1. Introduction

This work investigates zeta functions for \( d \)-dimensional shifts of finite type, \( d \geq 3 \). First, the three-dimensional case is studied. The trace operator \( T_{a_1,a_2,b_2} \) and rotational matrices \( R_{a_1,a_2,b_2} \) and \( R_{a_1,a_2,b_2} \) are introduced to study \( \mathbb{Z},a_1,a_2,b_2; \mathbb{Z},a_1,a_2,b_2 \) periodic patterns. The rotational symmetry of \( T_{a_1,a_2,b_2} \) induces the reduced trace operator \( \tau_{a_1,a_2,b_2} \) and then the associated zeta function \( \zeta_{a_1,a_2,b_2} = (\det(I - s\tau_{a_1,a_2,b_2}))^{-1} \). The zeta function \( \zeta \) is then expressed as \( \zeta = \prod_{a_1} \prod_{a_2} \prod_{b_2} \tau_{a_1,a_2,b_2} \), a reciprocal of an infinite product of polynomials. The results hold for any inclined coordinates, determined by unimodular transformation in \( GL_d(\mathbb{Z}) \). Hence, a family of zeta functions exists with the same integer coefficients in their Taylor series expansions at the origin, and yields a family of identities in number theory. The methods used herein are also valid for \( d \)-dimensional cases, \( d \geq 4 \), and can be applied to thermodynamic zeta functions for the three-dimensional Ising model with finite range interactions.

Keywords: Zeta function; shift of finite type; patterns generation problem; phase-transition; Ising model; cellular neural networks.

\[ \zeta(s) = \exp \left( \sum_{[L] \in \mathcal{L}} \frac{\Gamma_L(\phi)}{|L|} s^{|L|} \right), \]
generated by admissible local patterns \( \mathcal{B} \). \( \zeta_B^d \) is defined by

\[
\zeta_B^d = \exp \left( \sum_{k=1}^{\infty} \sum_{l=0}^{n-1} \frac{1}{nk} \Gamma_B \left( \begin{array}{l} n \\ l \end{array} \right) \right) .
\]

(2)

and the \( n \)th order zeta function \( \zeta_{B,n}(s) \) is

\[
\zeta_{B,n}(s) = \exp \left( \frac{1}{n} \sum_{k=1}^{\infty} \sum_{l=0}^{n-1} \frac{1}{nk} \Gamma_B \left( \begin{array}{l} n \\ l \end{array} \right) \right) \quad (3)
\]

for any \( n \geq 1 \), where \( \Gamma_B \left( \begin{array}{l} n \\ l \end{array} \right) \) is the number of \( \left[ \begin{array}{l} n \\ l \end{array} \right] \) periodic patterns that can be generated by \( \mathcal{B} \). The zeta function \( \zeta_B(s) \) is now given by

\[
\zeta_B(s) = \prod_{n=1}^{\infty} \zeta_{B,n}(s) .
\]

(4)

In deriving Eq. (3), the trace operator \( T_B(\mathcal{B}) \) and rotational matrix \( R_{\mathcal{B}} \) are introduced to accommodate the periodic patterns. Based on the rotational symmetry of the trace operator, the reduced trace operator \( \tau_B(\mathcal{B}) \) is defined. \( \zeta_{B,n} \) and \( \zeta_B \) can be expressed as

\[
\zeta_{B,n} = (dI - s^n \tau_B)^{-1}
\]

and

\[
\zeta_B = \prod_{n=1}^{\infty} (dI - s^n \tau_B)^{-1} .
\]

(5)

(6)

The latter is a reciprocal of an infinite product of polynomials. The results also hold when inclined coordinates are used for any unimodular transformation \( \gamma \in GL_2(\mathbb{Z}) \). Therefore, there exists a family of zeta functions with the same integer coefficients of their Taylor series expansions at \( s = 0 \) and the family of zeta functions yields a family of identities.

The two-dimensional thermodynamic zeta functions for the Ising model with finite range interactions are also studied.

It is clear that in many situations the three-dimensional problems are more related to our real world phenomena. In this work, the zeta functions of \( d \)-dimensional shifts of finite type are studied for \( d \geq 3 \), and the previous results of \( Z^2 \) are extended. For simplicity, only the zeta functions for three-dimensional shifts of finite type are introduced and the general case is studied in Sec. 5.

Let \( Z_{m \times m \times m} \) be the \( m \times m \times m \) cubic lattice in \( Z^3 \) and \( \mathcal{B} \) be the finite set of symbols (alphabets or colors), \( S_{m \times m \times m} = \left\{ \right\} \) is the set of all local patterns on \( Z_{m \times m \times m} \). Denote \( \mathcal{B} \subset S_{m \times m \times m} \) as a basic set of admissible local patterns and \( \mathcal{P}(\mathcal{B}) \) the set of all periodic patterns that are generated by \( \mathcal{B} \) on \( Z^3 \).

As in two other works [Lind, 1996; Ban et al., 2008a], the Hermite normal form [MacDuffie, 1956] can be used to parameterize \( \mathcal{L}_3 \) as

\[
\mathcal{L}_3 = \left\{ \begin{array}{c} \{ a_1 b_{12} b_{l3} \} \\
0 a_2 b_{23} \\
0 0 a_3 \end{array} \right\} Z^3 : a_i \geq 1, i = 1, 2, 3,
\quad 0 \leq b_{ij} \leq a_i - 1, i + 1 \leq j \leq 3 \}
\]

Given a basic set \( \mathcal{B} \), let \( L = \left\{ a_1 b_{12} b_{l3} \right\} Z^3 \in \mathcal{L}_3 \), denote \( \mathcal{P}_B(\left\{ a_1 b_{12} b_{l3} \right\}) \) as the set of all \( L \)-periodic patterns that are generated by \( \mathcal{B} \) on \( Z^3 \) and \( \Gamma_B(\left\{ a_1 b_{12} b_{l3} \right\}) \) as the number of \( \mathcal{P}_B(\left\{ a_1 b_{12} b_{l3} \right\}) \).

Then, the zeta function in Eq. (1) is

\[
\zeta_B^d = \exp \left( \sum_{i=1}^{\infty} \sum_{j=1+1}^{a_i-1} \sum_{k=0}^{a_i-1} \frac{1}{a_1 a_2 a_3} \right)
\]

(7)

Similar to Eqs. (3) and (4), the \((a_1, a_2, b_{12})\)th zeta function is defined by

\[
\zeta_{B,a_1,a_2,b_{12}}(s) = \exp \left( \frac{1}{a_1 a_2} \sum_{i=1}^{\infty} \sum_{j=1+1}^{a_i-1} \sum_{k=0}^{a_i-1} \frac{1}{a_3} \right)
\]

(8)

and the zeta function \( \zeta_{B}(s) \) is given by

\[
\zeta_B(s) = \prod_{n=1}^{\infty} \prod_{l=0}^{n-1} \zeta_{B,a_1,a_2,b_{12}}(s) .
\]

(9)

The trace operator \( T_{B,a_1,a_2,b_{12}}(\mathcal{B}) \) and rotational matrices \( R_{B,a_1,a_2,b_{12}} \) and \( R_{B,a_1,a_2,b_{13}} \) are introduced. After the rotational symmetry of \( T_{B,a_1,a_2,b_{12}} \) is demonstrated the reduced trace operator \( \tau_{B,a_1,a_2,b_{12}}(\mathcal{B}) \) can be defined. Finally, as in Eq. (5),
\(\zeta_{\delta_{a_1},a_2,b_2}(s)\) can be represented as a rational function:
\[
\zeta_{\delta_{a_1},a_2,b_2}(s) = (\det(I - s^{a_2} \tau_{a_1,a_2,b_2}))^{-1}.
\tag{10}
\]

Hence,
\[
\zeta_0(s) = \prod_{a_1=1}^{\infty} \prod_{a_2=1}^{\infty} \prod_{b_2=0}^{b_1} (\det(I - s^{a_2} \tau_{a_1,a_2,b_2}))^{-1}
\tag{11}
\]
is a reciprocal of an infinite product of polynomials. The proof of Eq. (10) in this paper is new and simpler than in an earlier work [Ban et al., 2008a], in which the proof is also valid for any \(d \geq 3\).

Additionally, for any \(\gamma \in GL_3(\mathbb{Z})\), the zeta function can also be represented in \(\gamma\)-coordinates. Therefore, a family of zeta functions exists that have the same integer coefficients in their Taylor series expansions at \(s = 0\).

According to [Lind, 1996] and [Ruelle, 1978], the thermodynamic zeta function with weight function \(\theta : X \to (0, \infty)\) is defined as
\[
\zeta_{\theta}(s) = \exp \left( \sum_{L \in \mathcal{L}_3} \frac{\sum_{\phi \in f_{\mathcal{L}_3}(\theta)} \prod_{\mathbf{k} \in \mathbb{Z}^3/L} \theta(\phi^k \mathbf{x})}{|L|^2} \right),
\tag{12}
\]
where \(f_{\mathcal{L}_3}(\phi)\) is the set of points fixed by \(\phi\) for all \(\mathbf{n} \in L\). Let \(\phi\) be a shift of finite type given by \(B\). As in the two-dimensional case [Ban et al., 2008a], the thermodynamic zeta function for the three-dimensional Ising model with finite range interactions can also be represented as a reciprocal of an infinite product of polynomials. The three-dimensional model can be applied to study three-dimensional phase-transition problems. Further results need to be investigated.


The rest of this article is organized as follows. Section 2 discusses periodic patterns, the trace operators and rotational matrices. Section 3 shows the rotational symmetry of trace operator and introduces the reduced trace operator. The rationality of \(\zeta_{\delta_{a_1},a_2,b_2}\) is then obtained for \(a_1, a_2 \geq 1, 0 \leq b_2 \leq a_1 - 1\). Section 4 studies the zeta function in \(\gamma\)-coordinates for \(\gamma \in GL_3(\mathbb{Z})\). Section 5 extends the previous result to \(d\)-dimensional, \(d \geq 4\), and to more symbols on a larger lattice. The thermodynamic zeta function for the three-dimensional Ising model with finite range interactions is also investigated.

2. Periodic Patterns, Trace Operator and Rotational Matrices

This section studies the properties of the periodic patterns and derives trace operator and rotational matrices. Furthermore, \(\Gamma_\mathcal{L}_3\left(\begin{smallmatrix} a_1 & b_1 & a_2 \\ 0 & a_2 & 0 \\ 0 & 0 & b_2 \end{smallmatrix}\right)\) can be expressed in terms of the trace of the products of the trace operator and rotational matrices.

