Asymptotic Behavior of Solutions of a Conservation Law without Convexity Conditions

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Received August 28, 1980

1. INTRODUCTION

In this paper we study the asymptotic behavior for the hyperbolic conservation law

\[ u_t + f(u)_x = 0, \quad t \geq 0, \quad -\infty < x < \infty, \quad (1.1) \]

\[ u(x, 0) = u^0(x), \quad -\infty < x < \infty, \quad (1.2) \]

with Riemann-like data for \(|x|\) large. The function \(f\) is a smooth nonlinear function of \(u\). In general, Eq. (1.1) does not have a continuous solution for all time. Shock curves appear after finite time. We will consider a piecewise continuous weak solution of (1.1) \([9, 10]\). It is well known that across a discontinuity line \(x = x(t)\), the solution satisfies the Rankine-Hugoniot condition (R-H) and the entropy condition (E) \([11]\),

(R-H) \[ x'(t) = \sigma(u_-, u_+), \]

(E) \[ \sigma(u_-, u_+) \leq \sigma(u, u) \quad \text{for all } u \text{ between } u_- \text{ and } u_+, \]

where \(u_- = u(x(t) \pm 0, t)\) and \(\sigma(u_1, u_2)\) is the shock speed defined as

\[ \sigma(u_1, u_2) = \frac{f(u_1) - f(u_2)}{u_1 - u_2}. \]

We will consider the solution of (1.1), (1.2) when the initial data \(u^0(x)\) are Riemann-like data for \(|x|\) large, or more specifically, when \(u^0(x)\) satisfies

\[ u^0(x) = u_l \quad \text{for} \quad x \leq -S, \]

\[ = u_r \quad \text{for} \quad x > S, \quad (1.3) \]
for some constants $u_l, u_r$ and $S, S > 0$. For the case when $f$ is convex (or concave) and $u_l = u_r$, the asymptotic behavior was discussed by Lax [8], Keylitz [7], DiPerna [4] and Dafermos [3]. For the case when $f$ is convex (or concave) and $u_l \neq u_r$, the asymptotic behavior with decay rates was recently obtained by Liu [9]. For more generally smooth $f$, the asymptotic behavior of solutions of (1.1)–(1.3) without decay rates was partially answered by Liu [10]. It is interesting to note that for the case when the initial data are periodic, for general $f$, the asymptotic behavior of solutions of (1.1), (1.2) with decay rates was also only partially answered by Greenberg and Tong [6] and Conlon [2]. In this paper we will investigate the asymptotic behavior of solutions of (1.1), (1.2) with initial conditions of the form (1.3). We assume that $f''(u)$ vanishes at a finite number of points and changes sign at these points. The main result which we obtain is that the solution approaches that of the corresponding Riemann problem at algebraic rates (we need the assumption that $f^{(n)}(u) \neq 0$ for some $n < \infty$ at points $f''(u) = 0$). In Section 2, we will consider the case when $f''(u)$ has only one zero. This case will illustrate the nature of the difficulties involved in the general one and will also be needed for the general case. In Section 3, we will consider the general case.

2. THE CASE WHEN $f''(u)$ HAS ONE ZERO

Without loss of generality we assume that the smooth function $f$ satisfies

$$f''(u) \geq 0 \quad \text{for} \quad u \leq 0. \quad (2.1)$$

We need some definitions and notation. The readers are referred to Ballou [1] for details.

**DEFINITION 2.1.** Let $\eta < 0$ be given and define $\eta^* = \eta^*(\eta)$ by

$$\eta^* = \sup\{u > \eta: \sigma(v, \eta) > \sigma(u, \eta) \forall v \in (\eta, u)\}.$$  

Let $\eta > 0$ be given and define $\eta_\ast = \eta_\ast(\eta)$ by

$$\eta_\ast = \inf\{u < \eta: \sigma(v, \eta) > \sigma(u, \eta) \forall v \in (u, \eta)\}.$$  

Let $\eta < 0$ be given and define $\eta^{**}$ as the unique number that satisfies $\eta = (\eta^{**})^\ast$. Similarly we can define $\eta^{**}$ for $\eta > 0$ as the unique number that satisfies $\eta = (\eta^{**})^\ast$. Note that $\eta^* = +\infty$ and $\eta_\ast = -\infty$ are possible.
DEFINITION 2.2. Under assumption (2.1), the solution of Eq. (1.1) with the following initial condition

\[ u_\alpha(x, 0) = u_\alpha^0(x) = u_l \quad \text{for} \quad x \leq -S, \]
\[ = a \quad \text{for} \quad -S < x \leq S, \]
\[ = u_r \quad \text{for} \quad S < x, \]

where \( a \) is a constant, is called \( u_\alpha(x, t) \).

The following lemmas on properties of special solutions are proved by direct constructions of these solutions.

**Lemma 2.1.** If \( u_l < u_r < u_\alpha^* \) or \( u_l > u_r > u_\alpha^* \), then there exists \( t_\alpha \) and \( x_\alpha \), \( t_\alpha \geq 0 \), such that for all \( t \geq t_\alpha \),

\[ u_\alpha(x, t) = u_l \quad \text{for} \quad x - \sigma t \leq x_\alpha, \]
\[ = u_r \quad \text{for} \quad x - \sigma t > x_\alpha, \]

where \( \sigma = \sigma(u_l, u_r) \) and

\[ x_\alpha = \frac{1}{u_r - u_l} \left[ \frac{u_r + u_l}{2} - a \right] (2S). \]

**Proof.** Since the case \( u_l > u_r > u_\alpha^* \) can be considered similarly, we prove the case \( u_l < u_r < u_\alpha^* \) only. We divide the proof into several cases:

