Unitary part of a contraction

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\textbf{ABSTRACT}

For a contraction \(A\) on a Hilbert space \(H\), we define the index \(j(A)\) (resp., \(k(A)\)) as the smallest nonnegative integer \(j\) (resp., \(k\)) such that \(\ker(I - A^j A^j)\) (resp., \(\ker(I - A^k A^k)\)) equals the subspace of \(H\) on which the unitary part of \(A\) acts. We show that if \(n = \dim H < \infty\), then \(j(A) \leq n\) (resp., \(k(A) \leq \lfloor n/2 \rfloor\)), and the equality holds if and only if \(A\) is of class \(S_n\) (resp., one of the three conditions is true: (1) \(A\) is of class \(S_n\), (2) \(n\) is even and \(A\) is completely nonunitary with \(\|A^{n-2}\| = 1\) and \(\|A^{n-1}\| < 1\), and (3) \(n\) is even and \(A = U \oplus A'\), where \(U\) is unitary on a one-dimensional space and \(A'\) is of class \(S_{n-1}\)).

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1. Introduction

Let \(H\) be a complex Hilbert space with inner product \((\cdot, \cdot)\) and norm \(\|\cdot\|\) and let \(A\) be a contraction (\(\|A\| \leq 1\)) on \(H\). There is a canonical decomposition of \(H\) as the direct sum \(H_1 \oplus H_2\) of reducing subspaces \(H_1\) and \(H_2\) for \(A\) so that \(A|H_1\) is unitary and \(A|H_2\) is completely nonunitary. (Recall that an operator \(B\) on \(K\) is completely nonunitary (c.n.u.) if there is no (nonzero) reducing subspace \(L\) of \(B\) such that \(B|L\) is unitary.) In this case, \(H_1\) and \(H_2\) are uniquely determined by \(A\): \(H_1 = \{x \in H : \|A^j x\| = \|x\| = \|A^j x\|\text{ for all }j \geq 0\}\) and \(H_2 = H \ominus H_1\), and \(U = A|H_1\) and \(A' = A|H_2\) are called the unitary part and completely nonunitary part of \(A\), respectively (cf. [3, Theorem I.3.2]). Another expression for the subspace \(H_1\) is \(\bigcap_{j=0}^\infty (\ker(I - A^{j} A^{j}) \cap \ker(I - A^{j} A^{j}))\). Thus it is natural to consider the smallest integer \(j\) (resp., \(k\)) for which \(\ker(I - A^{j} A^{j}) = H_1\) (resp., \(\ker(I - A^{k} A^{k}) \cap \ker(I - A^{k} A^{k}) = H_1\)). For this purpose, we give the following definitions:

**Definition 1.1.** For a contraction \(A\) on \(H\), let

\[
H_j(A) = \begin{cases} \ker(I - A^j A^j) & \text{if } 0 \leq j < \infty, \\ \bigcap_{j=0}^\infty \ker(I - A^j A^j) & \text{if } j = \infty, \end{cases}
\]

\[
j(A) = \min\{j : 0 \leq j \leq \infty, H_j(A) = H_\infty(A) \cap H_\infty(A^*)\},
\]

and

\[
k(A) = \min\{k : 0 \leq k \leq \infty, H_k(A) \cap H_k(A^*) = H_\infty(A) \cap H_\infty(A^*)\}.
\]

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The next example gives values of such indices for the $n$-by-$n$ Jordan block

$$J_n = \begin{bmatrix}
0 & 1 & & \\
0 & 0 & \ddots & \\
& \ddots & \ddots & 1 \\
& & & 0
\end{bmatrix}.$$  

Example 1.2. If $A = J_n$, then $j(A) = n$ and $k(A) = \lceil n/2 \rceil$, the ceiling of $n/2$ (that is, the smallest integer which is larger than or equal to $n/2$). This is because simple computations yield that

$$H_j(A) = \begin{cases}
\{0\} \oplus C^{n-j} & \text{if } 0 \leq j \leq n, \\
\{0\} & \text{if } n < j \leq \infty,
\end{cases}$$

$$H_j(A^*) = \begin{cases}
C^{n-j} \oplus \{0\} & \text{if } 0 \leq j \leq n, \\
\{0\} & \text{if } n < j \leq \infty,
\end{cases}$$

and $H_\infty(A) \cap H_\infty(A^*) = \{0\}$.