For clarity, two symbols on \(2 \times 2 \times 2\) lattice \(Z_{2x2x2}\) are examined first. For given positive integers \(N_1, N_2, N_3\), the rectangular lattice \(Z_{N_1 \times N_2 \times N_3}\) is defined by
\[
\{(n_1, n_2, n_3) : 0 \leq n_i \leq N_i - 1, 1 \leq i \leq 3\}.
\]
In particular,
\[
Z_{2 \times 2 \times 2} = \{(0,0,0),(0,0,1),(0,1,0),(0,1,1),
(1,0,0),(1,0,1),(1,1,0),(1,1,1)\}.
\]
Define the set of all global patterns on \(Z^3\) with two symbols \(\{0, 1\}\) by
\[
\Sigma_2^3 = \{0,1\}^{Z^3} = \{U : Z^3 \to \{0,1\}\}
\]
Here, \(Z^3 = \{(n_1, n_2, n_3) : n_1, n_2, n_3 \in \mathbb{Z}\}\), the set of all three-dimensional lattice points (vertices). The set of all local patterns on \(Z_{N_1 \times N_2 \times N_3}\) is defined by
\[
\Sigma_{N_1 \times N_2 \times N_3} = \{U|_{Z_{N_1 \times N_2 \times N_3}} : U \in \Sigma_2^3\},
\]
and...
and a local pattern of a global pattern $U$ on $\mathbb{Z}_{N_1 \times N_2 \times N_3}$ is denoted by

$$U_{N_1 \times N_2 \times N_3} = \{(u_{a_1 a_2 a_3})_{0 \le i \le N_1-1, 0 \le j \le N_2-1, 0 \le k \le N_3-1}\},$$

where $u_{a_1 a_2 a_3} \in \{0, 1\}$. To simplify the notation, the subscripts of $U_{N_1 \times N_2 \times N_3}$ and $(u_{a_1 a_2 a_3})_{0 \le i \le N_1-1, 0 \le j \le N_2-1, 0 \le k \le N_3-1}$ are omitted whenever such omission will not cause confusion.

Now, for any given $B \subset \Sigma_{2 \times 2 \times 2}$, $B$ is called a basic set of admissible local patterns. In short, $B$ is a basic set. A local pattern $U_{N_1 \times N_2 \times N_3} = (u_{a_1 a_2 a_3})$ is called $B$-admissible if for any vertex (lattice point) $(n_1, n_2, n_3)$ with $0 \le n_1 \le N_1 - 2$, $1 \le i \le 3$, there exists a $2 \times 2 \times 2$ admissible local pattern $(u_{i_1 i_2 i_3}) \in B$ such that

$$u_{a_1 + i_1 n_1 + i_2 n_2 + i_3 n_3} = \beta_{i_1 i_2 i_3}$$

for $0 \le k_1, k_2, k_3 \le 1$.

Given a lattice $L \in L_3$ with Hermite normal form,

$$L = \begin{bmatrix} a_1 & b_1 & b_1 a_3 \\ 0 & a_2 & b_3 \\ 0 & 0 & a_3 \end{bmatrix} \mathbb{Z}^3,$$

where $a_i \ge 1$ for $1 \le i \le 3$ and $0 \le b_i \le a_i - 1$ for $i + 1 \le j \le 3$. A global pattern

$$U = (u_{a_1 a_2 a_3})_{0 \le i \le N_1-1, 0 \le j \le N_2-1, 0 \le k \le N_3-1}$$

is called $L$-periodic if for every $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{Z}$

$$u_{a_1 + \alpha_1 p + \alpha_2 q + \alpha_3 r, n_1 + \alpha_1 n_2 + \alpha_2 n_3 + \alpha_3 n_3} = u_{a_1 a_2 a_3}$$

for all $p, q, r \in \mathbb{Z}$.

The periodicity of

$$\begin{bmatrix} a_1 & b_1 & b_1 a_3 \\ 0 & a_2 & b_3 \\ 0 & 0 & a_3 \end{bmatrix}$$

and

$$\begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}$$

are closely related as follows.

**Proposition 2.1.** For $a_i \ge 1, 1 \le i \le 3$, $0 \le b_i \le a_i - 1$, $i + 1 \le j \le 3$, let

$$s_1 = \frac{a_1}{(a_1, b_1)} \quad \text{and} \quad s_2 = \begin{bmatrix} a_1 & s_1 s_2 \\ (a_1, b_1) & (s_1 s_2, b_3) \end{bmatrix}.$$

Then, $s_1$ and $s_2$ are the least common divisors of $m$ and $n$ and $[p, q]$ is the least common multiple of $p$ and $q$.

**Proof.** By Eq. (14), the $\begin{bmatrix} a_1 & s_1 s_2 \\ (a_1, b_1) & (s_1 s_2, b_3) \end{bmatrix}$-periodic pattern is easily identified as for all $m_1, m_2 \in \mathbb{N}$. By taking $m_1 = s_1$ and $m_2 = s_2$, the result holds.

Now, the admissibility of $L$-periodic patterns is demonstrated to be verified on finite cubic lattices.

**Proposition 2.2.** An $L$-periodic pattern $U$ is $B$-admissible if and only if

$$U_{[\mathbb{Z}_{2 \times 2 \times 2}((a_1 a_2 a_3))]} = \mathbb{B}$$

for $0 \le \alpha_i \le a_i - 1, 1 \le i \le 3$.

**Proof.** Since $B \subset \Sigma_{2 \times 2 \times 2}$, it is sufficient to prove

$$\{U_{[\mathbb{Z}_{2 \times 2 \times 2}((a_1 a_2 a_3))]} : \alpha_1, \alpha_2, \alpha_3 \in \mathbb{Z}\} = \{U_{[\mathbb{Z}_{2 \times 2 \times 2}((a_1 a_2 a_3))]} : 0 \le \alpha_i \le a_i - 1, 1 \le i \le 3\}.$$

The proof follows easily from Eq. (14). The details are left to the reader.

According to Proposition 2.2, the admissibility of an $L$-periodic pattern $U$ is determined by

$$U_{[\mathbb{Z}_{[m]_n \times [n]_m \times m/n]}((a_1 a_2 a_3))} = (u_{a_1 a_2 a_3})$$

and $U_{[\mathbb{Z}_{[m]_n \times [n]_m \times m/n]}((a_1 a_2 a_3))}$ has the periodic property that is given by Eq. (14), which can be divided into two parts:

$$\begin{cases} u_{a_1 a_2 a_3} = u_{\alpha_1 a_2 a_3} \\ u_{a_1 a_2 a_3} = u_{(a_1 - 1) b_1 a_3} \end{cases}$$

for $0 \le \alpha_i \le a_i, 1 \le i \le 3$, where $[m]_n \equiv m \pmod{n}$.
for $0 \leq \alpha_1 \leq a_1$, $0 \leq \alpha_2 \leq a_2$.

Notably, $(u_{n_1,n_2}(\alpha_1,a_1),b_{n_1,n_2}(\alpha_2,a_2),\psi_{n_1,n_2}(\alpha_3))$ has the same structure Eq. (15) for all $0 \leq \alpha_3 \leq a_3$; this fact is useful in constructing the cylindrical ordering matrix. Then, the set of all local patterns in $\Sigma_{n_1+1,n_2+1,n_1+1}$ that satisfy the periodic property Eq. (15) is denoted by $\mathcal{P}_{n_1,n_2}(\alpha_1,a_1)$. However, Eq. (16) is important in allowing patterns in $\mathcal{P}_{n_1,n_2}(\alpha_1,a_1)$ to become $L$-periodic and it will be used to define the rotational matrices later.

Now, the counting function for $U_{n_1,n_2}(i,i)$ in $\Sigma_{n_1,n_2,n_3}$, $n_1, n_2, n_3 \geq 1$, is defined by

$$\psi(U_{n_1,n_2}(i,i)) = 1 + \sum_{\alpha_1=0}^{a_1} \sum_{\alpha_2=0}^{a_2} u_{n_1,n_2}(\alpha_1,a_1) \times 2^{n_2+i_1(n_1-i_3)+i_3(n_1-i_2)+i_2-n_1-1}. \quad (17)$$

Similar to Eq. (17), the counting function $\psi$ for patterns $U$ in $\mathcal{P}_{n_1,n_2}(i,i)$, $0 \leq i \leq n_1$, is defined by

$$\psi(U) = \psi(U_{n_1,n_2}(i,i)). \quad (18)$$

Notably, $\psi$ is bijective from $\mathcal{P}_{n_1,n_2}(i,i)$ to $\{i \mid 0 \leq i \leq 2^{n_1}\}$.

Given $n_1, n_2 \geq 1$, $0 \leq l \leq n_1 - 1$, $h \geq 1$, a local pattern $\Gamma$ in $\mathcal{P}_{n_1,n_2}(i,i)$ can be represented as

$$\Gamma = \Gamma_0 \oplus \Gamma_1 \oplus \cdots \oplus \Gamma_{h-1} \quad (19)$$

where $\Gamma_0 \in \mathcal{P}_{l,n_2}(i,i)$, $0 \leq i \leq h - 1$, and $\Gamma_0 \oplus \cdots \oplus \Gamma_{h-1}$ means that $\Gamma_0$ is put on the top (in the $z$-direction) of $\Gamma$. Therefore, the cylindrical ordering matrix $C_{n_1,n_2}(\alpha_1,a_1) = [C_{n_1,n_2}(\alpha_1,a_1)]_{i,j}^{n_1-n_2}$ of patterns in $\mathcal{P}_{n_1,n_2}(i,i)$ is defined by

$$C_{n_1,n_2}(\alpha_1,a_1) = \left( \begin{array}{c|c} \Gamma_0 & \cdots \Gamma_{h-1} \end{array} \right) \in \Sigma_{n_1,n_2,n_3} \quad (20)$$

In particular, for $h = 2$, $C_{n_1,n_2,2}$ can be applied to construct the associated trace operator. Notably, the set $C_{n_1,n_2,2, \alpha_3} = \{ \Gamma \mid |\Gamma| = 2 \\} \in \Sigma_{n_1,n_2,n_3}$ contains exactly one pattern.

$$u_{n_1,n_2}(\alpha_1,a_1) = \begin{cases} w_{|n_1-b_1|,a_2} & \text{if } \alpha_2 = a_2 \\ w_{|a_1+b_2|,a_2-b_2} & \text{if } 0 \leq \alpha_2 - b_2 \leq a_2 - 1 \\ w_{|n_1-b_2|,a_2-b_2} & \text{if } a_2 + 1 \leq \alpha_2 - b_2 \leq -1 \end{cases} \quad (16)$$

Now, given $B \subset \Sigma_{2,k}\times2$, the associated trace operator $T_{n_1,n_2}(B) = \{t_{n_1,n_2}(i,i,j)\}$, with $t_{n_1,n_2}(i,i,j) \in (0,1)$, can be defined by

$$t_{n_1,n_2}(i,i,j) = 1 \text{ if and only if the pattern in } C_{n_1,n_2,2}(i,i) \text{ is $B$-admissible}. \quad (21)$$

**Remark 2.2.** Given $L' = \{a_{i,j} \in Z^3 \mid \text{Eq. (15)} \}$ and (16) easily verify that

$$\{U_{n_1+1,n_2+1,n_1+1} : U \text{ is } L'-periodic\} = \{\mathcal{T}_0 = \mathcal{T}_0 \oplus \cdots \oplus \mathcal{T}_{a_1} \} \in \mathcal{P}_{n_1,n_2}(i,i). \quad (22)$$

Furthermore, given $B \subset \Sigma_{2,2}\times2$, from Proposition 2.2 and the construction of the transition matrix $T_{n_1,n_2}(i,i)(B)$,

$$\Gamma_B \begin{pmatrix} a_1 & b_1 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} = \text{tr}(T_{n_1,n_2}(i,i)(B)). \quad (23)$$

The shift maps and the related rotational matrices are considered below for general $L = \{a_{i,j} \in Z^3 \mid \text{Eq. (15)} \}$ in $\mathcal{P}_{n_1,n_2}(i,i)$, $u_{n_1,a_1} \in (0,1)$, is defined by

$$\sigma_{n_1,n_2}(i,i) = \left( \begin{array}{c|c|c} a_{i,j} & b_{i,j} & 0 \\ 0 & a_{i,j} & 0 \\ 0 & 0 & a_{j,k} \end{array} \right) \in Z^3. \quad (24)$$

Similarly, the shift (to the below) in the $y$-direction is defined by

$$\sigma_{n_1,n_2}(i,i) = \left( \begin{array}{c|c|c} a_{i,j} & b_{i,j} & 0 \\ 0 & a_{i,j} & 0 \\ 0 & 0 & a_{j,k} \end{array} \right) \in Z^3. \quad (25)$$
Notably, $\sigma_{y,1,n_1}$ and $\sigma_{y,1,n_2}$ are automorphisms on $P_{n_1,n_2,j}$. The following example illustrates $\sigma_{x,1,n_1,j}$ and $\sigma_{y,1,n_2,j}$.