A. \( u_l < u_r \leq 0 \)

(i) \( a \leq u_l \); then the solution \( u_\alpha(x, t) \) is

\[ u_\alpha(x, t) = u_l \quad \text{for} \quad x \leq x_1(t), \]
\[ = h_1 \left( \frac{x + S}{t} \right) \quad \text{for} \quad x_1(t) < x \leq x_2(t), \]
\[ = a \quad \text{for} \quad x_2(t) < x \leq x_3(t), \]
\[ = u_r \quad \text{for} \quad x_3(t) < x, \]
\[ u_\alpha(x, t) = u_l \quad \text{for} \quad x \leq x_1(t), \]
\[ = h_1 \left( \frac{x + S}{t} \right) \quad \text{for} \quad x_1(t) < x \leq x_4(t), \]
\[ = u_r \quad \text{for} \quad x_4(t) < x, \]
\[ u_\alpha(x, t) = u_l \quad \text{for} \quad x \leq x_4(t), \]
\[ = u_r \quad \text{for} \quad x_5(t) < x, \]

\[ 0 < t \leq t_1, \]
\[ t_1 < t \leq t_2, \]
\[ t_2 < t, \]
where \( h_1 \) is the inverse function of \( f' \) restricted in \( u < 0 \),

\[
x_1(t) = -S + f'(u_1)t, \\
x_2(t) = -S + f'(a)t, \\
x_3(t) = S + \sigma(a, u_r)t,
\]

\( x_4(t) \) satisfies

\[
x_4'(t) = \sigma \left( h_1 \left( \frac{x_4(t) + S}{t} \right), u_r \right), \quad x_4(t_1) = x_2(t_1) = x_3(t_1), \quad t_1 < t \geq t_1,
\]

and

\[
x_4(t_2) = x_1(t_2),
\]

\[
x_4(t) = x_\alpha + \sigma(u_1, u_r)t, \quad x_5(t_2) = x_4(t_2);
\]

\( t_1 \) is the time \( x_3(t) \) meets \( x_2(t) \) and \( t_2 \) is the time \( x_4(t) \) meets \( x_1(t) \). It is easy to calculate \( t_1 \); in fact,

\[
t_1 = \frac{2S}{f'(a) - \sigma(a, u_r)}.
\]

Note that \( f'(a) - \sigma(a, u_r) > 0 \) is the consequence of entropy condition (E) and assumption (2.1). To see that \( x_4(t) \) will meet \( x_1(t) \) at finite time \( t_2 \), we recall that

\[
f'(u_r) < x_4'(t) < f' \left( h_1 \left( \frac{x_4(t) + S}{t} \right) \right) = \frac{x_4(t) + S}{t},
\]

which is condition (E). (The strict inequalities are due to assumption (2.1).) We can calculate \( x_4''(t) \),

\[
x_4''(t) = -\frac{h'_1((x_4(t) + S)/t)(x_4'(t) - (x_4(t) + S)/t)^2}{(h_1((x_4(t) + S)/t) - u_r)}
\]

\[
\leq -\frac{\min \{-h'_1((x_4(t) + S)/t)(x_4'(t) - (x_4(t) + S)/t)^2\}}{(u_r - a)} = -\frac{a}{t} < 0. \quad (2.2)
\]

Thus for sufficiently large \( t \), \( x_4'(t) < f'(u_1) \). This would ensure that \( x_4(t) \) meets \( x_1(t) \) at finite time \( t_2 \). To find \( x_\alpha \), we proceed as in Liu [9]. We take \( A \) sufficiently large so that \( u_\alpha(x, t) = u_r \) for \( x = A + \alpha t \). It is easy to see from (1.1) and (R–H) that

\[
\eta(t) = \int_{-\infty}^{A + \alpha t} (u_\alpha(x, t) - u_1) \, dx
\]
is time invariant. For $t \geq t_\alpha = t_z$, $\eta(t) = (u_r - u_i)(A - x_\alpha)$. So

$$\eta(t) = \eta(0) = \int_{-\infty}^{A} (u_\alpha(x, 0) - u_i) \, dx$$

$$= \int_{-S}^{S} (\alpha - u_i) \, dx + \int_{-S}^{A} (u_r - u_i) \, dx.$$  

Thus

$$x_\alpha = \frac{1}{u_r - u_i} \left[ \frac{u_r + u_i}{2} - \alpha \right] (2S).$$

We prove the lemma in this subcase.

(ii) $u_i < \alpha \leq u_r$, then the solution $u_\alpha(x, t)$ is

$$u_\alpha(x, t) = u_i \quad \text{for } x \leq x_1(t),$$

$$= \alpha \quad \text{for } x_1(t) < x \leq x_2(t), \quad 0 < t \leq t_1,$$

$$= u_r \quad \text{for } x_2(t) < x, \quad t_1 < t,$$

where

$$x_1(t) = -S + \sigma(u_i, \alpha)t,$$

$$x_2(t) = S + \sigma(\alpha, u_r)t,$$

$$x_3(t) = x_\alpha + \sigma(u_i, u_r)t, \quad t > t_1,$$

$$x_1(t_1) = x_2(t_1) = x_3(t_1).$$

It is easy to see that $t_\alpha = t_1$ exists and $x_\alpha$ can be obtained as in A(i). This proves the lemma for this subcase.

(iii) $u_r < \alpha \leq 0$, then the solution $u_\alpha(x, t)$ is

$$u_\alpha(x, t) = u_i \quad \text{for } x \leq x_1(t),$$

$$= \alpha \quad \text{for } x_1(t) < x \leq x_2(t), \quad 0 < t \leq t_1,$$

$$= h_1 \left( \frac{x - S}{t} \right) \quad \text{for } x_2(t) < x \leq x_3(t),$$

$$= u_r \quad \text{for } x_3(t) < x,$$
\[ u_a(x, t) = u_l \quad \text{for} \quad x \leq x_4(t), \]
\[ = h_1 \left( \frac{x - S}{t} \right) \quad \text{for} \quad x_4(t) < x \leq x_3(t), \quad t_1 < t \leq t_2, \]
\[ = u_r \quad \text{for} \quad x_3(t) < x, \]
\[ u_a(x, t) = u_l \quad \text{for} \quad x \leq x_5(t), \]
\[ = u_r \quad \text{for} \quad x_5(t) < x, \quad t_2 < t, \]

where

\[ x_1(t) = x_3(t) = S + f'(u_r) t, \]
\[ x_2(t) = S + f'(u_r) t, \]
\[ x_4(t) \text{ satisfies} \]
\[ x_4'(t) = \sigma \left( u_l, h_1 \left( \frac{x_4(t) - S}{t} \right) \right), \]
\[ x_4(t_1) = x_1(t_1) = x_2(t_1), \quad t > t_1, \]

and

\[ x_5(t) = x_3(t) = x_a + \sigma(u_l, u_r) t = x_a + \sigma t, \quad x_5(t_2) = x_4(t_2) = x_3(t_2). \]

Using arguments similar to those in case A(i), we can prove that \( t_2 \) is finite. Thus we can choose \( t_\alpha = t_2 \) in this case. \( x_\alpha \) can be similarly determined.

