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ON THE REFLEXIVITY OF $C_0(N)$ CONTRACTIONS

PEI YUAN WU

Abstract. Let $T$ be a $C_0(N)$ contraction on a separable Hilbert space and let $J = S(q_1) \oplus S(q_2) \oplus \cdots \oplus S(q_k)$ be its Jordan model, where $q_1, q_2, \ldots, q_k$ are inner functions satisfying $q_j|q_{j-1}$ for $j = 2, 3, \ldots, k$, and $S(q_j)$ denotes the compression of the shift on $H^2 \ominus q_j H^2$, $j = 1, 2, \ldots, k$. In this note we show that $T$ is reflexive if and only if $S(q_j/q_{j-1})$ is.

In this note we only consider bounded linear operators defined on complex, separable Hilbert spaces. For each operator $T$, let $(T)'$, $(T)''$ and $\text{Alg } T$ denote the commutant, double commutant and the weakly closed algebra generated by $T$ and $I$, respectively. Let $Lat T$ denote the lattice of invariant subspaces of $T$ and $\text{Alg } Lat T$ denote the (weakly closed) algebra of operators which leave all the subspaces in $Lat T$ invariant. Recall that $T$ is reflexive if and only if $\text{Alg } Lat T = \text{Alg } T$. In [1] Deddens and Fillmore characterized reflexive operators on finite-dimensional spaces in terms of their Jordan canonical forms. Now we generalize their result to $C_0(N)$ contractions. More specifically, we prove the following

Theorem 1. If $T$ is a $C_0(N)$ contraction and $J = S(q_1) \oplus S(q_2) \oplus \cdots \oplus S(q_k)$ is its Jordan model, then $T$ is reflexive if and only if $S(q_j/q_{j-1})$ is.

A contraction $T$ $(\|T\| < 1)$ on a Hilbert space is of class $C_0(N)$ for some integer $N > 1$ if there exists an inner function $q$ such that $q(T) = 0$ and the defect indices $d_T \equiv \text{rank}(I - T^* T)^{1/2}$ and $d_{r_T} \equiv \text{rank}(I - TT^*)^{1/2}$, are both equal to some $M < N$. A $C_0(N)$ contraction is unitarily equivalent to the operator $T$ defined on $H = H^2 \ominus q_1 H^2$ by $T f = P(e^{i\theta} f)$ for $f \in H$, where $H^2\ominus q_1 H^2$ denotes the standard Hardy space of $C^N$-valued functions defined on the unit circle, $\Theta$ is the characteristic function of $T$, and $P$ denotes the (orthogonal) projection from $H^2\ominus q_1 H^2$ onto $H$ (cf. [5, Chapter VII]). Two operators $T_1$, $T_2$ are quasi-similar if there exist one-to-one operators $X$ and $Y$ with dense ranges (called quasi-affinities) such that $XT_1 = T_2 X$ and $YT_2 = T_1 Y$. A $C_0(N)$ contraction is quasi-similar to a uniquely determined Jordan operator (called its Jordan model) $J = S(q_1) \oplus S(q_2) \oplus \cdots \oplus S(q_k)$, where $q_1, q_2, \ldots, q_k$ are inner functions satisfying $q_j|q_{j-1}$, $j = 2, 3, \ldots, k$, and $S(q_j)$ denotes the operator defined on $H^2 \ominus q_j H^2$ by $S(q_j)f = P_j(e^{i\theta} f)$ for $f \in H^2 \ominus q_j H^2$, $P_j$ being the (orthogonal) projection from $H^2 \ominus q_j H^2$, $j = 1, 2, \ldots, k$ (cf. [4]). For $\xi$ and $\eta$ in $H^\infty$, $\xi \wedge \eta = 1$ denotes that $\xi$ and $\eta$ have no nontrivial common inner divisor.