In the following, we will show that if $A$ is a contraction on an $n$-dimensional space, then $j(A)$ and $k(A)$ can be at most $n$ and $\lceil n/2 \rceil$, respectively. The extremal cases can also be completely characterized. Recall that an $n$-dimensional operator $A$ is said to be of class $S_n$ if it is a contraction, its eigenvalues are all in the open unit disc $D \equiv \{ z \in \mathbb{C} : |z| < 1 \}$ and it satisfies $\text{rank}(I_n - A^*A) = 1$. One example of such operators is $J_n$. More generally, it is known [1, Corollary 1.3] that an $n$-by-$n$ matrix is of class $S_n$ if and only if it is unitarily equivalent to a matrix of the form $[a_{ij}]_{n,j=1}^n$, where $|a_{ij}| < 1$ for all $i$ and

$$a_{ij} = \begin{cases}
(1 - |a_{ii}|^2)^{1/2}(1 - |a_{jj}|^2)^{1/2}(\prod_{k=i+1}^{j-1}(-a_{kk})) & \text{if } i < j, \\
0 & \text{if } i > j.
\end{cases}$$

We show that a contraction $A$ on an $n$-dimensional space is such that $j(A) = n$ (resp., $k(A) = \lceil n/2 \rceil$) if and only if it is of class $S_n$ (resp., one of the following holds: (1) $A$ is of class $S_n$, (2) $n$ is even and $A$ is completely nonunitary with $\|A^n-1\| = 1$ and $\|A^{n-1}\| < 1$, and (3) $n$ is even and $A = U \oplus A'$, where $U$ is unitary on a one-dimensional space and $A'$ is of class $S_{n-1}$). These will be given in Sections 2 and 3, respectively.

Throughout this paper, we will frequently use the fact that, for a contraction $A$ and an integer $j \geq 0$, $x$ is in $H_j(A)$ if and only if $\|A^jx\| = \|x\|$.

We end this section with some basic properties of the $H_j(A)'s$, $j(A)$ and $k(A)$.

**Proposition 1.3.** Let $A$ be a contraction on $H$. Then

1. $H_0(A) = H_0(A^*) = H$,
2. $H_j(A) \downarrow H_\infty(A)$ as $j \to \infty$,
3. $\dim H_j(A) = \dim H_j(A^*)$ for all $j, 0 \leq j \leq \infty$,
4. $0 \leq k(A) \leq j(A) \leq \infty$,
5. the following conditions are equivalent: (a) $j(A) = 0$, (b) $k(A) = 0$, and (c) $A$ is unitary,
6. $j(A) = j(1 \leq j < \infty)$ if and only if $A = U \oplus A'$ on $H = H_1 \oplus H_2$, where $U$ is unitary, $H_j(A') = \{0\}$ and $H_{j-1}(A') \neq \{0\}$,
7. $j(A) = k(A'^*)$ and $k(A) = k(A'^*)$,
8. $j(A \oplus B) = \max\{j(A), j(B)\}$ and $k(A \oplus B) = \max\{k(A), k(B)\}$ for any other contraction $B$, and
9. for $A$ normal, $H_j(A) = H_\infty(A)$ for all $j \geq 1$ and thus $j(A) = k(A) \leq 1$.

For the proof of (3), we need the next lemma.

**Lemma 1.4.** For any operator $A$ on $H$, let $A^*A = A_1 + 0$ (resp., $AA^* = A_2 + 0$) on $H = \text{ran} A^* \oplus \ker A$ (resp., $H = \text{ran} A \oplus \ker A^*$). Then $A_1$ and $A_2$ are unitarily equivalent.

**Proof.** Since $\text{ran} A^* = \text{ran}(A^*A)^{1/2}$ (resp., $\text{ran} A = \text{ran}(AA^*)^{1/2}$), we may define $V : \text{ran} A^* \to \text{ran} A$ (resp., $W : \text{ran} A \to \text{ran} A^*$) by $V((A^*A)^{1/2}x) = Ax$ for $x$ in $H$ (resp., $W((AA^*)^{1/2}y) = A^*y$ for $y$ in $H$). Then $V$ and $W$ are surjective isometries satisfying

$$\langle V((A^*A)^{1/2}x, (AA^*)^{1/2}y) = \langle Ax, (AA^*)^{1/2}y = \langle x, A^*(AA^*)^{1/2}y \rangle$$

$$= \langle x, (A^*A)^{1/2}y \rangle = \langle (A^*A)^{1/2}x, W(AA^*)^{1/2}y \rangle$$

$$= \langle W^*(A^*A)^{1/2}x, (AA^*)^{1/2}y \rangle$$
for all \(x\) and \(y\). Thus \(V = W^*\). We have
\[
(V^* A_2 V)(A^* A)^{1/2} x = W A_2 A x = W (AA^*) A x
= W (AA^*)^{1/2} (AA^*)^{1/2} A x = A^* (AA^*) A^{1/2} x = A_1 (A^* A)^{1/2} x,
\]
which shows that \(V^* A_2 V = A_1\), completing the proof. \(\square\)

