4) = 0$, $\mathcal{H}_C^3(K_5) = 2$, $\mathcal{H}_C^3(K_6) = 5$, and $\mathcal{H}_C^3(K_10) = 9$. To reach this goal, we present some preliminary in the following section. In Section 3, we prove our main result.

2. Preliminary

Let $F$ be a faulty edge set. We define $K_n(F)$ be a graph with $E(K_n(F)) = F$ and $V(K_n(F)) = V(K_n)$. The following statement is proved in [3]:

$$\mathcal{H}_2^e(K_n) = 2n - 8 \text{ for } n \notin \{7, 9\} \text{ and } n \geq 4, \mathcal{H}_2^e(K_7) = 5, \text{ and } \mathcal{H}_2^e(K_9) = 9.$$ 

Yet, it is easy to check that $\mathcal{H}_2^e(K_5) = 0$ and $\mathcal{H}_2^e(K_4)$ is 2 (not 0). Thus, we have the following theorem.

**Theorem 1.** $\mathcal{H}_2^e(K_n) = 2n - 8 \text{ for } n \notin \{7, 9\} \text{ and } n \geq 5, \mathcal{H}_2^e(K_5) = 0, \mathcal{H}_2^e(K_4) = 2, \mathcal{H}_2^e(K_7) = 5, \text{ and } \mathcal{H}_2^e(K_9) = 9.$

**Proof.** Suppose that the lemma is false. Then $\deg_{K_n(F)}(w) \geq \left\lceil \frac{2n - 1}{2} \right\rceil$ for every vertices with $\deg_{K_n(F)}(w) \neq 0$. Obviously, there are at least $\left\lceil \frac{2n - 1}{2} \right\rceil + 1$ vertices with $\deg_{K_n(F)}(w) \neq 0$. Hence, $|F| \geq (\left\lceil \frac{2n - 1}{2} \right\rceil + 1)/2 > 2n - 10$ for $n \notin \{8, 10\}$ and $(\left\lfloor \frac{2n - 1}{2} \right\rfloor + 1)/2 > 2n - 11$ for $n \in \{8, 10\}$. It is a contradiction. The lemma is proved.

The following theorem can be found in [1].

**Theorem 2.** (See [1].) Let $D = (d_1, d_2, \ldots, d_n)$ be a non-increasing sequence with $d_1 \geq 1$ and $d_i \geq 0$ for $2 \leq i \leq n$. We set $D' = (d'_1, d'_2, \ldots, d'_n)$ such that $d'_i = (d_i - 1, d_i - 1, \ldots, d_{i+1}, d_{i+1} - 1, \ldots, d_n)$. Then there exists a graph $G$ with vertex set $\{x_1, x_2, \ldots, x_n\}$ such that $\deg_G(x_i) = d_i$ for $1 \leq i \leq n$ if and only if there exists a graph $G'$ with vertex set $\{y_1, y_2, \ldots, y_n\}$ such that $\deg_{G'}(y_j) = d'_j$ for $1 \leq j \leq n - 1$.

By the above theorem, we know that there is a graph $G$ with degree sequence $D$ if and only if there is a graph $G'$ with degree sequence $D'$. If $d'_i < 0$ for some $i$, then $D'$ is not the degree sequence of any graph, neither is $D$.

**Lemma 2.** Let $F$ be a subset of $E(K_9)$ with $|F| = 8$ and $\delta(K_9 - F) \geq 3$. Let $u$ and $v$ be any two distinct vertices in $K_9$ such that $\deg_{K_9(F)}(u) = 0$ and $\deg_{K_9(F)}(v) = 0$. Then there exists a vertex $w$ with $\deg_{K_9(F)}(w) \in \{2, 3\}$.

**Proof.** Let $\{x_1, x_2, \ldots, x_9 = u, x_9 = v\}$ be the vertex set of $K_9$ such that $\deg_{K_9(F)}(x_i) = d_i$ and $d_1 \geq d_2 \geq \cdots \geq d_9$. Obviously, $\sum_{i=1}^{9} d_i = 16$. Assume that the lemma is false. Then $\deg_{K_9(F)}(x_i) \in \{0, 1, 4, 5\}$ for $1 \leq i \leq 9$. By brute force, all such sequences are listed below: $(5, 5, 5, 1, 0, 0, 0, 0, 0), (5, 5, 4, 1, 1, 0, 0, 0, 0), (4, 4, 4, 1, 1, 1, 1, 0, 0)$, and $(4, 4, 4, 1, 1, 1, 1, 0, 0)$. By Theorem 2, we can check that such a graph does not exist. Hence, the lemma is proved.