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We start the proof of Theorem 1 by showing that for $C_0(N)$ contractions, the property of reflexivity is preserved under quasi-similarities. This generalizes Corollary 4.5 in [7].

**Theorem 2.** Let $T_1$ and $T_2$ be $C_0(N)$ contractions on $H_1$ and $H_2$, respectively. Assume that $T_1$ is quasi-similar to $T_2$. Then $T_1$ is reflexive if and only if $T_2$ is.

**Proof.** We may assume that $T_1$ and $T_2$ are defined on $H_1 = H_2^\perp \ominus \Theta_1 H_2^\perp$ and $H_2 = H_1^\perp \ominus \Theta_2 H_1^\perp$ by $T_1 f = P_1(e^{it}f)$ and $T_2 g = P_2(e^{it}g)$, respectively, where $f \in H_1$ and $g \in H_2$. Since $T_1$ and $T_2$ are quasi-similar to each other, there exist bounded analytic functions $\Phi$ and $\Psi$ such that $\Phi \Theta_1 = \Theta_2 \Psi$ and $(\det \Phi)(\det \Psi) \wedge (\det \Theta_1)(\det \Theta_2) = 1$ (cf. [3] and [2]). Let $\Phi^d$ denote the algebraic adjoint of $\Phi$. It can be easily verified that the operators $X: H_1 \to H_2$ and $Y: H_2 \to H_1$ defined by $X f = P_2(\Phi f)$ for $f \in H_1$ and $Y g = P_1((\det \Psi)\Phi^d g)$ for $g \in H_2$ implement the quasi-affinities intertwining $T_1$ and $T_2$ (cf. [2, Theorem 2]). Moreover, we have $XY = \eta(T_1)$ and $YX = \eta(T_2)$, where $\eta = (\det \Phi)(\det \Psi)$. Let $m_1$ and $m_2$ denote the minimal functions of $T_1$ and $T_2$, respectively. From the quasi-similarity of $T_1$ and $T_2$ we have $m_1 = m_2$.

Assume that $T_1$ is reflexive. Let $S \in \text{Alg Lat } T_2$ and $K \in \text{Lat } T_1$. Then $YS XK \subseteq \eta(T_1)\bar{K}$. $\eta \wedge (\det \Theta_1) = 1$ implies that $\eta \wedge m_1 = 1$ (cf. [5, Theorem VI.5.2]). In particular, $\eta$ and the minimal function of $T_1|K$ have no nontrivial common inner divisor. Thus $\eta(T_1|K)$ is a quasi-affinity (cf. [7, Theorem 2.3]) and therefore $\eta(T_1)\bar{K} = \eta(T_1|K)\bar{K} = K$. We have $YSXK \subseteq K$ for any $K \in \text{Lat } T_1$, which shows that $YS \in \text{Alg Lat } T_1 = \text{Alg } T_1$. Hence $YSX = \nu(T_1)^{-1}u(T_1)$ for some $u, v \in H^\infty$, where $v \wedge m_1 = 1$ (cf. [7, Theorem 3.2]). So $v(T_1)YSX = u(T_1)$ and we have $\eta(T_2)v(T_2)S\eta(T_2) = X\nu(T_2)SXY = X(\nu(T_1)YSX)Y = Xu(T_1)Y = u(T_2)XY = u(T_2)\eta(T_2)$. Since as above $\eta(T_2)$ is a quasi-affinity, this implies that $\eta(T_2)v(T_2)S = u(T_2)$. Note that $(\nu v) \wedge m_2 = 1$. We obtain $S = (\eta v)(T_2)^{-1}u(T_2) \in \text{Alg } T_2$. This shows that $T_2$ is reflexive, completing the proof.

As a by-product of the preceding proof, we have the following

**Theorem 3.** Let $T_1$ and $T_2$ be $C_0(N)$ contractions on $H_1$ and $H_2$, respectively. If $T_1$ is quasi-similar to $T_2$, then $\text{Lat } T_1 \cong \text{Lat } T_2$.