**Proof of Proposition 1.3.** (1), (2), (4), (5), (6) and (8) are trivial. (3) is an easy consequence of Lemma 1.4 and (7) follows from (6) and (3). We now prove (9). Indeed, if \(A\) is normal, then from \(I - A^{j+1}\) we obtain \(H_1(A) \subseteq H_j(A)\) for any \(j \geq 1\). Since \(H_j(A) \subseteq H_1(A)\) by (2), the equality \(H_j(A) = H_1(A)\) holds for all \(j \geq 1\). Our assertions follow immediately. \(\square\)

2. The index \(j(A)\)

The main result of this section is the following theorem.

**Theorem 2.1.** If \(A\) is a contraction on an \(n\)-dimensional space, then \(j(A) \leq n\). The equality holds if and only if \(A\) is of class \(S_n\).

For its proof, we need the next two lemmas.

**Lemma 2.2.** If \(A\) is a contraction on an \(n\)-dimensional space with \(\dim H_j(A) > n - j\) for some \(j\), \(1 \leq j \leq n\), then \(j(A) < j\).

**Proof.** Assume that \(A\) acts on the space \(H\) with \(\dim H = n\). Since there are \(j + 1\) subspaces in the sequence \(H_j(A) \subseteq H_{j-1}(A) \subseteq \cdots \subseteq H_1(A) \subseteq H_0(A) = H\), our assumption on \(H_j(A)\) implies, by the pigeonhole principle, the equality of \(H_k(A)\) and \(H_{k+1}(A)\) for some \(k\), \(0 \leq k < j\). If \(x\) is in \(H_{k+1}(A)\), then
\[
\|x\| = \|A^{k+1} x\| \leq \|Ax\| \leq \|x\|.
\]
which yields that \(\|A^k(Ax)\| = \|Ax\| = \|x\|\). Hence \(Ax\) is in \(H_k(A) = H_{k+1}(A)\) and thus \(A\) maps \(H_k(A)\) into itself isometrically. Therefore, \(A = A_1 \oplus A_2\) on \(H = H_k(A) \oplus H_k(A)^\perp\) with \(A_1\) unitary. This shows that \(H_k(A) \subseteq H_\infty(A) \cap H_\infty(A^*)\). Since the converse containment is trivial, we have \(H_k(A) = H_\infty(A) \cap H_\infty(A^*)\) and therefore \(j(A) \leq k < j\) as asserted. \(\square\)

We note that the above arguments are essentially contained in the proof of [2, Lemma 2.3].

**Lemma 2.3.** A contraction \(A\) on an \(n\)-dimensional space is of class \(S_n\) if and only if \(\dim H_j(A) = n - j\) for all \(j, 1 \leq j \leq n\).

This characterization of \(S_n\)-operators is from [2, Theorem 3.1].

**Proof of Theorem 2.1.** If \(H_n(A) = \{0\}\), then obviously \(H_\infty(A) \cap H_\infty(A^*) = \{0\}\) and \(j(A) \leq n\); otherwise, the assertion \(j(A) < n\) follows from Lemma 2.2.

For the remaining part, we need only check, in view of Lemma 2.3, that \(j(A) = n\) implies \(\dim H_j(A) = n - j\) for all \(j, 1 \leq j \leq n\). Indeed, if \(\dim H_j(A) < n - j\) for some \(j, 1 \leq j < n\), then, as in the proof of Lemma 2.2, the pigeonhole principle yields the equality of \(H_k(A)\) and \(H_{k+1}(A)\) for some \(k, j \leq k < n\). We then proceed as before to infer that \(A\) maps \(H_k(A)\) into itself isometrically and that \(H_k(A) = H_\infty(A) \cap H_\infty(A^*)\). This leads to \(j = j(A) \leq k < n\), a contradiction. Thus we must have \(\dim H_j(A) = n - j\) for all \(j\). If the strict inequality “>” holds for any \(j, 1 \leq j \leq n\), then Lemma 2.2 says that \(j(A) < j\), again a contradiction. Therefore, we have \(\dim H_j(A) = n - j\) for all \(j\). Thus \(A\) is of class \(S_n\) by Lemma 2.3. This completes the proof. \(\square\)