**Lemma 3.** Let $F$ be a subset of $E(K_{11})$ with $|F| = 12$ and $\delta(K_{11} - F) \geq 3$. Let $u$ and $v$ be any two distinct vertices in $K_{11}$ such that $\deg_{K_{11}(F)}(u) = 0$ and $\deg_{K_{11}(F)}(v) = 0$. Then there exists a vertex $w$ with $\deg_{K_{11}(F)}(w) \in \{2, 3, 4\}$.

**Proof.** Let $\{x_1, x_2, \ldots, x_{10} = u, x_{11} = v\}$ be the vertex set of $K_{11}$ such that $\deg_{K_{11}(F)}(x_i) = d_i$ and $d_1 \geq d_2 \geq \cdots \geq d_{11}$.
Lemma 5. The graph of the Kn×n, where n is a complete graph of Kn×n. Thus, we can easily obtain the following lemma.

**Lemma 4.** Let k ≥ 2. Let G be a hamiltonian connected graph. Then deleting any set S of k vertices from G, the resulting graph G−S contains at most k−1 connected components.

By the above lemma, we have a simple observation.

**Lemma 5.** Let k ≥ 2. Let G be a graph. If there is a set S of k vertices such that G−S contains k or more connected components, then G is not hamiltonian connected.

### 3. Main result

**Lemma 6.** Let n ≥ 4 and F ∈ E(Kn) with δ(Kn − F) ≥ 3. Then Kn − F is hamiltonian connected if |F| ≤ 2n − 10 for n ∈ {4, 5, 8, 10}; |F| = 0 for n = 4, |F| ≤ 2 for n = 5, and |F| ≤ 2n − 11 for n ∈ [8, 10].

**Proof.** We prove this lemma by induction on n. Yet, we should be very careful because the size of |F| is depending on n. Without loss of generality, we assume that |F| = 2n − 10 for n ∈ {4, 5, 8, 10}, |F| = 0 for n = 4, |F| = 2 for n = 5, and |F| = 2n − 11 for n ∈ [8, 10]. The induction bases are n = 4, n = 5, and n = 6. Suppose n = 4 and |F| = 0. It is easy to see that the complete graph Kn is hamiltonian connected. Suppose n = 5 and |F| = 2. To keep δ(Kn − F) ≥ 3, F forms two independent edges. By brute force, it is easy to check whether Kn − F is hamiltonian connected. Suppose that n = 6 and |F| = 2. Obviously, F is either two adjacent edges or two independent edges. Again, by brute force, we can check that Kn − F is hamiltonian connected.

Now, we assume that n ≥ 7. Let u and v be any two vertices of Kn. The lemma follows if we can find a hamiltonian path of Kn − F between u and v.

**Case 1.** degKn−F(u) ≠ 0 or degKn−F(v) ≠ 0. Without loss of generality, we assume that degKn−F(u) = k ≠ 0. Let i₁, ..., iₖ be the vertices such that (u, iₙ) ∈ F for 1 ≤ j ≤ k. Let F ′ = (F − {u, i₁, ..., u, iₖ}) ∪ {v, i₁, ..., v, iₖ}. Obviously, |F ′| < |F|. Now, we consider Kn−{u} as a complete graph of (n−1) vertices with faulty edge set F ′. Obviously, |F ′| ≤ 2(n − 1) − 8 for n ∈ [8, 10] and |F ′| ≤ 2(n − 1) − 9 for n ∈ [8, 10]. Moreover, δ(Kn−{u} − F ′) ≥ 2. Thus, we can apply Theorem 1 to obtain a hamiltonian cycle C in Kn−{u} − F ′. Without loss of generality, we write C as {v, x, ..., y, v}. Then, (u, x, ..., y, v) forms a hamiltonian path of Kn−F joining u to v.