**Example 2.4.** Let

$$U = (u_{n_1,n_2,0})$$

be a local pattern that lies on the plane $\{(z_1, z_2, 0) : z_1, z_2 \in Z\}$. Now, consider $\sigma_{x,3,2,1}$ and $\sigma_{y,3,2,1}$ which are acting on $U$. Then it is easy to see

$$\sigma_{x,3,2,1}(U) = (u_{n_1,n_2,0})$$

and

$$\sigma_{y,3,2,1}(U) = (u_{n_1,n_2,0})$$

Moreover, both $\sigma_{x,3,2,1}(U)$ and $\sigma_{y,3,2,1}(U)$ also belong to $P_{3,2,1}$.

From Eqs. (24) and (25), for $0 \leq r_i \leq n_i - 1$, $i = 1, 2$, the following can be straightforwardly verified:

$$\sigma_{y,1,n_1,n_2,j}^{r_1}(\sigma_{y,1,n_1,n_2}^{r_2}(u_{n_1,n_2,0})) = (u_{n_1,n_2,0})_{0 \leq n_1 \leq n_1, 0 \leq n_2 \leq n_2}$$

where

$$u_{n_1,n_2,0}^{(3)} = \begin{cases} u_{n_1+1,r_1-1,n_2+1,r_2} & \text{if } n_2 \leq \alpha_2 + r_2 \leq 2n_2 - 1, \\ u_{n_1+1,r_1,n_2+1,r_2} & \text{if } 0 \leq \alpha_2 + r_2 \leq n_2 - 1. \end{cases}$$

Furthermore,

$$\sigma_{y,1,n_1,n_2} \circ \sigma_{x,1,n_1,n_2} = \sigma_{x,1,n_1,n_2} \circ \sigma_{y,1,n_2}$$

and

$$\sigma_{x,1,n_1,n_2}^{r_1} = \sigma_{x,1,n_1,n_2}(\sigma_{y,1,n_1,n_2}^{r_2}) = \text{identity map.}$$

Hence,

$$\sigma_{x,1,n_1,n_2}^{-1} = \sigma_{x,1,n_1,n_2}^{-1} \circ \sigma_{y,1,n_1,n_2}^{r_2}$$

and

$$\sigma_{y,1,n_1,n_2}^{-1} = \sigma_{y,1,n_1,n_2}^{-1} \circ \sigma_{x,1,n_1,n_2}^{r_2}.$$ (29)

Therefore, for $0 \leq r_i \leq n_i - 1$, $i = 1, 2$,

$$\sigma_{x,1,n_1,n_2}(-r_2)(\sigma_{y,1,n_1,n_2}^{r_2}(u_{n_1,n_2,0})) = (u_{n_1,n_2,0})_{0 \leq n_1 \leq n_1, 0 \leq n_2 \leq n_2}$$

where

$$u_{n_1,n_2,0}^{(4)} = \begin{cases} u_{n_1+n_2-1,r_2-1,n_2-r_2} & \text{if } \alpha_2 - r_2 = n_2, \\ u_{n_1+n_2-1,r_2-1,n_2-r_2} & \text{if } 0 \leq \alpha_2 - r_2 \leq n_2 - 1, \\ u_{n_1+n_2-1,r_2-1,n_2-r_2} & \text{if } -n_2 + 1 \leq \alpha_2 - r_2 \leq -1. \end{cases}$$

Now, the two rotational matrices $R_{x,1,n_1,n_2,j}$ and $R_{y,1,n_1,n_2,j}$ are defined as follows.

**Definition 2.5.** The $2^{n_1} \times 2^{n_2}$ $x$-rotational matrix $R_{x,1,n_1,n_2,j} = [R_{x,1,n_1,n_2,j}]$, $R_{x,1,n_1,n_2,j} \in \{0, 1\}$, is defined by

$$R_{x,1,n_1,n_2,j} = \begin{cases} 1 & \text{if and only if } j = \psi(U) \text{ and } \sigma_x(i), \\ 0 & \text{otherwise}. \end{cases}$$

where $U \in P_{n_1,n_2,j,1}$. From Eq. (31), for convenience, denote by

$$j = \sigma_x(i).$$ (32)
Similarly, the $2^n \times 2^n$-g-rotational matrix $R_{g_{n_1,n_2};j} = [R_{g_{n_1,n_2};j}(i, j)]$, $R_{g_{n_1,n_2};j} \in \{0, 1\}$, is defined by

$$R_{g_{n_1,n_2};j} = 1 \quad \text{if and only if} \quad i = \psi(j(U)) \quad \text{and} \quad j = \psi(\sigma_{g_{n_1,n_2};j}(U)), \quad (33)$$

where $U \in P_{n_1,n_2;1}$. From Eq. (33), for convenience, denote by

$$j = \sigma_g(i). \quad (34)$$

Obviously, $R_{g_{n_1,n_2};j}$ and $R_{g_{n_1,n_2};j}^{(2)}$ are permutation matrices. By Eq. (28), $R_{g_{n_1,n_2};j} = R_{g_{n_1,n_2};j}^{(2)} R_{g_{n_1,n_2};j}^{(2)} \in I_{n \times n}$, where $I_n$ is the $n \times n$ identity matrix.

Example 2.6. Let $n_1 = 2$, $n_2 = 1$ and $l = 1$,

$$R_{g_{2,1};1} = R_{g_{2,1};1}^{(2)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then,

$$R_{g_{2,1};1}^{(2)} = R_{g_{2,1};1}^{(2)} R_{g_{2,1};1} = I_4 \quad \text{but} \quad R_{g_{2,1};1} \neq I_4.$$

The following proposition shows the permutation characters of $R_{g_{n_1,n_2};j}$ and $R_{g_{n_1,n_2};j}^{(2)}$.

Proposition 2.7. Let $M = [M_{ij}]_{2^n \times 2^n}$ be a matrix where $M_{ij}$ denotes a number or a pattern or a set of patterns. Then

$$(MR_{g_{n_1,n_2};j})_{i,j} = M_{\sigma^{-1}(i), \sigma^{-1}(j)} \quad \text{and} \quad (MR_{g_{n_1,n_2};j}^{(2)})_{i,j} = M_{\sigma^{-1}(i), \sigma^{-1}(j)}. \quad (35)$$

Furthermore, for any $r \geq 1$

$$\begin{aligned} (M R_{g_{n_1,n_2};j})_{i,j} &= M_{\sigma^{-r}(i), \sigma^{-r}(j)} \quad \text{and} \quad \left( M R_{g_{n_1,n_2};j}^{(2)} \right)_{i,j} = M_{\sigma^{-r}(i), \sigma^{-r}(j)}; \\
&= \sum_{q=1}^{r} M_{q} R_{g_{n_1,n_2};j} \sigma^{q-1}(i), \sigma^{q-1}(j); \\
&= M_{\sigma^{-r}(i), \sigma^{-r}(j)}. \quad (36) \end{aligned}$$

Proof. For any $1 \leq i, j \leq 2^n$, by Eq. (32),

$$(MR_{g_{n_1,n_2};j})_{i,j} = \sum_{q=1}^{r} M_{q} R_{g_{n_1,n_2};j} \sigma^{q-1}(i), \sigma^{q-1}(j); \\
= M_{\sigma^{-r}(i), \sigma^{-r}(j)}.$$ 

Similarly,

$$(MR_{g_{n_1,n_2};j}^{(2)})_{i,j} = \sum_{q=1}^{r} M_{q} R_{g_{n_1,n_2};j} \sigma^{q-1}(i), \sigma^{q-1}(j); \\
= M_{\sigma^{-r}(i), \sigma^{-r}(j)}.$$ 

Applying Eq. (35) $r$ times yields Eq. (36). The proof is complete.

Now, the following lemma can be obtained.

Lemma 2.8. Given $L = \begin{bmatrix} a_{11} & b_{12} & b_{13} \\ 0 & a_{22} & b_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \in \mathbb{Z}^3$, $U = \begin{bmatrix} a_{11} & b_{12} & b_{13} \\ 0 & a_{22} & b_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$ is $L$-periodic

$$U = U_0 \oplus \cdots \oplus U_{a_{33}-1} \in \mathbb{P}_{a_{11},a_{22},a_{33}+1} : \quad \begin{bmatrix} a_{11} & b_{12} & b_{13} \\ 0 & a_{22} & b_{23} \\ 0 & 0 & a_{33} \end{bmatrix} = \sigma^{-b_{23}} \sigma^{-b_{13}} \sigma_{g_{a_{11},a_{22},a_{33}}} \left( \sigma_{g_{a_{11},a_{22},a_{33}}}(U_0) \right). \quad (37)$$

Proof. From Eqs. (16) and (30),

$$U = U_0 \oplus \cdots \oplus U_{a_{33}-1} \in \mathbb{P}_{a_{11},a_{22},a_{33}+1} : \quad \begin{bmatrix} a_{11} & b_{12} & b_{13} \\ 0 & a_{22} & b_{23} \\ 0 & 0 & a_{33} \end{bmatrix} = \sigma^{-b_{23}} \sigma^{-b_{13}} \sigma_{g_{a_{11},a_{22},a_{33}}} \left( \sigma_{g_{a_{11},a_{22},a_{33}}}(U_0) \right).$$

Then, by the construction of $\mathbb{P}_{a_{11},a_{22},a_{33}+1}$, the last set is equal to

$$U \in \mathbb{P}_{a_{11},a_{22},a_{33}+1} : U \text{ satisfies Eqs. (15) and (16)},$$

Therefore, Eq. (37) follows. The proof is complete.