To conclude this section, we give two remarks. Firstly, the assertion \(j(A) \leq n\) in Theorem 2.1 can be slightly improved. Namely, if \(A\) is a finite-dimensional contraction, then \(j(A)\) is at most the degree of the minimal polynomial of \(A\). To prove this, let \(m\) denote this degree. Then, for any \(x\) in \(H_m(A)\), we have \(\|A^m x\| = \|x\|\). Thus \(\|A^{m-1} x\| = \cdots = \|Ax\| = \|x\|\), which shows that \(A^{m-1} x, \ldots, A x, x\) are all in \(H_1(A)\). Since \(A^m x\) is a linear combination of these vectors, it is also in \(H_1(A)\). We can then deduce successively that all the \(A^j x\)'s, \(j \geq m\), are in \(H_1(A)\). If \(K\) denotes the subspace generated by \(x, A x, A^2 x, \ldots\), \(K\) is an invariant subspace of \(A\) contained in \(H_1(A)\). Hence \(A_i K = K\) is isometric on \(K\). On a finite-dimensional space, \(A_1\) is unitary. Thus \(K\) is contained in \(H_\infty(A) \cap H_\infty(A^*)\). This shows that \(x \in H_\infty(A) \cap H_\infty(A^*)\) for any \(x\) in \(H_m(A)\) or \(H_m(A) \subseteq H_\infty(A) \cap H_\infty(A^*)\). Since the converse containment is trivial, we obtain \(H_m(A) = H_\infty(A) \cap H_\infty(A^*)\). Thus \(j(A) \leq m\) follows.
Secondly, for any finite-dimensional contraction $A$, the inequality
\[
\dim H_{j+1}(A) \leq \max\{\dim H_j(A) - 1, \dim (H_\infty(A) \cap H_\infty(A^*))\}
\]
holds for all $j \geq 0$. This is a consequence of [2, Lemma 2.3]. Indeed, if $\dim H_{j+1}(A) \leq \dim H_j(A) - 1$, then we are done. Otherwise, since $H_{j+1}(A) \subseteq H_j(A)$, we must have $H_{j+1}(A) = H_j(A)$. By [2, Lemma 2.3] or the arguments in the proof of Lemma 2.2, $A|H_{j+1}(A)$ is unitary. This implies that $H_{j+1}(A) \subseteq H_\infty(A) \cap H_\infty(A^*)$. Hence these two subspaces are equal and thus (a) also holds. Note that this can be used to give an alternative proof of $j(A) \leq n$ in Theorem 2.1.

3. The index $k(A)$

Our first result relates $k(A)$ to the norm-one index $k_A$ for a contraction $A$. Recall that if $A$ is a contraction, then its norm-one index $k_A$ is defined as $\sup\{k \geq 0: \|A^k\| = 1\}$ [cf. [2, p. 364]]. It is easily seen that (1) $0 \leq k_A \leq \infty$, (2) $k_A = 0$ if and only if $\|A\| < 1$, and (3) $k_A = \infty$ if and only if its spectral radius $r(A) (= \max\{|\lambda|: \lambda \in \sigma(A)|)$ equals 1. In fact, it was shown in [2, Proposition 2.1(c) or Theorem 2.2] that, for an $n$-dimensional contraction $A$, $k_A$ is either infinity or less than or equal to $n - 1$.

**Theorem 3.1.** If $A$ is a c.n.u. contraction, then $k(A) \leq [(k_A + 1)/2]$. If, in addition, $A$ is compact, then the equality $k(A) = [(k_A + 1)/2]$ holds.

The proof depends on the following lemma relating $H_j(A) \cap H_k(A^*)$ and $H_{j+k}(A)$.

**Lemma 3.2.** If $A$ is a contraction, then $A^{k_A}$ maps $H_j(A) \cap H_k(A^*)$ onto $H_{j+k}(A)$ isometrically for any $j$, $0 \leq j \leq n$, and $k$, $0 \leq k < n$, and, in particular, $j(A) \leq 2k(A)$.