(iv) \( u_{**}^* \geq u_l^*, \) \( 0 < \alpha \leq u_l^* \), then the solution \( u_a(x, t) \) is

\[ u_a(x, t) = u_l \quad \text{for} \quad x \leq x_1(t), \]
\[ = \alpha \quad \text{for} \quad x_1(t) < x \leq x_2(t), \quad 0 < t \leq t_1, \]
\[ = h_1 \left( \frac{x - S}{t} \right) \quad \text{for} \quad x_2(t) < x \leq x_3(t), \]
\[ = u_r \quad \text{for} \quad x_3(t) < x, \]
\[ u_a(x, t) = u_l \quad \text{for} \quad x \leq x_4(t), \]
\[ = h_1 \left( \frac{x - S}{t} \right) \quad \text{for} \quad x_4(t) < x \leq x_5(t), \quad t_1 < t \leq t_2, \]
\[ = u_r \quad \text{for} \quad x_5(t) < x, \]
\[ u_a(x, t) = u_l \quad \text{for} \quad x \leq x_5(t), \quad t_2 < t, \]
where
\[
\begin{align*}
x_1(t) &= -S + \sigma(u, t), \\
x_2(t) &= S + \sigma(a, a_t^*) t = S + f'(a^*) t, \\
x_3(t) &= S + f'(u^*) t;
\end{align*}
\]
\(x_4(t)\) satisfies
\[
\begin{align*}
x'_4(t) &= \sigma \left( u_t, h_1 \left( \frac{x_4(t) - S}{t} \right) \right), \\
x_4(t_1) &= x_1(t_1) = x_2(t_1), \quad t \geq t_1, \\
x_5(t) &= x_\alpha + \sigma t, \quad x_5(t_2) = x_4(t_2) = x_3(t_2).
\end{align*}
\]
Similar arguments as in A(i,iii) can be used to complete the proof of the lemma in this subcase.

(v) \(u_\alpha^{**} \geq u_\alpha^*, u_\alpha^* < \alpha \leq u_r^{**}\), then the solution \(u_\alpha(x, t)\) is
\[
\begin{align*}
u_\alpha(x, t) &= u_t \quad \text{for } x \leq x_1(t), \\
&= h_2 \left( \frac{x + S}{t} \right) \quad \text{for } x_1(t) < x \leq x_2(t), \\
&= \alpha \quad \text{for } x_2(t) < x \leq x_3(t), \quad 0 < t \leq t_1, \\
&= h_1 \left( \frac{x - S}{t} \right) \quad \text{for } x_3(t) < x \leq x_4(t), \\
&= u_r \quad \text{for } x_4(t) < x, \\
u_\alpha(x, t) &= u_t \quad \text{for } x \leq x_1(t), \\
&= h_2 \left( \frac{x + S}{t} \right) \quad \text{for } x_1(t) < x \leq x_5(t), \\
&= u_r(x, t) \quad \text{for } x_5(t) < x \leq x_5(t), \quad t_1 < t \leq t_2, \\
&= h_1 \left( \frac{x - S}{t} \right) \quad \text{for } x_5(t) < x \leq x_4(t), \\
&= u_r \quad \text{for } x_4(t) < x, \\
u_\alpha(x, t) &= u_t \quad \text{for } x \leq x_6(t), \\
&= u_r(x, t) \quad \text{for } x_6(t) < x \leq x_3(t), \quad t_2 < t \leq t_3, \\
&= h_1 \left( \frac{x - S}{t} \right) \quad \text{for } x_3(t) < x \leq x_4(t), \\
&= u_r \quad \text{for } x_4(t) < x.
\end{align*}
\]
$$u_\alpha(x, t) = u_I \quad \text{for } x \leq x_7(t),$$

$$= h_1 \left(\frac{x - S}{t}\right) \quad \text{for } x_7(t) < x \leq x_4(t), \quad t_3 < t \leq t_4,$$

$$= u_r \quad \text{for } x_4(t) < x,$$

$$u_\alpha(x, t) = u_I \quad \text{for } x \leq x_8(t), \quad t_4 < t,$$

where

$$x_1(t) = -S + \sigma(u_I, u^*_I)t = -S + f'(u^*_I)t,$$

$$x_2(t) = -S + f'(\alpha)t,$$

$$x_3(t) = S + \sigma(\alpha, \alpha^*_I)t = S + f'(\alpha^*_I)t,$$

$$x_4(t) = S + f'(u_r)t,$$

$x_5(t)$ satisfies

$$x'_5(t) = \sigma \left(h_2 \left(\frac{x_5(t) + S}{t}\right), h_2 \left(\frac{x_5(t) + S}{t}\right)\right)$$

with

$$x_5(t_1) = x_2(t_1) = x_3(t_1), \quad t \geq t_1,$$

$x_6(t)$ satisfies

$$x'_6(t) = \sigma(u_I, u_r(x_6(t), t))$$

with

$$x_6(t_2) = x_1(t_2) = x_5(t_2), \quad t \geq t_2,$$

$x_7(t)$ satisfies

$$x'_7(t) = \sigma \left(u_I, h_1 \left(\frac{x_7(t) - S}{t}\right)\right)$$

with

$$x_7(t_3) = x_6(t_3) = x_3(t_3), \quad t \geq t_3,$$

and finally

$$x_8(t) = x_\alpha + \sigma t, \quad x_8(t_4) = x_7(t_4) = x_4(t_4).$$
SOLUTION OF CONSERVATION LAW

The curve $x_4(t)$ has the property that $x_4'(t) < 0$. The function $u_+(x, t)$ can be constructed as follows: First, we draw a line through $(x, t)$ which is tangent to curve $x_4(t)$. This line is unique and contacts with $x_4(t)$ at one point, let it be $(\bar{x}, \bar{t})$. Then we let

$$u_+(x, t) = h_2 \left( \frac{\bar{x} + S}{\bar{t}} \right).$$

The existence of the curve $x_4(t)$ and the solution $u_+(x, t)$ can be found in Ballou [1]. The fact that times $t_1, t_2, t_3$ and $t_4$ exist and are finite can be established similarly. This completes the proof in this subcase.