Propositions 2.2, 2.7 and Lemma 2.8 yield the following main results for $\Gamma_{B}$

$$\Gamma_{B} = \begin{bmatrix} a_{11} & b_{12} & b_{13} \\ 0 & a_{22} & b_{23} \\ 0 & 0 & a_{33} \end{bmatrix} = \text{tr}(T_{a_{11},a_{22},a_{33}}^{(3)}(B) R_{g_{a_{11},a_{22},a_{33}}^{(2)} R_{g_{a_{11},a_{22},a_{33}}^{(2)}}). \quad (38)$$

Furthermore,

$$\sum_{b_{33}=0}^{a_{33}-1} \sum_{b_{23}=0}^{a_{22}-1} \Gamma_{B} = \begin{bmatrix} a_{11} & b_{12} & b_{13} \\ 0 & a_{22} & b_{23} \\ 0 & 0 & a_{33} \end{bmatrix} = \text{tr}(T_{a_{11},a_{22},a_{33}}^{(3)}(B) R_{g_{a_{11},a_{22},a_{33}}^{(2)}). \quad (39)$$

where

$$R_{a_{11},a_{22},a_{33}}^{(2)} = \sum_{b_{13}=0}^{a_{11}-1} \sum_{b_{12}=0}^{a_{11}-1} R_{g_{a_{11},a_{22},a_{33}}^{(2)} R_{g_{a_{11},a_{22},a_{33}}^{(2)}}. \quad (40)$$
3. Rationality of $\zeta_{a_1, a_2; b_1}(s)$

This section proves that $\zeta_{a_1, a_2; b_1}$ is a rational function. First, the rotational symmetry of $T_{a_1, a_2; b_1}$ is introduced.

**Theorem 3.1.** Given $B \subset \Sigma_{2 \times 2 \times 2}$. Denote by $T_{a_{1}, a_{2}; b_{1}}(B) = \{t_{a_{1}, a_{2}; b_{1}, z_{1} j}\}$. For $a_{1}, a_{2} \geq 1$, $0 \leq b_{1} \leq a_{1} - 1$, 

$$t_{a_{1}, a_{2}; b_{1}, \sigma_{b_{1}}^{-1}(i), \sigma_{b_{1}}^{-1}(j)} = t_{a_{1}, a_{2}; b_{1}, j}$$

and

$$t_{a_{1}, a_{2}; b_{1}, \sigma_{b_{1}}^{-1}(i), \sigma_{b_{1}}^{-1}(j)} = t_{a_{1}, a_{2}; b_{1}, 2j}$$

for all $1 \leq i, j \leq 2a_{1}$. Furthermore,

$$t_{a_{1}, a_{2}; b_{1}, \sigma_{b_{1}}^{-1}(i), \sigma_{b_{1}}^{-1}(j)}(\sigma_{b_{1}}^{-1}(i), \sigma_{b_{1}}^{-1}(j)) = t_{a_{1}, a_{2}; b_{1}, j}$$

for all $1 \leq i, j \leq 2a_{1}$. Moreover,

**Theorem 2.10.** Given a basic set $B \subset \Sigma_{2 \times 2 \times 2}$. For $a_{1}, a_{2} \geq 1$, $0 \leq b_{1} \leq a_{1} - 1$, 

$$\zeta_{a_{1}, a_{2}; b_{1}}(s) = \exp \left( \frac{1}{a_{1}a_{2}} \sum_{a_{1}j=1}^{a_{1}} \frac{1}{a_{1}} \log(T_{a_{1}, a_{2}; b_{1}}(B)) \right)$$

$$\times R_{a_{1}, a_{2}; b_{1}}(a_{1}a_{2})^{b_{1}}.$$

**Proof.** The results follow from Theorem 2.9. ■

To study the rationality of $\zeta_{a_1, a_2; b_1}$, we need more definitions and properties about the two shifts in Eqs. (32) and (34) as follows.

Given $a_{1}, a_{2} \geq 1$, $0 \leq b_{1} \leq a_{1} - 1$, for $1 \leq i \leq 2a_{1}$, the equivalent class $\zeta_{a_{1}, a_{2}; b_{1}}(i)$ of
i is defined by
\[ \mathcal{C}_{a_1,a_2,b_2}(i) \equiv \{ \sigma^{r_s}(\tau^{r_j}(i)) : 0 \leq r_1 \leq a_1 - 1, 0 \leq r_2 \leq a_2 - 1 \}. \]  
(46)

Clearly,
\[ \text{either } \mathcal{C}_{a_1,a_2,b_2}(i) \cap \mathcal{C}_{a_1,a_2,b_2}(j) = \emptyset. \]  
(47)

The cardinal number of \( \mathcal{C}_{a_1,a_2,b_2}(i) \) is denoted by \( \omega_{a_1,a_2,b_2}(i) \). Let \( i \) be the smallest element in its equivalent class, and the index set \( \mathcal{I}_{a_1,a_2,b_2} \) is defined by
\[ \mathcal{I}_{a_1,a_2,b_2} = \{ i : 1 \leq i \leq 2^{a_2} \}. \]
for all \( j \in \mathcal{C}_{a_1,a_2,b_2}(i) \).  
(48)

Therefore,
\[ \{ j : 1 \leq j \leq 2^{a_2} \} = \bigcup_{i \in \mathcal{I}_{a_1,a_2,b_2}} \mathcal{C}_{a_1,a_2,b_2}(i). \]  
(49)

The cardinal number of \( \mathcal{I}_{a_1,a_2,b_2} \) is denoted by \( \omega_{a_1,a_2,b_2} \).

Example 3.2.
\[ \begin{align*}
\mathcal{I}_{2,2,0} &= \{1 \} \\
\mathcal{I}_{2,2,2} &= \{2, 3, 5, 9 \} \\
\mathcal{I}_{2,2,6} &= \{4, 13 \} \\
\mathcal{I}_{2,2,8} &= \{6, 11 \} \\
\mathcal{I}_{2,2,7} &= \{7 \} \\
\mathcal{I}_{2,2,9} &= \{8, 12, 14, 15 \} \\
\mathcal{I}_{2,2,10} &= \{16 \} \\
\mathcal{I}_{2,2,11} &= \{1, 2, 4, 6, 7, 8, 16 \} \\
\mathcal{I}_{2,2,12} &= \{3 \} \\
\mathcal{I}_{2,2,13} &= \{4 \} \\
\mathcal{I}_{2,2,14} &= \{6, 7, 10, 11 \} \\
\mathcal{I}_{2,2,15} &= \{8, 12, 14, 15 \} \\
\mathcal{I}_{2,2,16} &= \{16 \} \\
\mathcal{I}_{2,2,17} &= \{1, 2, 4, 6, 8, 16 \}.
\end{align*} \]

The equivalent classes are invariant under the two shift maps. Therefore, the following proposition is directly obtained and the proof is omitted.

Proposition 3.3. Given \( a_1, a_2 \geq 1 \) and \( 0 \leq b_2 \leq a_1 - 1 \). Let \( N \equiv 2^{a_1 + a_2} \) and \( \mathcal{V} = (v_{1}, v_{2}, \ldots, v_{N}) \), for
\[ 1 \leq i \leq N, \]
\[ \sum_{r_1=0}^{a_1-1} \sum_{r_2=0}^{a_2-1} \mathcal{V}_{r_2}^{\sigma^{-r_2}(\tau^{r_1}(i))} \]
\[ = \mathcal{V}_{b_2}^{\omega_{a_1,a_2,b_2}(i)} \mathcal{V}_{b_1}^{\alpha_{a_1,a_2,b_2}(i)}, \]  
(50)

For the rationality of \( \zeta_{a_1,a_2,b_2} \), the reduced trace operator \( \tau_{a_1,a_2,b_2} \) is introduced as follows.

Definition 3.4. For \( a_1, a_2 \geq 1, 0 \leq b_2 \leq a_1 - 1 \), the reduced trace operator \( \tau_{a_1,a_2,b_2} \) of \( \mathcal{T}_{a_1,a_2,b_2} \) is a \( \chi_{a_1,a_2,b_2} \times \chi_{a_1,a_2,b_2} \) matrix and is defined by
\[ \tau_{a_1,a_2,b_2}(i, j) = \sum_{k \in \mathcal{I}_{a_1,a_2,b_2}} \mathcal{T}_{a_1,a_2,b_2}(i, k) \]  
(51)

for each \( i, j \in \mathcal{I}_{a_1,a_2,b_2} \).

The following theorem expresses the average of \( \Gamma_B \) in terms of the trace of the reduced trace operator \( \tau \) and plays a crucial role in proving the rationality of \( \zeta_{a_1,a_2,b_2} \). The proof here is simpler and more straightforward than the proofs in [Ban et al., 2008a] for \( d = 2 \).

Theorem 3.5. Given \( B \subset \Sigma_{2 \times 2} \). For \( a_1 \geq 1 \), \( 1 \leq i \leq \delta, 0 \leq b_2 \leq a_1 - 1 \), \( i + 1 \leq j \leq 3 \),
\[ \frac{1}{a_1 a_2} \sum_{b_1=0}^{a_1-1} \sum_{b_2=0}^{a_2-1} \Gamma_B \begin{pmatrix} a_1 & b_2 & b_1 \\ 0 & a_2 & 0 \end{pmatrix} = \text{tr}(\tau_{a_1,a_2,b_2}^\delta), \]  
(52)

where \( \Sigma(\tau_{a_1,a_2,b_2}) \) is the spectrum of \( \tau_{a_1,a_2,b_2} \) and \( \chi_{a_1,a_2,b_2}(\lambda) \) is the algebraic multiplicity of \( \tau_{a_1,a_2,b_2} \) with eigenvalue \( \lambda \).

Proof. For simplicity, let \( N = 2^{a_1 + a_2} \) and \( \mathcal{T}_{a_1,a_2,b_2} = [b_1, b_2] \). From Proposition 2.7 and Theorem 2.9,
\[ \mathcal{T}_{a_1,a_2,b_2} = \begin{pmatrix} \frac{a_1}{a_1 a_2} & \frac{a_2}{a_1 a_2} & \frac{a_2}{a_1 a_2} \\ 0 & \frac{a_1}{a_1 a_2} & 0 \\ 0 & 0 & \frac{a_1}{a_1 a_2} \end{pmatrix} \]
\[ \times \text{tr}(\mathcal{T}_{a_1,a_2,b_2}^{\delta}) \]
\[ = \frac{a_1}{a_1 a_2} \sum_{b_1=0}^{a_1-1} \sum_{b_2=0}^{a_2-1} \Gamma_B \begin{pmatrix} a_1 & b_2 & b_1 \\ 0 & a_2 & 0 \end{pmatrix}, \]  
(53)

for each \( i, j \in \mathcal{I}_{a_1,a_2,b_2} \).
Now, by Eq. (49), the last sum becomes
\[
\frac{1}{a_{12}} \sum_{t \in \mathbb{Z}_{12}} \sum_{q \in C_{12}^{1,2}} \sum_{k_1=0}^{a_1-1} \sum_{k_2=0}^{a_2-1} \sum_{j=1}^{N} \sum_{k_{j+1}=0}^{N} \left( a_{q,k_{j+1}} \sum_{i=1}^{a_{q,k_{j+1}}} \sum_{t_{q,k_{j+1}}} \left( t_{q,k_{j+1}} \right)^{i-1} \right) \times t_{q,k_1} t_{q,k_2} \cdots t_{q,k_{j-1}} \cdot \sigma_{t_{q,k_{j+1}}}^{x_2}(\sigma_{t_{q,k_{j+1}}}^{x_2}(q))^{(j)}.
\]

(53)

Fixed \( q \in C_{a_1,a_2,b_1}(t) \), there exist \( 0 \leq r_1 \leq a_1 - 1 \) and \( 0 \leq r_2 \leq a_2 - 1 \) such that \( q = \sigma_{t_{r_1}}(\sigma_{t_{r_2}}^2(i)) \).