**Proof.** We first check that $A^{k_A}$ maps $H_j(A) \cap H_k(A^*)$ to $H_{j+k}(A)$ for $0 \leq j, k < \infty$. Indeed, if $x$ is in $H_j(A) \cap H_k(A^*)$, then
\[
\|x\| = \|A^{k_A}x\| = \|A^{j+k}A^{k_A}x\| = \|A^j x\| = \|x\|.
\]
where the last two equalities follow from the assumptions that $x \in H_k(A^*)$ and $x \in H_j(A)$, respectively. It follows that $\|A^{j+k}(A^{k_A}x)\| = \|A^{k_A}x\|$ and thus $A^{k_A}x$ is in $H_{j+k}(A)$. (b) above also implies that $\|A^{k_A}x\| = \|x\|$ and thus $A^{k_A}: H_j(A) \cap H_k(A^*) \rightarrow H_{j+k}(A)$ is isometric. To show the surjectivity of the asserted map, let $y$ be in $H_{j+k}(A)$. Then
\[
\|y\| = \|A^k y\| = \|A^j (A^k y)\| = \|y\|,
\]
showing that $A^k y$ is in $H_j(A)$. We now prove that $x \equiv A^k y$ is also in $H_k(A^*)$. Indeed, this is so because
\[
A^k A^k x = A^k (A^{k_A}A^k y) = A^k y = x,
\]
where the second equality follows from $y \in H_{j+k}(A) \subseteq H_k(A)$. Finally, the equalities
\[
A^k x = A^{k_A} A^k y = y
\]
show that $A^{k_A}$ maps $x$ in $H_j(A) \cap H_k(A^*)$ to $y$. This proves our assertion for finite $j$ and $k$. The case for $j = \infty$ can be easily shown to be true from above.

To prove $j(A) \leq 2k(A)$, we decompose $A$ as $U \oplus A'$, where $U$ is unitary and $A'$ is c.n.u. Since $j(A) = j(A')$ and $k(A) = k(A')$ by Proposition 1.3(8) and (5), we need only check $j(A') \leq 2k(A')$. Assuming $k = k(A') < \infty$, we have $H_k(A') \subseteq H_{k+1}(A^*) = \{0\}$. Hence $H_{2k}(A') = \{0\}$ from above. Thus $j(A') \leq 2k = 2k(A')$ as desired. □

**Proof of Theorem 3.1.** Since $A$ is c.n.u., we have $H_\infty(A) \cap H_\infty(A^*) = \{0\}$. If $k(A) = \infty$, then $H_k(A) \cap H_k(A^*) \neq \{0\}$ for any $k$, $0 \leq k < \infty$. Thus $H_k(A) \neq \{0\}$ or $\|A^k\| = 1$ for all $k \geq 0$. It follows that $k(A) = \infty = [(k_A + 1)/2]$.

Now assume that $k \equiv k(A)$ is finite (resp., $k \equiv k(A)$ is finite and $A$ is compact). We have $H_{k-1}(A) \cap H_{k-1}(A^*) \neq \{0\}$ and $H_k(A) \cap H_k(A^*) = \{0\}$. Apply Lemma 3.2 to infer that $H_{2k-2}(A) \neq \{0\}$ and $H_{2k}(A) = \{0\}$. Hence $\|A^{2k-2}\| = 1$ (resp., $\|A^{2k-2}\| < 1$ because the compact $A^{2k}$ attains its norm). It follows that $k_A \geq 2k - 2$ (resp., $k_A = 2k - 1$ or $2k - 2$). Therefore, $k(A) \leq [(k_A + 2)/2]$ (resp., $k(A) = (k_A + 1)/2$ or $(k_A + 2)/2$). We conclude that $k(A) \leq [(k_A + 1)/2]$ (resp., $k(A) = [(k_A + 1)/2]$). □

Note that, in this theorem, the equality is in general false. One example is the c.n.u. normal contraction $A = \text{diag}(1/2, 2/3, \ldots, (n - 1)/n, \ldots)$ on $l^2$. In this case, $k(A) = 1$ and $k_A = \infty$.

**Corollary 3.3.** Let $A$ be a contraction on a finite-dimensional space. Then $k(A) = 1$ if and only if $A = U \oplus A'$, where $U$ is unitary, $A'$ is not missing, and $\|A^2\| < 1$ or $\|A'\| = 1$ and $\|A^{2}\| < 1$. 