(vi) $u_{**} \geq u_1^*, u_{**} < \alpha$, then the solution $u_\alpha(x, t)$ is

$$u_\alpha(x, t) = u_1$$

for $x \leq x_1(t)$,

$$= h_2 \left( \frac{x + S}{t} \right)$$

for $x_1(t) < x \leq x_3(t)$,

$$= a$$

for $x_3(t) < x \leq x_4(t)$,

$$= u_r$$

for $x_4(t) < x$,

$$u_\alpha(x, t) = u_1$$

for $x \leq x_1(t)$,

$$= h_2 \left( \frac{x + S}{t} \right)$$

for $x_1(t) < x < x_3(t)$,

$$= u_r$$

for $x_3(t) < x$,

$$u_\alpha(x, t) = u_1$$

for $x \leq x_1(t)$,

$$= h_2 \left( \frac{x + S}{t} \right)$$

for $x_1(t) < x \leq x_4(t)$,

$$= u_r$$

for $x_4(t) < x$,

$$u_\alpha(x, t) = u_1$$

for $x \leq x_1(t)$,

$$= u_r$$

for $x_1(t) < x < x_2(t)$,

$$= u_\alpha(x, t)$$

for $x_2(t) < x$,

$$u_\alpha(x, t) = u_1$$

for $x \leq x_7(t)$,

$$= u_r$$

for $x_7(t) < x \leq x_2(t)$,

$$= u_\alpha(x, t)$$

for $x_2(t) < x$,

$$u_\alpha(x, t) = u_1$$

for $x \leq x_8(t)$,

$$= u_r$$

for $x_8(t) < x$. 

where
\[ x_1(t) = -S + \sigma(u_1, u_1^*) t = -S + f'(u_1^*) t \]
\[ x_2(t) = -S + f''(u_1^*) t, \]
\[ x_3(t) = -S + f'(u_2^*) t, \]
\[ x_4(t) = S + \sigma(u, u_1) t, \]
\[ x_5(t) \text{ satisfies} \]
\[ x_5'(t) = \sigma \left( h_2 \left( \frac{x_5(t) + S}{t} \right), u_2 \right) \]
with
\[ x_5(t_1) = x_3(t_1) = x_4(t_1), \quad t \geq t_1, \]
\[ x_6(t) \text{ satisfies} \]
\[ x_6'(t) = \sigma \left( h_2 \left( \frac{x_6(t) + S}{t} \right), h_2 \left( \frac{x_6(t) + S}{t} \right) \right) \]
with
\[ x_6(t_2) = x_5(t_2) = x_5(t_2), \quad t \geq t_2, \]
\[ x_7(t) \text{ satisfies} \]
\[ x_7'(t) = \sigma(u_1, u_1(x_1(t), t)) \]
with
\[ x_7(t_3) = x_1(t_3) = x_6(t_3), \quad t \geq t_3, \]
and
\[ x_8(t) = x_\alpha + \sigma t, \quad x_8(t_4) = x_7(t_4) = x_7(t_4). \]
The function \( u_\alpha(x, t) \) is connected to the curve \( x_5(t) \) as in A(v). It is straightforward to verify that \( t_4 \) is finite; however, we do not go through this here.

(vii) \( u_1^{**} < u_1^*, 0 < \alpha \leq u_1^{**} \), then the solution \( u_\alpha(x, t) \) is
\[ u_\alpha(x, t) = u_1 \]
for \( x \leq x_1(t) \),
\[ = \alpha \]
for \( x_1(t) < x \leq x_2(t) \),
\[ = h_1 \left( \frac{x - S}{t} \right) \]
for \( x_2(t) < x \leq x_3(t) \),
\[ = u_r \]
for \( x_3(t) < x \),
\[ u_\alpha(x, t) = u_i \quad \text{for} \quad x \leq x_4(t), \]
\[ = h_1 \left( x - \frac{S}{t} \right) \quad \text{for} \quad x_4(t) < x \leq x_3(t), \quad t_1 < t \leq t_2, \]
\[ = u_r \quad \text{for} \quad x_1(t) < x, \]
\[ u_\alpha(x, t) = u_i \quad \text{for} \quad x \leq x_3(t), \]
\[ = u_r \quad \text{for} \quad x_3(t) < x, \quad t_2 < t, \]

where
\[ x_1(t) = -S + \sigma(u_i, \alpha) t, \]
\[ x_2(t) = S + f'(a_*) t, \]
\[ x_3(t) = S + f'(u_r) t, \]

\[ x_4(t) \] satisfies
\[ x'_4(t) = \left( u_i, h_1 \left( \frac{x_4(t) - S}{t} \right) \right) \]

with
\[ x_4(t_1) = x_1(t_1) = x_2(t_1), \quad t \geq t_1, \]

and
\[ x_4(t) = x_\alpha + \sigma t, \]

with \( x_3(t_2) = x_4(t_2) = x_3(t_2) \). This completes the proof for this subcase.

(viii) \( u_\alpha^{**} < u_\alpha^*, \ u_\alpha^{**} < \alpha \leq u_\alpha^* \), then the solution \( u_\alpha(x, t) \) is
\[ u_\alpha(x, t) = u_i \quad \text{for} \quad x \leq x_1(t), \]
\[ = \alpha \quad \text{for} \quad x_1(t) < x \leq x_2(t), \quad 0 \leq t \leq t_1, \]
\[ = u_r \quad \text{for} \quad x_2(t) < x, \]
\[ u_\alpha(x, t) = u_i \quad \text{for} \quad x \leq x_3(t), \]
\[ = u_r \quad \text{for} \quad x_3(t) < x, \quad t_1 < t, \]

where
\[ x_1(t) = -S + \sigma(u_i, \alpha) t, \]
\[ x_2(t) = S + \sigma(u, u_r) t, \]
\[ x_3(t) = x_\alpha + \sigma t \]

with \( x_1(t_1) = x_2(t_1) = x_3(t_1) \). This complete the proof for this subcase.
(ix) \( u_{r}^{*} < u_{l}^{*}, u_{l}^{*} < \alpha \), then the solution \( u(x, t) \) is

\[
\begin{align*}
\text{for } x \leqslant x_1(t), & \\
& = h_2 \left( \frac{x + S}{t} \right) \quad 0 < t \leqslant t_1, \\
& = \alpha \quad \text{for } x_2(t) < x \leqslant x_3(t), \\
& = u_r \quad \text{for } x_3(t) < x, \\
& \text{for } x < x_4(t), \\
& \text{for } x_4(t) < x < x_5(t), \\
& = u_r \\
\end{align*}
\]

where

\[
\begin{align*}
x_1(t) &= -S + \sigma(u_l, u_r^*) t = -S + f'(u_l^*) t, \\
x_2(t) &= -S + f'(\alpha) t, \\
x_3(t) &= S + \sigma(\alpha, u_r) t,
\end{align*}
\]

