$C_{11}$ Contractions are Reflexive
Author(s): Pei Yuan Wu
Published by: American Mathematical Society
Stable URL: http://www.jstor.org/stable/2042718
Accessed: 28/04/2014 17:08

Your use of the JSTOR archive indicates your acceptance of the Terms & Conditions of Use, available at http://www.jstor.org/page/info/about/policies/terms.jsp
JSTOR is a not-for-profit service that helps scholars, researchers, and students discover, use, and build upon a wide range of content in a trusted digital archive. We use information technology and tools to increase productivity and facilitate new forms of scholarship. For more information about JSTOR, please contact support@jstor.org.

American Mathematical Society is collaborating with JSTOR to digitize, preserve and extend access to Proceedings of the American Mathematical Society.

http://www.jstor.org
C_{11} CONTRACTIONS ARE REFLEXIVE

PEI YUAN WU

ABSTRACT. It is shown that a completely nonunitary C_{11} contraction defined on a separable Hilbert space with finite defect indices is reflexive.

In this note, only bounded linear operators defined on complex, separable Hilbert spaces will be considered. A contraction $T$ ($\|T\| < 1$) on a Hilbert space $\mathcal{H}$ is of class $C_{11}$ if $T^n_x \rightarrow 0$ and $T^*_n x \rightarrow 0$ for any $x \neq 0$. It is well known that such a contraction is quasi-similar to a unitary operator. Since unitary operators (even normal operators) are reflexive (cf. [3]), the question arises: Is the property of reflexivity preserved under the quasi-similarity? In other words, is a $C_{11}$ contraction reflexive? In the present note we show that the answer is affirmative if the $C_{11}$ contraction is completely nonunitary (c.n.u.) and has finite defect indices. We conjecture that the general case is also true.

Recall that a contraction $T$ is c.n.u. if there is no nontrivial reducing subspace on which $T$ is unitary. The defect indices of $T$ are, by definition,

$$d_T = \dim [(1 - T^*T)^{1/2}\mathcal{H}], \quad d_{r*} = \dim [(1 - TT^*)^{1/2}\mathcal{H}].$$

If $T$ is of class $C_{11}$, then $d_T = d_{r*}$. In the following discussion we shall make use of the functional model for contractions developed by Sz.-Nazy and Foiaş (cf. [4]). More specifically, if $T$ is a c.n.u. contraction with $d_T = d_{r*} = n < \infty$, then $T$ can be considered as defined on

$$H = \left[H_n^2 \oplus \Delta L_n^2\right] \oplus \left\{\Theta_T w \oplus \Delta w : w \in H_n^2\right\}$$

by $T(f \oplus g) = P(e^{\Theta_T f} \oplus e^{\Theta_T g})$ for $f \oplus g \in H$, where $L_n^2$ and $H_n^2$ denote the standard Lebesgue and Hardy spaces of $\mathbb{C}^n$-valued functions defined on the unit circle, $\Theta_T$ is the characteristic function of $T$, $\Delta = (1 - \Theta_T^*\Theta_T)^{1/2}$ and $P$ denotes the (orthogonal) projection onto $H$. Any operator $S$ in $\{T\}'$, the commutant of $T$, has the form $P[A B C]$, where $A$ is a bounded analytic function while $B$ and $C$ are bounded measurable functions satisfying $A \Theta_T = \Theta_T A_0$ and $B \Theta_T + C \Delta = \Delta A_0$ for some bounded analytic function $A_0$ (cf. [5]).

For an arbitrary operator $T$, $\{T\}'$, $\{T\}$" and Alg $T$ denote the commutant, double commutant and the weakly closed algebra generated by $T$ and $I$,
respectively, and Lat $T$, Lat$"T$ denote the lattice of invariant subspaces and the lattice of bi-invariant subspaces of $T$, respectively. Let Alg Lat $T$ and Alg Lat$"T$ denote the (weakly closed) algebras of operators which leave all the subspaces in Lat $T$ and Lat$"T$ invariant, respectively. An operator $T$ is reflexive if Alg Lat $T = Alg T$. Two operators $T_1$, $T_2$ are quasi-similar if there exist one-to-one operators $X$ and $Y$ with dense ranges (called quasi-affinities) such that $XT_1 = T_2X$ and $YT_2 = T_1Y$.