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This is an easy consequence of Proposition 1.3 and Theorem 3.1. It is comparable to the corresponding condition for \( j(A) = 1 \), namely, \( A = U \oplus A' \), where \( U \) is unitary, \( A' \) is not missing and \( \|A'\| < 1 \) (cf. Proposition 1.3(6)).

The theorem is the main result of this section.

**Theorem 3.4.** If \( A \) is a contraction on an \( n \)-dimensional space, then \( k(A) \leq \lceil n/2 \rceil \). For an operator \( A \) of class \( S_n \), we have the equality \( k(A) = \lfloor n/2 \rfloor \).

**Proof.** Let \( A = U \oplus A' \), where \( U \) is unitary and \( A' \) is c.n.u. Then \( k(A) = \max\{k(U), k(A')\} = k(A') \) by Proposition 1.3. Since \( k(A') = [(k_A' + 1)/2] \) by Theorem 3.1 and \( k_A' \leq n - 1 \) by [2, Proposition 2.1(c) or Theorem 2.2], we infer that \( k(A) \leq \lceil (n - 1 + 1)/2 \rceil = \lceil n/2 \rceil \) as asserted.

If \( A \) is of class \( S_n \), then \( k_A = n - 1 \) by [2, Theorem 3.1] and hence \( k(A) = \lceil ((n - 1) + 1)/2 \rceil = \lceil n/2 \rceil \) from above. □

We now give another proof for the first assertion in Theorem 3.4 which is more in line of the arguments in Section 2 and does not involve the norm-one index.

For the \( n \)-dimensional contraction \( A \), let \( k = \lceil n/2 \rceil \). We show that \( A^{k*} \) maps \( H_k(A) \cap H_k(A^*) \) onto itself isometrically.

Indeed, if \( x \) is any vector in \( H_k(A) \cap H_k(A^*) \), then

\[
\|x\| \geq \|A^{k*}x\| \geq \|A^{\lceil n-k \rceil}(A^kx)\| = \|A^{n-k}x\| = \|x\|
\]

where the last two equalities follow from the assumptions that \( x \in H_k(A^{*}) \) and \( x \in H_k(A) \subseteq H_{n-k}(A) \), respectively. It follows that \( \|A^{k*}(A^kx)\| = \|A^{k*}x\| = \|x\| \) and thus, in particular, \( A^kx \) is in \( H_n(A) \).

\[
A^{k*}(H_k(A) \cap H_k(A^*)) \subseteq H_n(A) = H_\infty(A) \cap H_\infty(A^*) \subseteq H_k(A) \cap H_k(A^*),
\]

where the equality in the middle is by Theorem 2.1. Therefore, \( A^{k*} \) maps \( H_k(A) \cap H_k(A^*) \) onto itself isometrically and thus we have equalities throughout (c). This yields \( H_k(A) \cap H_k(A^*) = H_\infty(A) \cap H_\infty(A^*) \) and thus \( k(A) \leq k = \lceil n/2 \rceil \).

The final result of this section is a characterization of \( n \)-dimensional contractions \( A \) with \( k(A) = \lceil n/2 \rceil \).

**Theorem 3.5.** Let \( A \) be a contraction on an \( n \)-dimensional space. Then \( k(A) = \lceil n/2 \rceil \) if and only if one of the following holds:

1. \( A \) is of class \( S_n \).
2. \( n \) is even and \( A \) is c.n.u. with \( \|A^{n-2}\| = 1 \) and \( \|A^{n-1}\| < 1 \), and
3. \( n \) is even and \( A = U \oplus A' \), where \( U \) is unitary on a one-dimensional space and \( A' \) is of class \( S_{n-1} \).

**Proof.** Assume that \( k(A) = \lceil n/2 \rceil \). Let \( A = U \oplus A' \), where \( U \) is unitary on a space of dimension \( n - m \) and \( A' \) is c.n.u. on a space of dimension \( m \), \( 0 \leq m \leq n \). Since

\[
\lceil n/2 \rceil = k(A) = \max\{k(U), k(A')\} = k(A')
\]

\[
\lceil (k_A' + 1)/2 \rceil \leq \lceil m/2 \rceil \leq \lceil n/2 \rceil,
\]

where the first inequality is by [2, Proposition 2.1(c) or Theorem 2.2], we have equalities throughout. In particular, \( \lceil (k_A' + 1)/2 \rceil = \lceil m/2 \rceil \) and \( \lceil m/2 \rceil = \lceil n/2 \rceil \) hold. The latter implies that either \( m = n \) or \( n = m = n - 1 \):

(i) If \( n = m = 2k + 1 \) is odd, then \( \lceil (k_A' + 1)/2 \rceil = k + 1 \), which implies that \( k_A' = 2k \) or \( 2k + 1 \). In the former case, we have \( k_A' = n - 1 \) and hence \( A = A' \) is of class \( S_n \) by [2, Theorem 3.1]. For the latter, we have \( k_A' = n \), which is impossible by [2, Proposition 2.1(c) or Theorem 2.2]. Thus, in this case, (1) holds.