\( x_4(t) \) satisfies

\[
x_4'(t) = \sigma \left( h_2 \left( \frac{x_4(t) + S}{t} \right), u_r \right)
\]

with

\[
x_4(t_1) = x_2(t_1) = x_3(t_1), \quad t \geq t_1,
\]

and finally

\[
x_5(t) = x_\alpha + \sigma t
\]

with \( x_5(t_2) = x_1(t_2) = x_4(t_2) \). This complete the proof for subcase A.

B. \( u_l < 0 \leqslant u_r < u_l^* \)

(i) \( \alpha < u_l \), this case is similar to A(i).

(ii) \( u_l < \alpha \leqslant u_r \), this case is similar to A(ii,iv,v).

(iii) \( u_r < \alpha \leqslant u_l^* \), this case is also similar to A(ii) (two shocks).

(iv) \( u_l^* < \alpha \), then the solution \( u(x, t) \) is
u_a(x, t) = u_i \quad \text{for} \quad x \leq x_1(t),

= h_2 \left( \frac{x + S}{t} \right) \quad \text{for} \quad x_1(t) < x \leq x_2(t), 0 < t \leq t_1,

= a \quad \text{for} \quad x_2(t) < x \leq x_3(t),

= u_r \quad \text{for} \quad x_3(t) < x,

u_a(x, t) = u_i \quad \text{for} \quad x \leq x_1(t),

= h_2 \left( \frac{x + S}{t} \right) \quad \text{for} \quad x_1(t) < x \leq x_4(t), \quad t_1 < t \leq t_2,

= u_r \quad \text{for} \quad x_4(t) < x,

u_a(x, t) = u_i \quad \text{for} \quad x \leq x_3(t), \quad t_2 < t,

= u_r \quad \text{for} \quad x_5(t) < x,

\text{where}

x_1(t) = -S + \sigma(u_i, u^*_i) t,

x_2(t) = -S + f'(a) t,

x_3(t) = S + \sigma(a, u_r) t,

x_4(t) \text{ satisfies}

x_4'(t) = \sigma \left( h_2 \left( \frac{x_4(t) + S}{t} \right), u_r \right)

\text{with}

x_4(t_1) = x_3(t_1) = x_2(t_1),

\text{and}

x_5(t) = x_a + \sigma t

\text{with } x_4(t_2) = x_5(t_2) = x_1(t_2). \text{ This completes the proof for this subcase B.}

Q.E.D.

**Lemma 2.2.** If $0 < u_i \leq u_r$, then there exists $t_a$, $X_j(a)$ and $X_r(a)$, $t_a \geq 0$, such that

\begin{align*}
&u_a(x, t_a) = u_i \quad \text{for} \quad x \leq X_j(a), \\
&= u_a(x, t_a) \quad \text{for} \quad X_j(a) < x \leq X_r(a), \\
&= u_r \quad \text{for} \quad X_r(a) < x,
\end{align*}
Remark. We have a similar theorem for $u_r \leq u_l < 0$.

Proof. We divide the range of $a$ into several cases:

(i) $0 \leq a$, this case is trivial.

(ii) $u_l \leq a < 0$, then the solution $u_a(x, t)$ is

\[
u_a(x, t) = u_l = a \quad \text{for} \quad x \leq x_1(t),\]

\[= h_2 \left( \frac{x - S}{t} \right) \quad \text{for} \quad x_1(t) < x \leq x_2(t), \quad 0 < t \leq t_1,\]

\[= u_r \quad \text{for} \quad x_2(t) < x < x_4(t),\]

\[u_a(x, t) = u_l \quad \text{for} \quad x \leq x_5(t),\]

\[= h_2 \left( \frac{x - S}{t} \right) \quad \text{for} \quad x_5(t) < x \leq x_4(t), \quad t_1 < t \leq t_2,\]

\[= u_r \quad \text{for} \quad x_4(t) < x,\]

where

\[x_1(t) = S + \sigma(u_l, \alpha)t,\]

\[x_2(t) = S + \sigma(\alpha, \alpha^*)t = S + f'(\alpha^*)t,\]

\[x_3(t) = S + f''((u_l)^*)t,\]

\[x_4(t) = S + f'(u_r)t,\]

$x_5(t)$ satisfies

\[x_5'(t) = \sigma \left( u_l, h_2 \left( \frac{x_5(t) - S}{t} \right) \right)\]

with

\[x_4(t_1) = x_4(t_1) = x_2(t_1), \quad t \geq t_1,\]

and finally $x_5(t_2) = x_3(t_2)$. It is obvious that we can choose $t = t_2$, $X_l(\alpha) = x_5(t_2)$, $X_r(\alpha) = x_4(t_2)$ to complete the proof for this subcase.
(iii) \((u_r)_** \leq \alpha < u_*\), then the solution \(u_\alpha(x, t)\) is
\[
\begin{align*}
\text{for } x & \leq x_1(t), & u_\alpha(x, t) &= u_1 \\
\text{for } x_1(t) < x & \leq x_2(t), & = h_1 \left( \frac{x + S}{t} \right) \\
\text{for } x_2(t) < x & \leq x_3(t), & = \alpha \\
\text{for } x_3(t) < x & \leq x_4(t), & = h_2 \left( \frac{x - S}{t} \right) \\
\text{for } x_4(t) & < x, & = u_r \\
\end{align*}
\]

where
\[
\begin{align*}
x_1(t) &= -S + \sigma(u, u_r) t = -S + f''(u_r) t, \\
x_2(t) &= -S + f'(\alpha) t, \\
x_3(t) &= S + \sigma(u, \alpha^*) t = S + f'(\alpha^*) t, \\
x_4(t) &= S + f'(u_r) t, \\
\end{align*}
\]