**Proof.** Let $X: H_1 \to H_2$ and $Y: H_2 \to H_1$ be the intertwining quasi-affinities given in the proof of Theorem 2. For $K_1 \in \text{Lat } T_1$ and $K_2 \in \text{Lat } T_2$ consider the mappings $K_1 \to XK_1$ and $K_2 \to YK_2$. As before we have

$$YXK_1 = \eta(T_1)K_1 = \eta(T_1|K_1)K_1 = K_1.$$ Similarly, $X\bar{Y}K_2 = K_2$. We infer that these mappings implement the lattice isomorphisms between $\text{Lat } T_1$ and $\text{Lat } T_2$ and hence $\text{Lat } T_1 \cong \text{Lat } T_2$.

As a consequence of Theorem 2, to prove Theorem 1 it suffices to consider Jordan operators. The next lemma will be needed in the proof of the necessity part.

**Lemma 4.** Let $T$ be an operator on a Hilbert space $H$. Let $S \in \text{Alg Lat } T \cap \{T\}'$ and $T_1 = T|SH$. Assume that $\text{Alg Lat } T_1 \cap \{T_1\}' = \text{Alg } T_1$. If $T$ is reflexive, so is $T_1$.

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PROOF. Let $S_1 \in \text{Alg Lat } T_1$. Consider $S_1S$ as an operator on $H$. For any $K \in \text{Lat } T$, $SK \subseteq K \cap \overline{SH}$. Since $K \cap \overline{SH} \in \text{Lat } T_1$, we have $S_1SK \subseteq S_1(K \cap \overline{SH}) \subseteq K \cap \overline{SH} \subseteq K$. This shows that $S_1S \in \text{Alg Lat } T = \text{Alg } T$. Hence $S_1TS = S_1ST = TS_1S$. It follows that $S_1T_1 = T_1S_1$ on $\overline{SH}$, that is, $S_1 \in \{T_1\}'$. We conclude that $S_1 \in \text{Alg Lat } T_1 \cap \{T_1\}' = \text{Alg } T_1$ and hence $T_1$ is reflexive.

To prove the sufficiency part, we essentially follow the same line of arguments as given by Deddens and Fillmore [1] for reflexive linear transformations. The next two lemmas are analogous to part of Theorem 2 and its Corollary in [1], respectively.

**Lemma 5.** Let $T = S(\psi_1) \oplus \cdots \oplus S(\psi_k)$ be a Jordan operator defined on $H = (H^2 \oplus \varphi_1 H^2) \oplus \cdots \oplus (H^2 \oplus \varphi_k H^2)$ and let $\psi = \psi_1/\psi_2$. If $S \in \text{Alg Lat } T$, then there exist an outer $\eta \in H^\infty$ and $\delta \in H^\infty$ such that $\eta(T)S = \delta(T) + D$, where $D$ is an operator on $H$ satisfying

$$D \left[ (\xi H^2 \oplus \varphi_1 H^2) \oplus (H^2 \oplus \varphi_2 H^2) \oplus \cdots \oplus (H^2 \oplus \varphi_k H^2) \right]$$

$$\subseteq (\varphi_2 H^2 \oplus \varphi_1 H^2) \oplus 0 \oplus \cdots \oplus 0 \quad \text{for any } \xi|\psi.$$