(ii) If \( n = m = 2k \) is even, then \( \lceil (k_A' + 1)/2 \rceil = k \), which implies that \( k_A' = 2k - 1 \) or \( 2k - 2 \). For the former, we have \( k_A' = n - 1 \) and thus \( A = A' \) is of class \( S_{n-1} \), that is, (1) holds. For the latter, \( k_A' = n - 2 \), which yields (2).

(iii) If \( n = 2k \) is even and \( m = n - 1 \), then \( \lceil (k_A' + 1)/2 \rceil = k \), and hence \( k_A' = 2k - 1 \) or \( 2k - 2 \). The former says that \( k_A' = m \), which is impossible by [2, Proposition 2.1(c) or Theorem 2.2]. The latter implies that \( k_A' = m - 1 \) and hence \( A' \) is of class \( S_{n-1} \) by [2, Theorem 3.1]. Thus, in this case, (3) holds.

That (1) and (3) give \( k(A) = \lceil n/2 \rceil \) is easily seen from Theorem 3.4 while (2) yields this equality is by Theorem 3.1. □

We conclude this paper by remarking that, analogous to the situation for \( j(A) \) in Section 2, we can (1) slightly improve \( k(A) \leq \lceil n/2 \rceil \) in Theorem 3.4 by proving that, for a finite-dimensional contraction \( A \), \( k(A) \leq \lfloor m/2 \rfloor \), where \( m \) is the degree of the minimal polynomial of \( A \), and (2) relate the dimensions of \( H_{k+1} \cap H_{k+1}(A^*) \) and \( H_k(A) \cap H_k(A^*) \) as

\[
\dim(H_{k+1}(A) \cap H_{k+1}(A^*)) \leq \max\{\dim(H_k(A) \cap H_k(A^*)) - 2, \dim(H_{\infty}(A) \cap H_{\infty}(A^*))\}
\]

for a finite-dimensional contraction \( A \) and \( k \geq 0 \). To prove (1), we let \( l = \lfloor m/2 \rfloor \). Then \( 2l \geq m \) and hence \( H_{2l}(A) \subseteq H_m(A) \). Since \( H_m(A) = H_{\infty}(A) \cap H_{\infty}(A^*) \) as was proved in Section 2, we have \( H_{2l}(A) = H_{\infty}(A) \cap H_{\infty}(A^*) \). Thus \( H_l(A) \cap H_l(A^*) = H_{\infty}(A) \cap H_{\infty}(A^*) \) by Lemma 3.2. Therefore, \( k(A) \leq l = \lfloor m/2 \rfloor \). For the proof of (2), note that \( \dim(H_k(A) \cap H_k(A^*)) = \dim(H_{2k}(A) \cap H_{2k}(A^*)) \).
all \( k \geq 0 \) by Lemma 3.2. Hence
\[
\dim(H_{k+1}(A) \cap H_{k+1}(A^*)) \\
= \dim H_{2k+2}(A) \\
\leq \max\{ \dim H_{2k+1}(A) - 1, \dim(H_{\infty}(A) \cap H_{\infty}(A^*)) \} \quad \text{(by (a))} \\
\leq \max\{ \max\{ \dim H_{2k}(A) - 1, \dim(H_{\infty}(A) \cap H_{\infty}(A^*)) \} - 1, \dim(H_{\infty}(A) \cap H_{\infty}(A^*)) \} \quad \text{(by (a))} \\
= \max\{ \dim H_{2k}(A) - 2, \dim(H_{\infty}(A) \cap H_{\infty}(A^*)) \} \\
= \max\{ \dim(H_k(A) \cap H_k(A^*)) - 2, \dim(H_{\infty}(A) \cap H_{\infty}(A^*)) \}.
\]
As before, this can be used to give an alternative proof of \( k(A) \leq \lceil n/2 \rceil \) in Theorem 3.4.

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References