\(x_3(t)\) satisfies
\[
x'_3(t) = \sigma \left( h_1 \left( \frac{x_3(t) + S}{t} \right), h_1 \left( \frac{x_3(t) + S}{t} \right) \right)_{**}
\]

with \(x_3(t_1) = x_2(t_1) = x_3(t_1),\) \(t \geq t_1,\) and \(t_2\) is the time when \(x_3(t_2) = x_1(t_2).\) The curve \(x_3(t)\) and function \(u_\alpha(x, t)\) is similar to the case \(A(v).\) Choose \(t_\alpha = t_2, X_1(\alpha) = x_3(t_2),\) and \(X_\alpha(\alpha) = x_4(t_2)\) to complete the proof.

(iv) \(\alpha < (u_r)_**,\) then the solution \(u_\alpha(x, t)\) is
\[
\begin{align*}
\text{for } x & \leq x_1(t), & u_\alpha(x, t) &= u_1 \\
\text{for } x_1(t) < x & \leq x_3(t), & = h_2 \left( \frac{x + S}{t} \right) \\
\text{for } x_3(t) < x & \leq x_4(t), & = \alpha \\
\text{for } x_4(t) & < x, & = u_r \\
\end{align*}
\]
where

\[ x_1(t) = -S + \sigma(u_1, u_*) t, \]
\[ x_2(t) = -S + f'(u_*) t = -S + \sigma(u_*) u_*, \]
\[ x_3(t) = -S + f'(u) t, \]
\[ x_4(t) = S + \sigma(a, u_*) t, \]

\( x_5(t) \) satisfies

\[ x_5'(t) = \sigma \left( h_2 \left( \frac{x_5(t) + S}{t} \right), u_r \right) \]

with

\[ x_5(t_1) = x_3(t_1) = x_4(t_1), \quad t \geq t_1, \]

\( x_6(t) \) satisfies

\[ x_6'(t) = \sigma \left( h_2 \left( \frac{x_6(t) + S}{t} \right), h_2 \left( \frac{x_6(t) + S}{t} \right)^* \right) \]

with

\[ x_6(t_2) = x_3(t_2) = x_4(t_2), \quad t \geq t_2, \]

and finally \( x_6(t_3) = x_1(t_3) \). Choose \( t_* = t_3, \) \( X_1(\alpha) = x_1(t_3), \) and \( X_r(\alpha) = x_2(t_3) \) to complete the proof. Q.E.D

**Lemma 2.3.** If \( u_i < u_i^* \leq u_r \), let

\[ x_i(t) = \sup \{ x : u_a(x', t) = u_i \ \forall x' < x \}, \]
then there exists \( x_\alpha, t_\alpha \geq 0 \) such that

(i) \( x'_i(t) \geq \sigma(u_i, u_i^*) \) for all \( t > 0 \) except at finite points of \( t \).

(ii) \( x_i(t) \leq x_\alpha + \sigma(u_i, u_i^*)t \).

(iii) \( \lim_{t \to \infty} [x_\alpha + \sigma(u_i, u_i^*)t - x_i(t)] = 0 \).

(iv) \( u_i(x, t) \geq u_i^* \) for \( x > x_\alpha + \sigma(u_i, u_i^*)t, t \geq t_\alpha \).

**Remark.** We have a similar lemma for the case \( u_i > u_i^* \geq u_r \).

**Proof.** (i) \( \alpha \leq (u_r)^* \), then the solution \( u_\alpha(x, t) \) is

\[
\begin{align*}
  u_\alpha(x, t) &= u_l \quad \text{for } x \leq x_1(t), \\
  &= h_1 \left( \frac{x + S}{t} \right) \quad \text{for } x_1(t) < x \leq x_3(t), \quad 0 < t \leq t_1, \\
  &= \alpha \quad \text{for } x_3(t) < x \leq x_4(t), \\
  &= u_r \quad \text{for } x_4(t) < x, \\
  u_\alpha(x, t) &= u_l \quad \text{for } x \leq x_1(t), \\
  &= h_1 \left( \frac{x + S}{t} \right) \quad \text{for } x_1(t) < x \leq x_5(t), \quad t_1 < t \leq t_2, \\
  &= \alpha \quad \text{for } x_5(t) < x < x_6(t), \\
  &= u_r \quad \text{for } x_6(t) < x, \\
  u_\alpha(x, t) &= u_l \quad \text{for } x \leq x_1(t), \\
  &= h_1 \left( \frac{x + S}{t} \right) \quad \text{for } x_1(t) < x \leq x_6(t), \quad t_2 < t \leq t_3, \\
  &= u_r(x, t) \quad \text{for } x_6(t) < x \leq x_9(t), \\
  &= u_r \quad \text{for } x_9(t) < x, \\
  u_\alpha(x, t) &= u_l \quad \text{for } x \leq x_7(t), \\
  &= u_r(x, t) \quad \text{for } x_7(t) < x \leq x_9(t), \quad t_3 < t, \\
  &= u_r \quad \text{for } x_9(t) < x,
\end{align*}
\]

where

\[
\begin{align*}
  x_1(t) &= -S + f''(u_l)t, \\
  x_2(t) &= -S + f''(u_r^*)t, \\
  x_3(t) &= -S + f''(\alpha)t, \\
  x_4(t) &= S + \sigma(\alpha, u_r)t,
\end{align*}
\]
$x_s(t)$ satisfies

$$x'_s(t) = \sigma \left( h_1 \left( \frac{x_s(t) + S}{t} \right) , u_r \right)$$

with

$$x_s(t_1) = x_3(t_1) = x_4(t_1), \quad t \geq t_1,$$

and

$$x_6(t)$$ satisfies

$$x'_6(t) = \sigma \left( h_1 \left( \frac{x_6(t) + S}{t} \right) , h_1 \left( \frac{x_5(t) + S}{t} \right) \right)$$

with

$$x_6(t_2) = x_2(t_2) = x_5(t_2), \quad t \geq t_2,$$

and

$$x_7(t) = x_1(t_3) + \sigma(u_l, u^*_3)(t - t_3) = x_6(t_3) + f''(u^*_3)(t - t_3), \quad t \geq t_3,$$

$$x_8(t) = x_2(t_2) + \sigma(u_r^*, u_r)(t - t_2)$$

$$= x_2(t_2) + f''(u_r)(t - t_2).$$

Obviously,

$$x_9(t) = x_1(t), \quad 0 < t \leq t_3,$$

$$= x_7(t), \quad x_3 < t;$$

choose $x_\alpha = x_1(t_3) - f''(u^*)t_3$, $t_\alpha = t_3$ to complete the proof for this subcase.