**Proof.** Let $T_j = S(\psi_j)$, $H_j = H^2 \oplus \varphi_j H^2$ and let $P_j$ denote the (orthogonal) projection from $H^2$ onto $H_j$, $j = 1, 2, \ldots, k$. For brevity of notation, we identify $H_j$ as a subspace of $H$ in the natural way. Let $e = P_1(1) \in H_1$ and $h = Se \in H_1$, since $S$ leaves $H_1$ invariant. Let

$$h(\lambda) = \eta(\lambda)h_e(\lambda)$$

$$= \eta(\lambda)\exp \left[ \frac{1}{2\pi} \int_0^{2\pi} e^{it} + \lambda e^{-it} k(t) \, dt \right] \quad \text{for } |\lambda| < 1,$n

where $\eta$ and $h_e$ are the inner and outer parts of $h$, and $k(t) = \log|h_e(t)|$ a.e. Fix $M > 0$ and let $\alpha = \{t: |h_e(t)| > M\}$. Let

$$\eta(\lambda) = \exp \left[ \frac{1}{2\pi} \int_\alpha e^{it} + \lambda e^{-it} (-k(t)) \, dt \right] \quad \text{for } |\lambda| < 1,$n

and $\delta = \eta h$. Then it is easily seen that $\eta, \delta \in H^\infty$ and $\eta(T)Se = \delta(T)e$. Let $D = \eta(T)S - \delta(T)$. Then $De = 0$.

We first check that $D(\overline{H_2} \oplus \cdots \oplus \overline{H_k}) = \{0\}$. Let $f \in H^\infty$ and consider the element $P_j(f)$ in $H_j$, $j = 2, 3, \ldots, k$. Let $W$ and $U$ be the invariant subspaces for $T$ generated by $P_j(f)$ and $e \oplus P_j(f) \in H_1 \oplus H_j$, respectively. Let $g \in W \cap U \subseteq H_j$. Then there exists a sequence of polynomials $(p_n)$ such that $p_n(T)(e \oplus P_j(f)) \to 0 \oplus g$ as $n \to \infty$. Hence $P_j(p_n) = p_n(T)e \to 0$ and $P_j(p_n) = p_n(T)P_j(f) \to g$, which imply that $P_j(p_n) = P_jP_n(p_n) \to 0$ and $(f(T)P_j(p_n) \to g$. It follows that $g = 0$, whence $W \cap U = \{0\}$. Since $De = 0$, we have $D(P_j(f)) = D(e \oplus P_j(f)) \in W \cap U = \{0\}$. Therefore $D(P_j(f)) = 0$. Note that $\{P_j(f): f \in H^\infty\}$ is dense in $H_j$. We conclude that $D\overline{H_j} = \{0\}$ for $j = 2, 3, \ldots, k$. Hence $D(\overline{H_2} \oplus \cdots \oplus \overline{H_k}) = \{0\}$, as asserted.
Next we show that $D(tH^2 \ominus q_1H^2) \subseteq \varphi_2H^2 \ominus q_1H^2$ for any $\xi|\psi$. Let $W_1 = \xiH^2 \ominus q_1H^2$ and $U_1 = \{P_1(\xi f) + P_2(f); f \in H^2\}$. For $g = \xi f \in W_1$, $Dg = D(P_1(\xi f) + P_2(f)) \subseteq W_1 \cap U_1$. Thus to complete the proof it suffices to show that $W_1 \cap U_1 \subseteq \varphi_2H^2 \ominus q_1H^2$. Let $w \in W_1 \cap U_1$. There exists a sequence $\{f_n\} \subseteq H$ such that $P_1(\xi f_n) + P_2(f_n) \to w \ominus 0$ as $n \to \infty$. Assume that $f_n = g_n + q_2h_n$, where $g_n \in H^2 \ominus q_2H^2$ and $h_n \in H^2$ for each $n$. We infer that $P_1(\xi g_n + \xi q_2h_n) \to w$ and $g_n \to 0$. Thus $w - P_1(\xi q_2h_n) = (w - P_1(\xi g_n + \xi q_2h_n)) + P_1(\xi g_n) \to 0$. It follows that $w \in \varphi_2H^2 \ominus q_1H^2$, completing the proof.

**Lemma 6.** Let $T = S(q_1) \oplus \cdots \oplus S(q_k)$ be a Jordan operator defined on $H = (H^2 \ominus q_1H^2) \oplus \cdots \oplus (H^2 \ominus q_kH^2)$ and let $\psi = q_1/q_2$. Then $T$ is reflexive if and only if $S(\psi)$ is.