We start the proof with the following:

**Lemma 1.** Let $T$ be a normal operator on a separable Hilbert space. Then $(T)' \cap \text{Alg Lat}"T = \{(T)"\}$.

**Proof.** By the spectral theorem, we may assume that $T = M_{E_1} \oplus M_{E_2} \oplus \cdots$ acting on $H = L^2(E_1, \mu) \oplus L^2(E_2, \mu) \oplus \cdots$, where $E_1 \supseteq E_2 \supseteq \cdots$ are Borel subsets of the complex plane, $\mu$ is a finite positive Borel measure and $M_{E_j}$ denotes the operator of multiplication by independent variable on $L^2(E_j, \mu)$, $j = 1,2, \cdots$ (cf. [2, p. 916]). Let $S \in (T)' \cap \text{Alg Lat}"T$. Since for normal operators bi-invariant subspaces are exactly reducing subspaces, $S \in \{(T)\}' \subseteq \text{Alg Lat}"T$ implies that $S = \varphi_1 \oplus \varphi_2 \oplus \cdots$, where $\varphi_j \in L^\infty(E_j, \mu)$, $j = 1,2, \cdots$. Consider the reducing (hence bi-invariant) subspace

$$K = \{f_1 \oplus f_2 \oplus \cdots \in H: \chi_{E_i}f_i = f_j \text{ for all } i < j\}.$$  

We have $SK \subseteq K$, which implies that $\chi_{E_i}\varphi_i f_i = \varphi_j f_j$ for all $f_i \in L^2(E_i, \mu)$, $f_j \in L^2(E_j, \mu)$, $i < j$. In particular, if $f_i = \chi_{E_i}$ and $f_j = \chi_{E_j}$, we have $\varphi_i = \varphi_j$ on $E_j$. Hence

$$S = \varphi_1 \oplus \chi_{E_2} \varphi_1 \oplus \chi_{E_3} \varphi_1 \oplus \cdots = \varphi_1(T) \in \{(T)\}'$$

This shows that $(T)' \cap \text{Alg Lat}"T \subseteq \{(T)\}'$. Since the other inclusion is trivial, this completes the proof.

The next lemma characterizes the operators in $(T)'$ for a c.n.u. $C_{11}$ contraction $T$ with finite defect indices.

**Lemma 2.** Let $T$ be a c.n.u. $C_{11}$ contraction with $d_T = d_{T^*} = n < \infty$ defined on

$$H = \left[H_n^2 \oplus \overline{\Delta L_n^2}\right] \oplus \{\Theta w \oplus \Delta w: w \in H_n^2\}.$$  

Then $(T)' = \{P_A^T \Xi B_0 \cap C\}: A\Theta_T = \Theta_T A_0$, $B\Theta_T + C\Delta = \Delta A_0$ for some bounded analytic function $A_0$, and $C$ is scalar-valued.

**Proof.** Let $S = P_A^T \Xi B_0 \cap C$ be an operator in $(T)'$, where $A\Theta_T = \Theta_T A_0$ and $b\Theta_T + C\Delta = \Delta A_0$ for some bounded analytic function $A_0$, and let $U$ be the operator of multiplication by $e^u$ on $\overline{\Delta L_n}^2$. It was shown in [6, Lemma 3.1] that $C \in \{U\}'$. As in the proof of Lemma 1, $C = \varphi(U)$ for some $\varphi \in L^{\infty}$, that is, $C$ is scalar-valued.
For the other inclusion, let $S = P[0 \oplus C] \in \{T\}'$ be such that $C$ is scalar-valued, and let $S' = P[0 \oplus C] \in \{T\}'$. Note that the linear manifold $K = \{ P(0 \oplus g): g \in \Delta L_n^2 \}$ is dense in $H$. Indeed, since $\Theta_T$ is an outer function, for any $f \in H_n^2$ there exists a sequence $\{w_j\}$ of elements in $H_n^2$ such that $\Theta_T w_j \rightarrow f$ in norm. Hence

$$P(0 \oplus -\Delta w_j) = P(\Theta_T w_j \oplus 0) \rightarrow P(f \oplus 0).$$

It follows that

$$P(f \oplus g) = P(f \oplus 0) + P(0 \oplus g) \in \bar{K}$$

for any $f \in H_n^2$ and $g \in \Delta L_n^2$. Thus $\bar{K} = H$, as asserted. Let $Y = S[K]$ and $Y' = S'[K]$ be operators (not necessarily bounded) defined on $K$. It is easily seen that $YY' = Y'Y$. By the denseness of $K$, this implies that $SS' = S'S$ whence $S \in \{T\}''$.