(ii) $u_r^* < \alpha \leq u_l$, then the solution $u_\alpha(x, t)$ is

$$u_\alpha(x, t) = u_l$$

for $x \leq x_1(t),$

$$= h_1 \left( \frac{x + S}{t} \right)$$

for $x_1(t) < x \leq x_2(t),$

$$= \alpha$$

for $x_2(t) < x \leq x_3(t), \quad 0 < t \leq t_1,$

$$= h_2 \left( \frac{x - S}{t} \right)$$

for $x_3(t) < x \leq x_4(t),$

$$= u_r$$

for $x_4(t) < x,$
\[ u_o(x, t) = u_l \quad \text{for} \quad x \leq x_1(t), \]
\[ = h_1 \left( \frac{x + S}{t} \right) \quad \text{for} \quad x_1(t) < x \leq x_3(t), \]
\[ = u_k(x, t) \quad \text{for} \quad x_3(t) < x \leq x_4(t), \]
\[ = h_2 \left( \frac{x - S}{t} \right) \quad \text{for} \quad x_4(t) < x, \]
\[ u_o(x, t) = u_l \quad \text{for} \quad x \leq x_6(t), \]
\[ = u_k(x, t) \quad \text{for} \quad x_6(t) < x \leq x_3(t), \]
\[ = h_2 \left( \frac{x - S}{t} \right) \quad \text{for} \quad x_3(t) < x \leq x_4(t), \]
\[ = u_r \quad \text{for} \quad x_4(t) < x, \]

where

\[ x_1(t) = -S + f'(u_l) t, \]
\[ x_2(t) = -S + f'(a) t, \]
\[ x_3(t) = S + f'(\alpha^*) t, \]
\[ x_4(t) = S + f'(u_r) t, \]

\( x_5(t) \) satisfies

\[ x_5'(t) = \sigma \left( h_1 \left( \frac{x_2(t) + S}{t} \right), h_1 \left( \frac{x_5(t) + S}{t} \right) \right) \]

with

\[ x_5(t_1) = x_2(t_1) = x_3(t_1), \quad t \geq t_1 \]

and

\[ x_6(t) = x_1(t_2) + f'(u_{l*})(t - t_2) = x_3(t_2) + f'(u_{l*})(t - t_2). \]

It is obvious that we have

\[ x_6(t) = x_1(t), \quad 0 \leq t \leq t_2, \]
\[ x_6(t) = x_6(t), \quad t_2 < t. \]

Choose \( x_\alpha = x_1(t_2) - f'(u_{l*}) t_2, \ t_\alpha = t_2 \) to complete the proof for this subcase.
(iii) \( u_t < \alpha < 0 \), then the solution \( u_\alpha(x, t) \) is

\[
\begin{align*}
    u_\alpha(x, t) &= u_t & \text{for } x \leq x_1(t), \\
    &= \alpha & \text{for } x_1(t) < x \leq x_2(t), \\
    &= h_2 \left( \frac{x - S}{t} \right) & \text{for } x_2(t) < x \leq x_4(t), \\
    &= u_r & \text{for } x_4(t) < x, \\
\end{align*}
\]

for \( x \leq x_5(t) \),

where

\[
\begin{align*}
    x_1(t) &= -S + \sigma(u_t, \alpha)t, \\
    x_2(t) &= S + \sigma(\alpha, \alpha^*) = S + f''(\alpha^*)t, \\
    x_3(t) &= S + f''(u_\alpha^*)t, \\
    x_4(t) &= S + f''(u_r)t, \\
    x_5(t) &= x_3(t), \\
\end{align*}
\]

\( x_5(t) \) satisfies

\[
x_5'(t) = \sigma \left( u_t, h_2 \left( \frac{x_5(t) - S}{t} \right) \right)
\]

with

\[
x_5(t_1) = x_1(t_1) = x_2(t_1), \quad t \geq t_1.
\]

It is obvious that

\[
\begin{align*}
    x_5(t) &= x_1(t) & \text{for } 0 \leq t \leq t_1, \\
    &= x_5(t) & \text{for } t_1 < t.
\end{align*}
\]

Choose \( x_\alpha = S, \ t_\alpha = 0 \) to complete the proof.

(iv) \( 0 \leq \alpha \leq u_t^* \), this case is similar to the above case.

(v) \( u_t^* < \alpha \), this case is trivial.

Thus we complete the proof for the lemma. \( \text{Q.E.D.} \)
Before we present our main theorems, we state an ordering principle due originally to Douglis [5]. See also Wu [13], Ballou [1] and Keyfitz [7].

**ORDERING PRINCIPLE.** Let the function $f$ be a smooth function, and let $u(x, t)$ and $v(x, t)$ be piecewise smooth weak solutions satisfying condition (E) to the Cauchy problem (1.1), (1.2), where the initial data are $u(x, 0)$ and $v(x, 0)$, respectively. Then $u(x, 0) \leq v(x, 0)$ $\forall x \in (-\infty, \infty)$ implies $u(x, t) \leq v(x, t)$ $\forall t \geq 0$, $\forall x \in (-\infty, \infty)$.

**DEFINITION 2.3.** For solutions of (1.1), (1.2), (1.3), $u(x, t)$, let

$$x_l(t) = \sup \{x : u(x', t) = u_l, \forall x' \leq x\},$$

$$x_r(t) = \inf \{x : u(x', t) = u_r, \forall x' \geq x\}.$$ 

Now we state our main theorems.