**Case 2.** degKn−F(u) = 0 and degKn−F(v) = 0. By Lemmas 1, 2, and 3, there exists a vertex w such that degKn−{w}−F(x) = 2. Obviously, (x, w) ∈ F, degKn−{w}−F(x) = 3, and degKn−{w}−F(x) = n − 4. We claim that x is the only vertex in Kn−w with degKn−{w}−F(x) = 2. If otherwise, let z be another vertex in Kn−w with degKn−{w}−F(z) = 2. Then |F| ≥ degKn−{w}−F(x) + degKn−{w}−F(z) = 1 − 2n − 9. This is impossible because |F| ≤ 2n − 10. Thus, x is the only vertex in Kn−w such that degKn−{w}−F(x) = 2. Thus, δ(Kn−F − {u, x}) ≥ 3.

Let F ′ = F − {(x, i) | i ∈ V(Kn)}. We consider Kn−{u, x} as a complete graph of (n−2) vertices with faulty edge set F ′. Obviously, |F ′| ≥ 1 ≤ 2 for n = 7, |F ′| = n − 7 ≤ 2(n − 2) − 10 for n ∈ {10, 12}, and |F ′| = n − 7 ≤ 2(n − 2) − 11 for n ∈ [10, 12]. By induction, we have a hamiltonian path P of Kn−{u, x} − F ′ joining w to v. So (u, x, w, P, v) forms a hamiltonian path of Kn−F joining u to v.

Now, we consider δ(Kn−{w}−F) ≤ 2. Thus, we have a hamiltonian path of Kn−F − {w}−F joining u to v.

**Theorem 3.** Let n ≥ 4. Then HC2(Kn) = 2n − 10 for n ∈ {4, 5, 8, 10}, HC2(K4) = 0, HC2(K5) = 2, HC2(K6) = 5, and HC2(K8) = 9.

**Proof.** Let F be any subset of E(Kn) with δ(Kn − F) ≥ 3. Since δ((Kn − F) − F) ≥ 3, |F| = 0 for n = 4 and |F| ≤ 2 for n = 5. Hence, HC2(K4) = 0 and HC2(K5) = 2.

Suppose n = 8. Let V(K8) = {x₁, x₂, ..., x₈}. We set R = {x₁, x₂, ..., x₈}, S = {x₈}, and P = {(u, v) | u, v ∈ R}. We can check that δ((Kn−F) − F) ≥ 3, |F| = 6 and (Kn−F) − S has four connected components. By Lemma 5, Kn−F is not hamiltonian connected. See Fig. 1(a) for illustration. Thus, HC2(K8) = 6. By Lemma 6, HC2(K8) = 5.

Suppose n = 10. Let V(K10) = {x₁, x₂, ..., x₁₀}. We set R = {x₁, ..., x₅}, S = {x₆, ..., x₁₀}, and P = {(u, v) | u, v ∈ R}. We can check that δ((Kn−F) − F) ≥ 3, |F| = 7 and (Kn−F) − S has four connected components. By Lemma 5, Kn−F is not hamiltonian connected. See Fig. 1(b) for illustration. Thus, HC2(K10) = 6. By Lemma 6, HC2(K10) = 5.
Then, $\delta(K_{10} - F) \geq 3$, $|F| = 10$, and $(K_{10} - F) - S$ has five connected components. By Lemma 5, $K_{10} - F$ is not hamiltonian connected. See Fig. 1(b) for illustration. Thus, $\mathcal{HC}^3_e(K_{10}) < 10$. By Lemma 6, $\mathcal{HC}^3_e(K_{10}) = 9$. 

Suppose that $n \in \{6, 7, 9\} \cup \{i \mid i \geq 11\}$. Let $V(K_n) = \{x_1, x_2, \ldots, x_n\}$. We set $R = \{x_1, x_2\}$, $S = \{x_3, x_4, x_5\}$, $T = \{x_6, \ldots, x_n\}$, and $F = \{(u, v) \mid u \in R, v \in R \cup T\}$. Obviously, $\delta(K_n - F) \geq 3$, $|F| = 2(n - 5) + 1 = 2n - 9$, and $(K_n - F) - S$ has three connected components. See Fig. 1(c) for illustration for case $n = 9$. By Lemma 5, $K_n - F$ is not hamiltonian connected. Thus, $\mathcal{HC}^3_e(K_n) < 2n - 9$. By Lemma 6, $\mathcal{HC}^3_e(K_n) = 2n - 10$.

The theorem is proved. \(\square\)

References