Then, by Theorem 3.1,
\[
\sum_{k_1=0}^{a_1-1} \sum_{k_2=0}^{a_2-1} \sum_{j=1}^{N} \sum_{k_{j+1}=0}^{N} \left( a_{q,k_{j+1}} \sum_{i=1}^{a_{q,k_{j+1}}} \sum_{t_{q,k_{j+1}}} \left( t_{q,k_{j+1}} \right)^{i-1} \right) \times t_{q,k_1} t_{q,k_2} \cdots t_{q,k_{j-1}} \cdot \sigma_{t_{q,k_{j+1}}}^{x_2}(\sigma_{t_{q,k_{j+1}}}^{x_2}(q))^{(j)}.
\]

(54)

Therefore, Eq. (53) is equal to
\[
\frac{1}{a_{12}} \sum_{t \in \mathbb{Z}_{12}} \sum_{q \in C_{12}^{1,2}} \sum_{k_1=0}^{a_1-1} \sum_{k_2=0}^{a_2-1} \sum_{j=1}^{N} \sum_{k_{j+1}=0}^{N} \left( a_{q,k_{j+1}} \sum_{i=1}^{a_{q,k_{j+1}}} \sum_{t_{q,k_{j+1}}} \left( t_{q,k_{j+1}} \right)^{i-1} \right) \times t_{q,k_1} t_{q,k_2} \cdots t_{q,k_{j-1}} \cdot \sigma_{t_{q,k_{j+1}}}^{x_2}(\sigma_{t_{q,k_{j+1}}}^{x_2}(q))^{(j)}.
\]

(55)

According to Proposition 3.3, Eq. (55) is equal to
\[
\sum_{t \in \mathbb{Z}_{12}} \sum_{q \in C_{12}^{1,2}} \sum_{k_1=0}^{a_1-1} \sum_{k_2=0}^{a_2-1} \sum_{j=1}^{N} \sum_{k_{j+1}=0}^{N} \left( a_{q,k_{j+1}} \sum_{i=1}^{a_{q,k_{j+1}}} \sum_{t_{q,k_{j+1}}} \left( t_{q,k_{j+1}} \right)^{i-1} \right) \times t_{q,k_1} t_{q,k_2} \cdots t_{q,k_{j-1}} \cdot \sigma_{t_{q,k_{j+1}}}^{x_2}(\sigma_{t_{q,k_{j+1}}}^{x_2}(q))^{(j)}.
\]

(56)

For any \( q_{a_1-1} \in C_{a_1,a_2,b_1}(k_{a_1-1}) \), there exist \( 0 \leq r_1 \leq a_1 - 1 \) and \( 0 \leq r_2 \leq a_2 - 1 \) such that \( q_{a_1-1} = \sigma_{t_{r_1}}(\sigma_{t_{r_2}}(k_{a_1-1})) \).

Then, by Theorem 3.1,
\[
\sum_{q \in C_{12}^{1,2}} \sum_{t \in \mathbb{Z}_{12}} \left( a_{q,k_{j+1}} \sum_{i=1}^{a_{q,k_{j+1}}} \sum_{t_{q,k_{j+1}}} \left( t_{q,k_{j+1}} \right)^{i-1} \right) \times t_{q,k_1} t_{q,k_2} \cdots t_{q,k_{j-1}} \cdot \sigma_{t_{q,k_{j+1}}}^{x_2}(\sigma_{t_{q,k_{j+1}}}^{x_2}(q))^{(j)}.
\]

(57)

Therefore,
The proof is complete. ■

Eq. (57) follows from Eq. (58) and Theorem 3.5.

Example 3.7. Consider

\[ \mathcal{B} = \{ U_{2 \times 2} = (u_{ij})_{2 \times 2} : u_{0,0,j} = u_{1,0,j} = u_{i,j} = 0, j \neq 0, 1 \} \]

Clearly, the set \( \mathcal{P}(\mathcal{B}) \) of all \( \mathcal{B} \)-admissible and periodic patterns is

\[ \{ U = (u_{ij})_{2 \times 2} : u_{i,j} = 0, i,j \in \mathbb{Z} \} \]

Then, it is easy to verify that

\[ \Gamma_{\mathcal{B}} \begin{pmatrix} a_1 & b_1 & b_2 \\ 0 & a_2 & b_3 \\ 0 & 0 & a_3 \end{pmatrix} = 2^{a_2} \]

for \( a_i \geq 1, 1 \leq i \leq 3, 0 \leq b_i \leq a_i - 1, i + 1 \leq j \leq 3 \).

Therefore,

\[ \zeta_{a_1,a_2,b_3}(s) = (1 - 2^{a_2})^{-1} \]

and

\[ \zeta(s) = \prod_{a_1=1}^{\infty} \prod_{a_2=1}^{\infty} \prod_{b_3=0}^{a_2-1} (1 - 2^{a_2})^{-a_2}^{-1} \]

However, Eqs. (60) and (61) can be obtained from Eqs. (57) and (58). The trace operator

\[ T_{a_1,a_2,b_3} = T_{a_1,a_2,b_0}(\mathcal{B}) \]

The following example is used to demonstrate the application of the above result.
Therefore, \( \zeta(s) = (\det(I - s^{a_1 a_2} \tau_{a_1 a_2}))^{-1} \) and
\[
\zeta(s) = \prod_{a_1=1}^{\infty} \prod_{a_2=1}^{\infty} \left(1 - 2s^{a_1 a_2}\right)^{-1}.
\]
Equations (60) and (61) are recovered.

4. Zeta Functions in Inclined Coordinates

This section presents the zeta function with respect to inclined coordinates, determined by applying the unimodular transformations in \( GL_3(\mathbb{Z}) \). \( \mathbb{Z}^3 \) is known to be invariant under the unimodular transformation in \( GL_3(\mathbb{Z}) \). Indeed, Lind [1996] proved that the zeta function \( \zeta_{\alpha}^{\gamma} \) is independent of a choice of basis for \( \mathbb{Z}^3 \). Recall that
\[
GL_3(\mathbb{Z}) = \{ \gamma = \begin{pmatrix} \gamma_{ij} \end{pmatrix}_{1 \leq i,j \leq d} : \gamma_{ij} \in \mathbb{Z} \text{ for } 1 \leq i, j \leq d \text{ and } |\det(\gamma)| = 1 \}.
\]

This section presents the construction of the trace operator \( T_{\gamma a_1 a_2 a_3}(B) \) and the reduced trace operator \( \tau_{\gamma a_1 a_2 a_3}(B) \), and then determines \( \zeta_{\gamma a_1 a_2 a_3}(s) \). Finally, \( \zeta_{\gamma a_1 a_2 a_3} \) is obtained as
\[
\zeta_{\gamma a_1 a_2 a_3}(s) = \prod_{a_1=1}^{\infty} \prod_{a_2=1}^{\infty} \prod_{a_3=1}^{\infty} \left(1 - 2s^{a_1 a_2 a_3}\right)^{-1} 
\times (\det(I - s^{a_1 a_2} \tau_{\gamma a_1 a_2 a_3}))^{-1}.
\]

For simplicity, only \( B \subset \Sigma_{2x2x2} \) with two symbols are considered. The general cases can be treated analogously.

For a given \( \gamma = \begin{pmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{pmatrix} \in GL_3(\mathbb{Z}) \), the lattice points in \( \gamma \)-coordinates are
\[
(1,0,0) = (\gamma_{11}, \gamma_{12}, \gamma_{13}), \\
(0,1,0) = (\gamma_{21}, \gamma_{22}, \gamma_{23}), \text{ and} \\
(0,0,1) = (\gamma_{31}, \gamma_{32}, \gamma_{33}),
\]
and the unit vectors are
\[
\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma_{11} \\ \gamma_{12} \\ \gamma_{13} \end{pmatrix}, \\
\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma_{21} \\ \gamma_{22} \\ \gamma_{23} \end{pmatrix}, \text{ and} \\
\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \gamma_{31} \\ \gamma_{32} \\ \gamma_{33} \end{pmatrix}.
\]

Notably, when \( \gamma = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \), standard rectangular coordinates are used and the subscript \( \gamma \) is omitted.

The matrix \( M_\gamma \) is defined by
\[
M_\gamma = \begin{pmatrix} a_1 & b_{12} & b_{13} \\ 0 & a_2 & b_{23} \\ 0 & 0 & a_3 \end{pmatrix} \gamma.
\]
Let \( L_\gamma = M_\gamma \mathbb{Z}^3 \). Then,
\[
L_\gamma = \begin{pmatrix} a_1 & b_{12} & b_{13} \\ 0 & a_2 & b_{23} \\ 0 & 0 & a_3 \end{pmatrix} \mathbb{Z}^3 \quad (63)
\]
is easily verified.

A global pattern \( U_{\gamma} = (u_{a_1 a_2 a_3})_{a_1, a_2, a_3 \in \mathbb{Z}} \) is called \( L_\gamma \)-periodic or \( \mathbb{Z}^3 \)-periodic if for every \( a_1, a_2, a_3 \in \mathbb{Z} \)
\[
u_{u_{a_1 a_2 a_3}}(0, b_{12}, b_{13}, 0, b_{23}, a_{12}, a_{23}, a_{12} + a_{23} + b_{12}, 0) = u_{a_1 a_2 a_3}, \quad (64)
\]
for all \( p, q, r \in \mathbb{Z} \). Therefore, the \( (a_1, a_2, b_{13}) \)th zeta function of \( C_\gamma^{\gamma}(s) \) with respect to \( \gamma \) is defined by
\[
\zeta_{\gamma a_1 a_2 a_3}(s) = \exp \left( \sum_{a_{12} = 0}^{a_{12} = a_{23}} \sum_{b_{12} = 0}^{b_{12} = a_{12}} \sum_{b_{23} = 0}^{b_{23} = a_{23}} \frac{a_{13}}{a_{13}} \right) \nu_{u_{a_1 a_2 a_3}}(1, b_{12}, b_{13}, 0, b_{23}, a_{12}, a_{23}, a_{12} + a_{23} + b_{12}, 0) \gamma.
\]
and the zeta function $\zeta_\gamma$ with respect to $\gamma$ is defined by
\[ \zeta_\gamma(s) = \prod_{a_i=1}^{\infty} \prod_{a_j=1}^{\infty} \prod_{k=1}^{1-a_i-a_j} \zeta_{3,3}^{a_i+a_j+k}, \]
(66)

The following introduces the cylindrical ordering matrix, the trace operator and the rotational matrices. The proofs of the results as in previous sections are omitted.

Fix a $\gamma \in GL_3(\mathbb{Z})$. Let $\mathcal{Z}_{3,3} \times \mathcal{Z}_{3,3} \times \mathcal{Z}_{3,3}$ be the $n_1 \times n_2 \times n_3$ lattice with the basis
\[ \gamma_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \]
and $\gamma_3 = \gamma_2 \gamma_1$. The total number of lattice points on $\mathcal{Z}_{3,3} \times \mathcal{Z}_{3,3} \times \mathcal{Z}_{3,3}$ is $n_1 \cdot n_2 \cdot n_3$.

Since the basic set $B \subset \Sigma_{2,2,2,2}$, the $L_3$-periodic patterns that are $B$-admissible must be verified on $\mathcal{Z}_{2,2,2,2}$. Let $(n_1, n_2, n_3) = (m_1, m_2, m_3)$,
\[ \mathcal{Z}_{2,2,2,2}((n_1, n_2, n_3)) = \{(m_1 + k_1, m_2 + k_2, m_3 + k_3) : 0 \leq k_1, k_2, k_3 \leq 1 \}. \]

Now, the admissibility is demonstrated to be verified on finite lattice as follows.