As a preliminary step toward showing that $C_{11}$ contractions are reflexive, the next result says that they satisfy $\{T\}' \cap \text{Alg Lat } T = \text{Alg } T$.

**Theorem 3.** Let $T$ be a c.n.u. $C_{11}$ contraction with $d_T = d_{r_T} = n < \infty$ defined on $H = [H^2 \oplus \Delta L_n^2] \oplus \{\Theta_T w \oplus \Delta w: w \in H_n^2\}$.

1. If $\Theta_T(e^{it})$ is isometric for $t$ in a set of positive Lebesgue measure, then $\{T\}' \cap \text{Alg Lat } T = \text{Alg } T = \{T\}''$.
2. If $\Theta_T(e^{it})$ is not isometric for almost all $t$, then $\{T\}' \cap \text{Alg Lat } T = \text{Alg } T = \{u(T): u \in H\}$.

**Proof.** We first show that $\{T\}' \cap \text{Alg Lat } T \subseteq \{T\}''$. Let $S = P[0 \oplus C] \in \{T\}' \cap \text{Alg Lat } T$. Let $U, V$ be the operators of multiplication by $e^{it}$ on $\Delta L_n^2$, $\Delta^2 L_n^2$, respectively, where $\Delta^2 = (1 - \Theta_T \Theta_T^\dagger)^{1/2}$, and let $X: H \rightarrow \Delta^2 L_n^2$ be the quasi-affinity $X(f \oplus g) = -\Delta^2 f + \Theta_T g$ (cf. [6, Lemma 3.4]). Since $\Theta_T \Delta^2 = \Delta^2 \Theta_T$, we may consider $\Theta_T$ as a multiplication operator from $\Delta^2 L_n^2$ to $\Delta^2 L_n^2$. For any $K \in \text{Lat}'' U$, let $H_0 = X^{-1}(\overline{\Theta_T K})$. Since operators in $\{V\}''$ are of the form $\varphi(V)$ where $\varphi \in L^\infty$, it is easily seen that $\overline{\Theta_T K} \in \text{Lat}'' V$. By Corollary 3.6 of [6], $H_0 \in \text{Lat}'' T$. Hence $SH_0 \subseteq H_0$. Thus for any $f \oplus g \in H_0$, $XS(f \oplus g) \in \overline{\Theta_T K}$. As in the proof of Theorem 3.5 in [6], it can be shown that

$$XS(f \oplus g) = \Theta_T C \Theta_T^{-1}(\Delta^2 f + \Theta_T g).$$

Note that $\Theta_T$ admits scalar multiples. Let $\delta$ be an outer scalar multiple of $\Theta_T$ and let $\Omega$ be a contractive analytic function such that $\Theta_T \Omega = \Omega \Theta_T = \delta I_{C^\dagger}$. Since $\Theta_T^{-1} = \delta^{-1} \Omega$, we conclude that

$$\Theta_T C \Omega(-\Delta^2 f + \Theta_T g) \in \delta \overline{\Theta_T K}$$

for any $f \oplus g \in H_0$. By Corollary 3.6 of [6], $XH_0$ is dense in $\overline{\Theta_T K}$. Therefore,

$$\Theta_T C \delta K = \Theta_T C \Omega \Theta_T K \subseteq \delta \overline{\Theta_T K}.$$

Hence for any $x \in K$, there exists a sequence $\{x_n\}$ of elements in $K$ such that
\[ \delta \Theta_T x_n \to \Theta_T C \delta x, \]
which implies that \[ \delta^2 x_n = \delta \Omega \Theta_T x_n \to \Omega \Theta_T C \delta x = \delta^2 C x. \]
Since \( \delta^2 x_n \in K \) for all \( n \), \( \delta^2 C x \in K \). This shows that \( \delta^2 C K \subseteq K \), and hence by Lemma 1 we conclude that \( \delta^2 C \in \{ U \}' \cap \text{Alg Lat}'' U = \{ U \}'' \). Thus \( \delta^2 C \) is scalar-valued, and so is \( C \). By Lemma 2, \( S \in \{ T \}'' \). If \( \Theta_T (e^{it}) \) is isometric for \( t \) in a set of positive Lebesgue measure, then \( \{ T \}'' = \text{Alg T} \) (cf. [6, Theorem 3.8]). This shows that \( \{ T \}' \cap \text{Alg Lat} T \subseteq \text{Alg T} \). Since the other inclusion is trivial, we have proved (1).