**Proof. Necessity.** Note that $T|\varphi_2(T)H$ is unitarily equivalent to $S(\psi)$. (An explicit proof can be found in [6, pp. 315–316].) Since $\varphi_2(T) \in \text{Alg Lat } T \cap \{T\}'$ and $\text{Alg Lat } S(\psi) \cap \{S(\psi)\}' = \text{Alg } S(\psi)$, the reflexivity of $T$ implies that of $S(\psi)$ by Lemma 4.

**Sufficiency.** Let $T$, $H$ and $P$ be as in the proof of Lemma 5 and let $S \in \text{Alg Lat } T$. By Lemma 5, there exist an outer $\eta \in H^\infty$ and $\delta \in H^\infty$ such that $\eta(T)S = \delta(T) + D$, where $D$ satisfies

$$D[\xiH^2 \ominus q_1H^2] \subseteq \varphi_2H^2 \ominus q_1H^2 \ominus 0 \ominus \cdots \ominus 0$$

for any $\xi|\psi$. Let $D_1 = D[H^2 \ominus \psi H^2]$ and $D_2 = D[\psi H^2 \ominus q_1H^2] \ominus H_2 \ominus \cdots \ominus H_k$. Since $D(\psi H^2 \ominus q_1H^2) \subseteq \varphi_2H^2 \ominus q_1H^2 = \{0\}$ and $D(H_2 \ominus \cdots \ominus H_k) = \{0\}$, we have $D_2 = 0$. On the other hand, for any $\xi|\psi$ consider the subspace $\xiH^2 \ominus \psi H^2$ in $\text{Lat } S(\psi)$. Note that $\xiH^2 \ominus \psi H^2 \subseteq \xiH^2 \ominus q_1H^2$. Hence from the property of $D$ we infer that $D_1(\xiH^2 \ominus \psi H^2) \subseteq \varphi_2H^2 \ominus q_1H^2$. Thus the operator $D'$ defined on $H^2 \ominus \psi H^2$ by $D'f = \varphi_2D_1f$ for $f \in H^2 \ominus \psi H^2$ is in $\text{Alg Lat } S(\psi)$. By the reflexivity of $S(\psi)$, there exists $\rho \in H^\infty$ such that $D'f = \rho(S(\psi)f)$ for all $f \in H^2 \ominus \psi H^2$. It follows that $D_1f = \varphi_2(\rho(\eta f)) = P_1(\varphi_2\rho f)$, where $P$ denotes the projection from $H^2$ onto $H^2 \ominus \psi H^2$. For any $h \in H$, $h = f + g$ where $f \in H^2 \ominus \psi H^2$ and $g = g_1 \oplus \cdots \oplus g_k \in (\psi H^2 \ominus q_1H^2) \ominus H_2 \ominus \cdots \ominus H_k$. We deduce that $(\varphi_2\rho)(T)h = (\varphi_2\rho)(T)(f + g_1) = P_1(\varphi_2\rho f + \varphi_2\rho g_1) = P_1(\varphi_2\rho f) = D_1f$. Consequently, $Dh = D_1f + D_2g = (\varphi_2\rho)(T)h$. This shows that $D = (\varphi_2\rho)(T)$ and hence $\eta(T)S = \delta(T) + (\varphi_2\rho)(T)$. Since $\eta$ is outer, we conclude that $S \in \{T\}'' = \text{Alg } T$ (cf. [7, Theorem 3.2]). Thus $T$ is reflexive, completing the proof.

Now Theorem 1 follows from Theorem 2 and Lemma 6. The condition in Theorem 1 was first formulated by C. Foiaş for general $C_0$ contractions in a private communication to the author. He also proved the necessity part. However our presentation here is more simplified.

**References**

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