**Proposition 4.1.** Given $\gamma = \begin{pmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \end{pmatrix} \in GL_3(\mathbb{Z})$. An $L_3$-periodic pattern $U$ is $B$-admissible if and only if
\[ U|_{\mathcal{Z}_{2,2,2,2}}((\alpha_1, \alpha_2, \alpha_3)) \in B \]
for $0 \leq \alpha_i \leq 1$, $1 \leq i \leq 3$. For $\alpha_1, \alpha_2, \alpha_3 \geq 1$, it is easy to check that there exist positive integers $\tilde{a}_1(\gamma), \tilde{a}_2(\gamma)$ and $\tilde{a}_3(\gamma)$ such that
\[ \bigcup_{i=1}^{n_1-1} \bigcup_{j=1}^{n_2-1} \mathcal{Z}_{2,2,2,2}((\xi_1 + \alpha_1, \xi_2 + \alpha_2, \xi_3 + \alpha_3)) \subseteq \mathcal{Z}_{3,3} \times \mathcal{Z}_{3,3} \times \mathcal{Z}_{3,3}, \]
for some $\xi_1, \xi_2, \xi_3 \in \mathbb{Z}$.

According to Proposition 4.1, the admissibility of an $L_3$-periodic pattern $U$ is determined by $U|_{\mathcal{Z}_{3,3} \times \mathcal{Z}_{3,3} \times \mathcal{Z}_{3,3}} = (\{u(\alpha_1, \alpha_2, \alpha_3)\}, 0 \leq \alpha_1, \alpha_2, \alpha_3 \leq 1)$ and $U|_{\mathcal{Z}_{3,3} \times \mathcal{Z}_{3,3} \times \mathcal{Z}_{3,3}}$ has the th periodic condition that is given by Eq. (64), which can be divided into two parts: (i) for $0 \leq \alpha_i \leq \tilde{a}_i - 1$, $1 \leq i \leq 3$ and $p, q \in \mathbb{Z}$, if $0 \leq \alpha_1 + \alpha_2 p + \alpha_3 q \leq \tilde{a}_1 - 1$ and $0 \leq \alpha_2 + \alpha_3 q \leq \tilde{a}_2 - 1$,
\[ u(\alpha_1 + \alpha_2 p + \alpha_3 q, \alpha_2 + \alpha_3 q) = u(\alpha_1, \alpha_2, \alpha_3); \]
(ii) for $0 \leq \alpha_i \leq \tilde{a}_i - 1$, $1 \leq i \leq 3$, $p, q \in \mathbb{Z}$ and $r \in \mathbb{Z} \setminus \{0\}$, if $0 \leq \alpha_1 + \alpha_2 p + \alpha_3 q + r \leq \tilde{a}_1 - 1$, $0 \leq \alpha_2 + \alpha_3 q + r \leq \tilde{a}_2 - 1$ and $0 \leq \alpha_3 + \alpha_3 r \leq \tilde{a}_3 - 1$,
\[ u(\alpha_1 + \alpha_2 p + \alpha_3 q + r, \alpha_2 + \alpha_3 q + r, \alpha_3 + \alpha_3 r) = u(\alpha_1, \alpha_2, \alpha_3). \]

Then, for $h \geq 1$, the set of all local patterns on $\mathcal{Z}_{n_1 \gamma_1, n_2 \gamma_2, n_3 \gamma_3}$ that satisfy Eq. (67) with $0 \leq \alpha_j \leq h - 1$ is denoted by $\mathcal{P}_{\gamma_1, \gamma_2, \gamma_3}^{h}$.

Similar to Eq. (18), the counting function $\overline{\psi}_\gamma$ for patterns $U_\gamma$ in $\mathcal{P}_{\gamma_1, \gamma_2, \gamma_3}^{h}$ is defined by
\[ \overline{\psi}_\gamma(U_\gamma) = \sum_{\gamma_1 \in \mathcal{P}_{\gamma_1, \gamma_2, \gamma_3}^{h} \gamma_1} \sum_{\gamma_2 \in \mathcal{P}_{\gamma_2, \gamma_3}^{h} \gamma_2} \sum_{\gamma_3 \in \mathcal{P}_{\gamma_3}^{h} \gamma_3} u(\gamma_1, \gamma_2, \gamma_3), \]

A local pattern $U_\gamma$ in $\mathcal{P}_{\gamma_1, \gamma_2, \gamma_3}^{h}$ can be represented as
\[ U_\gamma = U_{\gamma_1 \gamma_2 \gamma_3} \oplus U_{\gamma_1 \gamma_2 \gamma_3} \oplus \cdots \oplus U_{\gamma_1 \gamma_2 \gamma_3} \gamma_1 \gamma_2 \gamma_3, \]
where $U_{\gamma_1} \in \mathcal{P}_{\gamma_1, \gamma_2, \gamma_3}^{h}$, $0 \leq i \leq h - 1$, and $U_{\gamma_1} \oplus U_{\gamma_2} \gamma_1$ means that $U_{\gamma_1}$ is put on the top (in the $\gamma_3$-direction) of $U_{\gamma_2}$. For $0 \leq i \leq j \leq h - 1$, let $U_{\gamma_1 \gamma_2} = U_{\gamma_1} \oplus U_{\gamma_2} \gamma_1$. Therefore, for $h \geq \tilde{a}_i$, the cylindrical ordering matrix $C_{\gamma_1, \gamma_2, \gamma_3}^{h} \gamma_1 \gamma_2 \gamma_3 = C_{\gamma_1, \gamma_2, \gamma_3}^{h} \gamma_1 \gamma_2 \gamma_3$ of patterns in $\mathcal{P}_{\gamma_1, \gamma_2, \gamma_3}^{h}$ is defined by
\[ C_{\gamma_1, \gamma_2, \gamma_3}^{h} \gamma_1 \gamma_2 \gamma_3 \gamma_1 \gamma_2 \gamma_3 = \overline{\psi}_\gamma(U_{\gamma_1 \gamma_2 \gamma_3} \gamma_1 \gamma_2 \gamma_3) = i \quad \text{and} \quad \overline{\psi}_\gamma(U_{\gamma_1 \gamma_2 \gamma_3} \gamma_1 \gamma_2 \gamma_3) = j. \]
In particular, for $h = h_1$, $C_{\gamma_1, \gamma_2, \gamma_3}^{h_1} \gamma_1 \gamma_2 \gamma_3 \gamma_1 \gamma_2 \gamma_3$ can be used to construct the associated trace operator. Notably the set $C_{\gamma_1, \gamma_2, \gamma_3}^{h_1} \gamma_1 \gamma_2 \gamma_3 \gamma_1 \gamma_2 \gamma_3$ either contains exactly one pattern or is an empty set.

Now, given $B \subset \Sigma_{2,2,2}$, the associated trace operator $T_{\gamma_1, \gamma_2, \gamma_3}^{h} = \{t_{\gamma_1, \gamma_2, \gamma_3}^{h} \gamma_1 \gamma_2 \gamma_3 \gamma_1 \gamma_2 \gamma_3\}$, with $t_{\gamma_1, \gamma_2, \gamma_3}^{h} \gamma_1 \gamma_2 \gamma_3 \gamma_1 \gamma_2 \gamma_3 \in \{0, 1\}$, can be defined by $t_{\gamma_1, \gamma_2, \gamma_3}^{h} \gamma_1 \gamma_2 \gamma_3 \gamma_1 \gamma_2 \gamma_3 = 1$ if and only if
\[ C_{\gamma_1, \gamma_2, \gamma_3}^{h} \gamma_1 \gamma_2 \gamma_3 \gamma_1 \gamma_2 \gamma_3 \neq 0 \quad \text{and} \quad \text{the pattern is } C_{\gamma_1, \gamma_2, \gamma_3}^{h} \gamma_1 \gamma_2 \gamma_3 \gamma_1 \gamma_2 \gamma_3 \text{ is } B\text{-admissible}. \]
Similarly, the shift (to the below) in the rotational matrix $R_{\gamma_1,\alpha,\beta_1,\beta_2}$ is defined by
\[
\sigma_{\gamma_1,\alpha,\beta_1,\beta_2}(u_{(\alpha_1,\alpha_2,\alpha_3)}) = (u_{(\alpha_1,\alpha_2,\alpha_3)})_{\beta_1 \alpha_2} - (u_{(\alpha_1,\alpha_2,\alpha_3)})_{\gamma_1 \alpha_2}.
\]
where
\[
\begin{align*}
(u_{(\alpha_1,\alpha_2,\alpha_3)})_{\beta_1} & = \begin{cases} u_{(\alpha_1,\alpha_2,\alpha_3)} & \text{if } 0 \leq \alpha_1 \leq \alpha_2 - 2, \\
0 & \text{otherwise},
\end{cases} \\
(u_{(\alpha_1,\alpha_2,\alpha_3)})_{\gamma_1} & = \begin{cases} u_{(\alpha_1,\alpha_2,\alpha_3)} & \text{if } 0 \leq \alpha_1 \leq \alpha_2 - 2, \\
0 & \text{otherwise}.
\end{cases}
\end{align*}
\]
Similarly, the shift (to the below) in the $\gamma_2$-direction is defined by
\[
\sigma_{\gamma_2,\alpha,\beta_1,\beta_2}(u_{(\alpha_1,\alpha_2,\alpha_3)}) = (u_{(\alpha_1,\alpha_2,\alpha_3)})_{\alpha_2} - (u_{(\alpha_1,\alpha_2,\alpha_3)})_{\gamma_2}.
\]
where
\[
\begin{align*}
(u_{(\alpha_1,\alpha_2,\alpha_3)})_{\alpha_2} & = \begin{cases} u_{(\alpha_1,\alpha_2,\alpha_3)} & \text{if } 0 \leq \alpha_2 \leq \alpha_3 - 2, \\
0 & \text{otherwise},
\end{cases} \\
(u_{(\alpha_1,\alpha_2,\alpha_3)})_{\gamma_2} & = \begin{cases} u_{(\alpha_1,\alpha_2,\alpha_3)} & \text{if } 0 \leq \alpha_2 \leq \alpha_3 - 2, \\
0 & \text{otherwise}.
\end{cases}
\end{align*}
\]
Moreover,
\[
R_{\gamma_1,\alpha,\beta_1,\beta_2} = \sum_{b_1=0}^{\alpha_1-1} \sum_{b_2=0}^{\alpha_2-1} R_{\alpha_1,a_1,b_1}^{b_1} R_{\alpha_2,a_2,b_2}^{b_2} R_{\gamma_1,a_1,a_2}^{a_1},
\]
The main results for $\Gamma_{\gamma_1,\alpha,\beta_1,\beta_2}$ is as in Theorems 2.9 and 2.10 as follows.