For the rest of the proof we assume that \( \Theta_T (e^{it}) \) is not isometric for almost all \( t \). Let

\[ Y: \Delta \Delta L^2_n \to L^2 \oplus L^2(E_2) \oplus \cdots \oplus L^2(E_k) \]

be the unitary transformation which intertwines \( V \) and \( M \oplus M_{E_2} \oplus \cdots \oplus M_{E_k} \), where \( E_2 \supseteq \cdots \supseteq E_k \) are Borel subsets of the unit circle and \( M, M_{E_2}, \ldots, M_{E_k} \) denote the operators of multiplication by \( e^{it} \) on \( L^2, L^2(E_2), \ldots, L^2(E_k) \), respectively (cf. [4, pp. 272–273]). For any \( x \in K \cong Y^{-1}(H^2 \oplus 0 \oplus \cdots \oplus 0) \), consider the element

\[ f \oplus g \equiv P(0 \oplus \Omega x) = (0 \oplus \Omega x) - (\Theta_T w \oplus \Delta w) \]
in \( H \), where \( w \in H_n^2 \). Since \( \delta \in H^\infty \),

\[ - \Delta \delta f + \Theta_T g = - \Delta \delta (- \Theta_T w) + \Theta_T (\Omega x - \Delta w) = \Theta_T \Omega x \]

\[ = \delta x \in \delta K = Y^{-1}(\delta H^2 \oplus 0 \oplus \cdots \oplus 0) \subseteq K. \]

It follows that \( f \oplus g \in X^{-1} K \). Since \( X^{-1} K \) is dense in \( T \), \( S(f \oplus g) \in X^{-1} K \). Hence

\[ XS(f \oplus g) = XP \begin{bmatrix} A & 0 \\ B & C \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} = XP \begin{bmatrix} A & 0 \\ B & C \end{bmatrix} \begin{bmatrix} 0 \\ \Omega x \end{bmatrix} = XP(0 \oplus C \Omega x) \]

\[ = \Theta_T C \Omega x = C \delta x \in K \]

for any \( x \in K \). In particular, for \( h \in H^2 \) consider

\[ x = Y^{-1}(h \oplus 0 \oplus \cdots \oplus 0). \]

Then

\[ C \delta x = Y^{-1}(C \delta h \oplus 0 \oplus \cdots \oplus 0) \in K = Y^{-1}(H^2 \oplus 0 \oplus \cdots \oplus 0), \]

which implies that \( C \delta h \in H^2 \) for any \( h \in H^2 \). Since \( \delta \) is outer, \( \delta H^2 \) is dense in \( H^2 \). From above we conclude that \( CH^2 \subseteq H^2 \) whence \( C \in H^\infty \).

Note that the linear manifold \( \{ P(0 \oplus g) : g \in \Delta L_n^2 \} \) is dense in \( H \) (cf. the proof of Lemma 2). Hence

\[ SP(0 \oplus g) = P(0 \oplus C g) = C(T) P(0 \oplus g) \]

for any \( g \in \Delta L_n^2 \) implies that \( S = C(T) \) on \( H \). Thus \( S = C(T) \in \text{Alg T} \), which proves (2).

**Lemma 4.** Let \( T \) be an operator on \( H \) satisfying \( \{ T \}' \cap \text{Alg Lat} T = \text{Alg T} \). If \( T \) is quasi-similar to a normal operator, then \( T \) is reflexive.
PROOF. It was proved by Apostol [1] that if $T$ is quasi-similar to a normal operator, then there exists a sequence $\{H_n\}$ of invariant subspaces for $T$ which are basic and such that $T_n \equiv T|H_n$ is similar to a normal operator for each $n$. Let $S \in \text{Alg Lat } T$. Then $SH_n \subseteq H_n$. Let $S_n = S|H_n$ for each $n$. Since $T_n$, being similar to a normal operator, is reflexive, we have $S_n \in \text{Alg Lat } T_n = \text{Alg } T_n$. It follows that $ST = TS$ on $H_n$ for all $n$. Since $\{H_n\}$ spans $H$, $ST = TS$ on $H$. Thus $S \in \{T\}' \cap \text{Alg Lat } T = \text{Alg } T$. This shows that $T$ is reflexive.

THEOREM 5. If $T$ is a c.n.u. $C_{11}$ contraction with finite defect indices, then $T$ is reflexive.

PROOF. This follows from Theorem 3 and Lemma 4.

REFERENCES


DEPARTMENT OF APPLIED MATHEMATICS, NATIONAL CHIAO TUNG UNIVERSITY, HSINCHU, TAIWAN, REPUBLIC OF CHINA