**Theorem 4.3.** Given a basic set $B \subset \Sigma_{2,3,2,2}$, for $\alpha_1 \geq 1, 1 \leq i \leq 3, 0 \leq b_j \leq \alpha_i - 1$, $i + 1 \leq j \leq 3$,
\[
\begin{align*}
\Gamma_{B} & = \begin{pmatrix} a_1 & b_2 & b_3 \\
0 & a_2 & b_3 \\
0 & 0 & a_3 \end{pmatrix}, \\
\text{tr}(T_{\gamma_1,\alpha,\beta_1,\beta_2}^{B}) & = \text{tr}(T_{\gamma_1,\alpha,\beta_1,\beta_2}^{B}) R_{\gamma_1,\alpha,\beta_1,\beta_2}^{a_1}
\end{align*}
\]
and
\[
\begin{align*}
\sum_{b_1=0}^{\alpha_1-1} \sum_{b_2=0}^{\alpha_2-1} \Gamma_{B} & = \begin{pmatrix} a_1 & b_2 & b_3 \\
0 & a_2 & b_3 \\
0 & 0 & a_3 \end{pmatrix}, \\
\text{tr}(T_{\gamma_1,\alpha,\beta_1,\beta_2}^{B}) & = \text{tr}(T_{\gamma_1,\alpha,\beta_1,\beta_2}^{B}) R_{\gamma_1,\alpha,\beta_1,\beta_2}^{a_1}
\end{align*}
\]
Furthermore,
\[
\chi_{\gamma_1,\alpha,\beta_1,\beta_2}(s) = \exp \left( \sum_{a_1=1}^{\infty} \frac{1}{a_1 \alpha_2} \sum_{a_2=1}^{\infty} \sum_{a_3} \text{tr}(T_{\gamma_1,\alpha,\beta_1,\beta_2}^{a_1}) \right) R_{\gamma_1,\alpha,\beta_1,\beta_2}^{a_1 a_2 a_3}.
\]
The equivalent class \( C_{\alpha_1, \alpha_2, \beta_{12}}(i) \), the cardinal number \( \chi_{\alpha_1, \alpha_2, \beta_{12}} \) of \( C_{\alpha_1, \alpha_2, \beta_{12}}(i) \), the index set \( \mathcal{I}_{\alpha_1, \alpha_2, \beta_{12}} \) and the cardinal number of \( \chi_{\alpha_1, \alpha_2, \beta_{12}} \) can be defined as in Sec. 3 and are omitted here.

**Definition 4.4.** For \( a_1, a_2 \geq 1, 0 \leq b_{12} \leq a_1 - 1 \), the reduced trace operator \( \tau_{\gamma_{a_1, a_2, b_{12}}} = [\tau_{\gamma_{a_1, a_2, b_{12}}}, i] \) of \( T_{\gamma_{a_1, a_2, b_{12}}} = [\gamma_{a_1, a_2, b_{12}}, i] \) is a \( \chi_{\gamma_{a_1, a_2, b_{12}}} \times \chi_{\gamma_{a_1, a_2, b_{12}}} \) matrix defined by
\[
\tau_{\gamma_{a_1, a_2, b_{12}}, i, j} = \sum_{k \in C_{\gamma_{a_1, a_2, b_{12}}}} t_{\gamma_{a_1, a_2, b_{12}}, i, k}
\]
for each \( i, j \in \mathcal{I}_{\gamma_{a_1, a_2, b_{12}}} \).

By the argument as in Sec. 3, the rotational symmetry of \( T_{\gamma_{a_1, a_2, b_{12}}} \) can be obtained, yielding then the rationality of the \((a_1, a_2, b_{12})\)-th zeta function \( \zeta_{\gamma_{a_1, a_2, b_{12}}} \). The results are stated as follows.

**Theorem 4.5.** Given \( B \subset \Sigma_{2 \times 2 \times 2} \) and \( \gamma \in GL_3(\mathbb{Z}) \). For \( a_i \geq 1, 1 \leq i \leq 3, 0 \leq b_{12} \leq a_i - 1, i + 1 \leq j \leq 3, \)
\[
\frac{1}{a_1 a_2} \sum_{k \in C_{\gamma_{a_1, a_2, b_{12}}}} t_{\gamma_{a_1, a_2, b_{12}}, i, k} = \sum_{\lambda \in \Sigma_{\gamma_{a_1, a_2, b_{12}}}} \chi_{\gamma_{a_1, a_2, b_{12}}} (\lambda) \lambda^{a_1},
\]
where \( \Sigma_{\gamma_{a_1, a_2, b_{12}}} \) is the spectrum of \( \tau_{\gamma_{a_1, a_2, b_{12}}} \) and \( \chi_{\gamma_{a_1, a_2, b_{12}}} (\lambda) \) is the algebraic multiplicity of \( \tau_{\gamma_{a_1, a_2, b_{12}}} \) with eigenvalue \( \lambda \).

Moreover, \( \zeta_{\gamma_{a_1, a_2, b_{12}}} (s) \) is defined as
\[
\zeta_{\gamma_{a_1, a_2, b_{12}}} (s) = (\det (I - s^{a_1} \tau_{\gamma_{a_1, a_2, b_{12}}}))^{-1} = \prod_{\lambda \in \Sigma_{\gamma_{a_1, a_2, b_{12}}}} (1 - \lambda s^{a_1})^{-1}.
\]
and
\[
\zeta_{\gamma} (s) = \prod_{a_1 = 1}^{\infty} \prod_{a_2 = 1}^{\infty} \prod_{0 \leq b_{12} \leq a_1 - 1} (\det (I - s^{a_1} \tau_{\gamma_{a_1, a_2, b_{12}}}))^{-1},
\]
for \( \zeta_{\gamma_{a_1, a_2, b_{12}}} = T_{\gamma_{a_1, a_2, b_{12}}} = T_{\gamma_{a_1, a_2, b}} \) and \( \zeta_{\gamma_{a_1, a_2, b_{12}}} = T_{\gamma_{a_1, a_2, b_{12}}} = T_{\gamma_{a_1, a_2, b}} \) for \( a_1, a_2 \geq 1, 0 \leq b_{12} \leq a_1 - 1 \). Moreover, after the zero columns and rows of \( T_{\gamma_{a_1, a_2, b_{12}}} \) were deleted, \( T_{\gamma_{a_1, a_2, b_{12}}} \) is reduced to \( T_{\gamma_{a_1, a_2, b_{12}}} \) reduced to \( T_{\gamma_{a_1, a_2, b_{12}}} \) reduced to \( T_{\gamma_{a_1, a_2, b_{12}}} \). Clearly
\[
\tau_{\gamma_{a_1, a_2, b_{12}}} = I_{a_2} \]
and
\[
\tau_{\gamma_{a_1, a_2, b_{12}}} = I_{a_2},
\]
where
\[
\lambda_n = \frac{1}{n} \sum_{d \mid n} \phi(d) 2^{a_1/d},
\]
and \( \phi(d) \) is the Euler totient function. Note that \( \lambda_n \) is the number of necklaces that can be made from \( n \) beads of two colors when the necklaces can be rotated but not turned over [Plouffe & Sloane, 1995].

**Proof.** Since \( \tau_{\gamma_{a_1, a_2, b_{12}}} \) has integer entries for any \( a_1, a_2 \geq 1, 0 \leq b_{12} \leq a_1 - 1 \), the result follows.}

Now, that \( \zeta_{\gamma_{a_1, a_2, b_{12}}} \) are meromorphic extensions of \( \zeta_{\gamma} \) is obtained as follows.

**Theorem 4.7.** Given \( B \subset \Sigma_{2 \times 2 \times 2} \). For any \( \gamma \in GL_3(\mathbb{Z}) \),
\[
\zeta_{\gamma_{a_1, a_2, b_{12}}} (s) = \zeta_{\gamma} (s)
\]
for \( |s| < \exp (\gamma (B)) \), where
\[
g(B) = \lim_{|z| \rightarrow \infty} \frac{\log \Gamma (L)}{|z|}.
\]
Moreover, \( \zeta_{\gamma_{a_1, a_2, b_{12}}} \) has the same (integer) coefficients in its Taylor series expansions at \( s = 0 \), for all \( \gamma \in GL_3(\mathbb{Z}) \).

**Proof.** By [Lind, 1996], \( \zeta_{\gamma} \) has radius of convergence \( \exp (\gamma (B)) \) and is analytic in \( |s| < \exp (\gamma (B)) \). Since \( \zeta_{\gamma_{a_1, a_2, b_{12}}} \) is a rearrangement of \( \zeta_{\gamma} \), Eq. (84) holds. From [Lind, 1996] or Corollary 4.6, \( \zeta_{\gamma_{a_1, a_2, b_{12}}} \) has the same integer coefficients in its Taylor series expansions at \( s = 0 \). The proof is complete.}

**Remark 4.8.** From Theorem 4.5, for any \( B \subset \Sigma_{2 \times 2 \times 2} \), there exists a family of zeta functions \( \{ \zeta_{\gamma_{a_1, a_2, b_{12}}} : \gamma \in GL_3(\mathbb{Z}) \} \). For certain \( B \), the other \( \gamma \in GL_3(\mathbb{Z}) \) may give a different description to \( \zeta_{\gamma} \); see Example 3.7 and the following Example 4.9.

These different descriptions of \( \zeta_{\gamma} \) may be useful in studying zeta functions.

**Example 4.9.** Consider the basic set \( B \) in Example 3.7 and \( \gamma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \). It is easy to verify that
\[
\tau_{\gamma_{a_1, a_2, b_{12}}} = \tau_{\gamma_{a_1, a_2, b}} = \tau_{\gamma_{a_1, a_2, b}}
\]
for \( a_1, a_2 \geq 1, 0 \leq b_{12} \leq a_1 - 1 \). Moreover, after the zero columns and rows of \( \tau_{\gamma_{a_1, a_2, b_{12}}} \) were deleted, \( \tau_{\gamma_{a_1, a_2, b_{12}}} \) is reduced to \( \tau_{\gamma_{a_1, a_2, b_{12}}} \). Clearly
\[
\tau_{\gamma_{a_1, a_2, b_{12}}} = I_{a_2}
\]
and
\[
\tau_{\gamma_{a_1, a_2, b_{12}}} = I_{a_2},
\]
where
\[
\lambda_n = \frac{1}{n} \sum_{d \mid n} \phi(d) 2^{a_1/d},
\]
and \( \phi(d) \) is the Euler totient function. Note that \( \lambda_n \) is the number of necklaces that can be made from \( n \) beads of two colors when the necklaces can be rotated but not turned over [Plouffe & Sloane, 1995].
Hence,
\[ \zeta_{\gamma_{a_1,b_1}} = (1 - s^{a_1})^{\gamma_{a_1}} \] (86)
and
\[ \zeta_{\gamma} = \prod_{a_1=1}^{\infty} \prod_{a_2=1}^{\infty} (1 - s^{a_1 a_2})^{-a_2 \gamma_{b_2}}. \] (87)

It can be proved that \( g(B) = \log 2 \). Therefore, from Example 3.7 and Theorem 4.7,
\[ \prod_{a_1=1}^{\infty} \prod_{a_2=1}^{\infty} (1 - s^{a_1 a_2})^{-a_2 \gamma_{b_2}} = \prod_{a_1=1}^{\infty} \prod_{a_2=1}^{\infty} (1 - 2^{a_1 a_2})^{-a_2} \] (88)

5. Further Results

This section briefly describes the results for \( \mathbb{Z}^d \), \( d \geq 4 \), and more symbols on larger lattices. The thermodynamic zeta function for the three-dimensional Ising model with finite range interactions is also studied.

5.1. Higher-dimensional shifts of finite type

This subsection considers the zeta functions for shifts of finite type on \( \mathbb{Z}^d \), \( d \geq 4 \). Only brief statements are made here.

As in [Lind, 1996], \( \mathcal{L}_d \) can be parameterized by using Hermite normal form [MacDuffie, 1956]:
\[ \mathcal{L}_d = \left\{ \begin{array}{c} a_1 \ b_2 \ b_3 \ \cdots \ b_d \\ 0 \ a_2 \ b_3 \ \cdots \ b_d \\ 0 \ 0 \ a_3 \ \cdots \ b_d \\ \vdots \ \\ 0 \ 0 \ 0 \ \cdots \ a_d \end{array} \right\} \mathbb{Z}^d : a_i \geq 1, 1 \leq i \leq d, 0 \leq b_j \leq a_i, -1 \leq j \leq d \} \] (89)

Let the lattice \( \mathcal{L}_d \) be the smallest cubic lattice that contains \( \mathcal{L}_d \). Fix a basic set \( B \subset \{0,1\}^d \). For \( a_i \geq 1 \), \( 1 \leq i \leq d - 1 \), \( 0 \leq b_j \leq a_i - 1 \), \( 1 \leq j \leq d - 1 \), the \( (a_i, b_i) \)-th zeta function is defined by
\[ \zeta_{\mathcal{B}(a_i, b_i)}(s) \equiv \exp \left( \frac{1}{a_1 \cdots a_d - 1} \sum_{a_{i,1}=1}^{a_i} \sum_{b_{i,1}=0}^{b_{i,1}} \frac{1}{a_{d,i} b_{i,1}^s} \right) \] (90)

and
\[ \zeta(s) \equiv \prod_{d-1}^{d-1} \prod_{a_{d-1}}^{a_k} \prod_{b_{d-1}}^{b_{k,1}} \zeta_{\mathcal{B}(a_i, b_i)}(s). \] (91)

As in Secs. 2 and 3, the cylindrical ordering matrix, the trace operator, the rotational matrices and the reduced trace operator can be defined. The method in Secs. 2 and 3 can also be applied to verify that \( \zeta_{\mathcal{B}(a_i, b_i)} \) is a rational function. Therefore, \( \zeta_{\mathcal{B}} \) is an infinite product of rational functions. Furthermore, given any \( \gamma \in \text{GL}_d(\mathbb{Z}) \), the result also holds in \( \gamma \)-coordinates. Hence, a family of zeta functions exists with the same integer coefficients in their Taylor series expansions at \( s = 0 \), and they have the same integer coefficients in their Taylor series expansions at \( s = 0 \).

5.2. More symbols on larger lattice

This subsection extends the results of the previous sections and subsections to any finite number of symbols and any finite lattice. For simplicity, only the zeta functions for three-dimensional shifts of finite type are discussed. Given a set of symbols \( \mathcal{S}_p = \{0, 1, \ldots, p - 1\} \), \( p \geq 2 \), a set of finite lattice points \( \mathcal{L} \subset \mathbb{Z}^d \) and a basic set \( \mathcal{B}(\mathcal{L}) \subset \mathcal{S}_p \). Let \( \mathcal{L}_{m \times m \times m} \) be the smallest cubic lattice that contains \( \mathcal{L} \) and \( \mathcal{B}(\mathcal{L}_{m \times m \times m}) \) be the set of all admissible patterns that are generated by \( \mathcal{B}(\mathcal{L}) \). Then, it is easy to verify that
\[ \mathcal{P}(\mathcal{B}(\mathcal{L}_{m \times m \times m})) = \mathcal{P}(\mathcal{B}(\mathcal{L})). \]
Therefore, only \( B \subset \mathcal{S}_n \) for \( m \geq 2 \), need to be considered. The definitions of cylindrical order-

lattice Ising model with finite range interactions. This subsection will extend the results to the

\[ \mathcal{L}_3 \]

of \( \mathcal{L}_3 \), the set of all \( B \)-admissible and \( L \)-periodic patterns is denoted by \( \mathcal{P}_B(L) \).

Then, the partition function for \( B \) with \( L \)-periodic patterns is defined as

\[
\mathcal{Z}_B(L) = \mathcal{Z}_B \begin{pmatrix} a_1 & b_{12} & b_{13} \\ a_2 & a_3 \end{pmatrix}
\]

\[
= \sum_{U \in \mathcal{P}_B(L)} \exp \left[ \sum_{0 \leq a_1, a_2, a_3} \left( K_1 u_{a_1+1, a_2, a_3} + K_2 u_{a_1, a_2+1, a_3} + K_3 u_{a_1, a_2, a_3+1} + h \right) \right]
\]

where \( K_i = J_i/k_B T \), \( 1 \leq i \leq 3 \), \( k_B \) is Boltzmann’s constant and \( T \) is the temperature. Therefore, the

thermodynamic zeta function is defined by

\[
\zeta_{\text{therm}}(L) \equiv \exp \left( \sum_{U \in \mathcal{L}_3} \mathcal{Z}_B(U)^{\frac{s}{|U|}} \right).
\]

As Eqs. (8) and (9), for any \( a_1, a_2, a_3 \geq 1 \), \( 0 \leq b_{12} \leq a_1 - 1 \), the \((a_1, a_2, b_{12})\)-th thermodynamic zeta function \( \zeta_{\text{therm}}(a_1, a_2, b_{12}) \) is defined as

\[
\zeta_{\text{therm}}(a_1, a_2, b_{12}) \equiv \exp \left( \sum_{a_3=1}^{a_1-1} \sum_{b_{13}=0}^{a_2-1} \mathcal{Z}_B \begin{pmatrix} a_1 & b_{12} & b_{13} \\ a_2 & a_3 \end{pmatrix} \right)
\]

For simplicity, only the case of the nearest neighbor interactions is considered. Let the \( \mathbb{Z}^3 \) lattice Ising model be with the external field \( H \), the coupling constant \( J_1 \) in the \( x \)-direction, the coupling constant \( J_2 \) in the \( y \)-direction and the coupling constant \( J_3 \) in the \( z \)-direction. Each site \((a_1, a_2, a_3)\) of \( \mathbb{Z}^3 \) lattice has a spin \( u_{a_1, a_2, a_3} \) taking two possible values, +1 or −1. Assume that the state space is given by \( B \subset \{0, 1\}^{Z^3} \). Given a state

\[
U = (u_{a_1, a_2, a_3}) \in \{0, 1\}^{Z^3},
\]

denote by \( U_{B \times a_1 \times a_3} = (u_{a_1, a_2, a_3})_{0 \leq a_1, a_2, a_3 \leq 3} \).

Now, the Hamiltonian (energy) \( \mathcal{E}(U_{a_1 \times a_2 \times a_3}) \) is defined by

\[
\mathcal{E}(U_{a_1 \times a_2 \times a_3}) = -J_1 \sum_{0 \leq a_1 \leq a_2 - 1} u_{a_1, a_2, a_3} u_{a_1+1, a_2, a_3} - J_2 \sum_{0 \leq a_1 \leq a_2} u_{a_1, a_2, a_3} u_{a_1, a_2+1, a_3} - J_3 \sum_{0 \leq a_1 \leq a_2} u_{a_1, a_2, a_3} u_{a_1+1, a_2, a_3} - H \sum_{0 \leq a_1 \leq a_2} u_{a_1, a_2, a_3} - H
\]

(92)

Given \( L = \begin{pmatrix} a_1 & b_{12} & b_{13} \\ a_2 & a_3 \end{pmatrix} \in \mathcal{L}_3 \), the set of all \( B \)-admissible and \( L \)-periodic patterns is denoted by \( \mathcal{P}_B(L) \).

Then, the partition function for \( B \) with \( L \)-periodic patterns is defined as

\[
\mathcal{Z}_B(L) = \mathcal{Z}_B \begin{pmatrix} a_1 & b_{12} & b_{13} \\ a_2 & a_3 \end{pmatrix}
\]

\[
= \sum_{U \in \mathcal{P}_B(L)} \exp \left[ \sum_{0 \leq a_1, a_2, a_3} \left( K_1 u_{a_1+1, a_2, a_3} + K_2 u_{a_1, a_2+1, a_3} + K_3 u_{a_1, a_2, a_3+1} + h \right) \right]
\]

(93)
and the thermodynamic zeta function $\zeta_{\text{Ising}}(s)$ is given by

$$
\zeta_{\text{Ising}}(s) = \prod_{\vec{a}, \vec{b} \in \{\pm 1\}} \prod_{a=1}^{\infty} \prod_{b=1}^{\infty} \zeta_{\text{Ising},\vec{a},\vec{b}}(s).
$$

Since the spin $u_{a_1,a_2,a_3} \in \{+1,-1\}$, the cylindrical ordering matrix $C_{\text{Ising},a_1,a_2,b_1,b_2} = [C_{\text{Ising},a_1,a_2,b_1,b_2}]_{a_1,a_2}$ is obtained by replacing all symbols "0" in $C_{\text{Ising},a_1,a_2,b_1,b_2}$ with the symbols "−1". Notably, exactly one pattern exists in $C_{\text{Ising},a_1,a_2,b_1,b_2}$, and the pattern is given by $U_{\text{Ising},a_1,a_2,b_1,b_2} = (u_{a_1,a_2,a_3})$.

Define

$$
Z(U_{\text{Ising},a_1,a_2,b_1,b_2}) = \sum_{0 \leq a_1 \leq a_0 \leq -1 \leq b_0 \leq a_2} u_{a_1,a_2,0}(K_1u_{a_1+1,a_2,0} + K_2u_{a_1,a_2+1,0} + K_3u_{a_1,a_2,1} + h).
$$

Then, the trace operator $T_{\text{Ising},a_1,a_2,b_1,b_2} = [t_{\text{Ising},a_1,a_2,b_1,b_2}]_{a_1,a_2}$ is defined by

$$
t_{\text{Ising},a_1,a_2,b_1,b_2} = \left\{ \begin{array}{ll}
0 & \text{if } U_{\text{Ising},a_1,a_2,b_1,b_2} \text{ is not } B\text{-admissible,} \\
Z(U_{\text{Ising},a_1,a_2,b_1,b_2}) & \text{if } U_{\text{Ising},a_1,a_2,b_1,b_2} \text{ is } B\text{-admissible.}
\end{array} \right.
$$

Therefore, the associated reduced operator $T_{\text{Ising},a_1,a_2}$ can be defined as in Definition 3.4. Since all arguments for the rationality of $\zeta_{\text{Ising},\vec{a},\vec{b}}$ are similar to those in Secs. 2 and 3, only the final result is stated, as follows.

**Theorem 5.1.** For $a_1, a_2 \geq 1, 0 \leq b_2 \leq a_1 - 1$,

$$
\zeta_{\text{Ising},a_1,a_2}(s) = \frac{1}{(\det(I - s^{a_1}2\tau_{\text{Ising},a_1,a_2})))^{-1}}
$$

and

$$
\zeta_{\text{Ising}}(s) = \prod_{a_1=1}^{\infty} \prod_{a_2=1}^{\infty} a_1 - 1
$$

$$
\times (\det(I - s^{a_1}2\tau_{\text{Ising},a_1,a_2}))^{-1}.
$$

Notably, this result also holds in $\gamma$-coordinates for $\gamma \in GL_2(\mathbb{Z})$.

**References**