**THEOREM 2.4.** If $u_l < u_r < u_l^*$, then there exists $t_0$ and $x_0$, $t_0 \geq 0$, such that for all $t \geq t_0$,

$$u(x, t) = u_l \quad \text{for} \quad x \leq x_0 + \sigma t, \quad x_l(t) = u_l$$

$$= u_r \quad \text{for} \quad x > x_0 + \sigma t, \quad x_r(t) = u_r,$$

where $\sigma = \sigma(u_l, u_r)$ and

$$x_0 = \frac{1}{u_r - u_l} \int_{-N}^{N} \left( \frac{u_r + u_l}{2} - u^0(x) \right) dx, \quad N \geq S.$$ 

**Remark.** We have a similar theorem for $u_l > u_r > u_l^*$. 

**Proof.** Let

$$M = \sup \{u^0(x) : -S \leq x \leq S\}, \quad (2.3)$$

$$m = \inf \{u^0(x) : -S \leq x \leq S\}. \quad (2.4)$$

From Lemma 2.1, let the solutions corresponding to $\alpha = M$ and $\alpha = m$ be respectively $u_M(x, t)$ and $u_m(x, t)$ with corresponding $x_M, x_m, t_M, t_m$. It is easy to see that $x_M \leq x_m$. If $x_M = x_m$, which is the case $M = m$, then we are done. So assume $x_M < x_m$. Using the Ordering Principle, we have

$$u_m(x, t) \leq u(x, t) \leq u_M(x, t) \quad \forall x, \forall t \geq 0.$$ 

Thus for all $t \geq \tilde{t} = \max \{t_M, t_m\}$, we have

$$u(x, t) = u_l \quad \text{for} \quad x \leq x_m + \sigma t, \quad x_l(t) = u_l$$

$$= u_r \quad \text{for} \quad x > x_m + \sigma t, \quad x_r(t) = u_r.$$
and
\[ u_t = u_m(x, t) \leq u(x, t) \leq u_m(x, t) = u_r \quad \text{for } x_M + \sigma t < x \leq x_m + \sigma t. \]

From Definition 2.3, for \( t \geq \bar{t} \), it is easy to see that
\[ x_M + \sigma t \leq x_i(t) \leq x_r(t) \leq x_m + \sigma t. \]

Furthermore \( x_i(t) \) and \( x_r(t) \) are Lipschitz continuous curves with slopes
\[ x_i'(t) = \sigma(u_i, u(x_i(t) + 0, t)), \]
and
\[ x_r'(t) = \sigma(u_r, u(x_r(t) - 0, t)), \]
respectively, and with bounded second derivatives. Since \( u(x_i(t) + 0, t) \) and \( u(x_r(t) - 0, t) \) are both between \( u_i \) and \( u_r \) for \( t \geq \bar{t} \), we have from (E)
\[ x_i'(t) \geq \sigma(u_i, u_r) = \sigma \geq x_r'(t). \]

Thus \( x_i'(t) - \sigma \geq 0 \) and \( x_r'(t) - \sigma \leq 0 \) or \( (x_i(t) - \sigma t) \) is nondecreasing and \( x_r(t) - \sigma t \) is nonincreasing for \( t \geq \bar{t} \). But \( x_i(t) - \sigma t \leq x_r(t) - \sigma t \) for \( t \geq \bar{t} \). Thus if there exists \( t_0 \geq \bar{t} \) such that \( x_i(t_0) = x_r(t_0) \), then we have \( x_i(t) = x_r(t) \) and \( x_i'(t) = \sigma \geq x_r'(t) \) for all \( t \geq t_0 \). If this is the case, then we are done. Now suppose the opposite, that is, \( x_i(t) < x_r(t) \) for all \( t \geq \bar{t} \); then \( x_i(t) - \sigma t \) is nondecreasing and bounded and \( x_r(t) - \sigma t \) is nonincreasing and bounded for all \( t \geq \bar{t} \). Hence
\[
\begin{align*}
\lim_{t \to \infty} (x_i(t) - \sigma t) &= X_i, & \lim_{t \to \infty} x_i'(t) &= \sigma, \\
\lim_{t \to \infty} (x_r(t) - \sigma t) &= X_r, & \lim_{t \to \infty} x_r'(t) &= \sigma,
\end{align*}
\]
with \( X_i \leq X_r \). From the entropy condition (E),
\[ f'(u_i) > \sigma(u_i, u_r) > f'(u_r), \]
we can choose sufficiently small \( \delta \) such that
\[ f'(u) > \sigma > f'(v) \quad \text{for all } u \in (u_i, u_i + \delta), \ v \in (u_r - \delta, u_r). \quad (2.7) \]

From (2.5) and (2.6), we can choose sufficiently large \( t_\delta \), such that \( u(x_i(t) + 0, t) \in (u_r - \delta, u_r) \) and \( u(x_r(t) - 0, t) \in (u_i, u_i + \delta) \) for all \( t \geq t_\delta \). Now for \( t \geq t_\delta \), through \( (x_i(t) + 0, t) \) and \( (x_r(t) - 0, t) \) we draw characteristics backward in time. They would intersect along a discontinuity line whose slope is approximately \( \sigma \) due to (2.7) and (R–H). (Note that they
cannot terminate to a contact discontinuity before they meet.) But it is obvious that this discontinuity line violates (E). Q.E.D.

**Theorem 2.5.** If $0 < u_i \leq u_r$, let

$$p_i(t) = \min_x \int_{-\infty}^{x} (u(y, t) - u_i) \, dy,$$

$$q_i(t) = \max_x \int_{x}^{\infty} (u(y, t) - u_r) \, dy,$$

then

(i) $p_i'(t) \geq 0$, $q_i'(t) \leq 0$,

(ii) there exists $t_0$, $t_0 \geq 0$, such that for all $t \geq t_0$, $p_i(t) - p_i(t_0) - p_i(\infty)$, $q_i(t) = q_i(t_0) = q_i(\infty)$.

**Proof:** (i) follows from Liu [9, Theorem 1(i)]. Let $M$ and $m$ be as defined in (2.3), (2.4). From Lemma 2.2 and the Ordering Principle, we know that

$$u_M(x, t) \geq u(x, t) \geq u_m(x, t), \quad t \geq 0, \quad -\infty < x < \infty,$$

and for $t \geq t_0 = \max\{t_M, t_m\}$,

$$u_M(x, t) \geq (u_*)^*, \quad u_m(x, t) \geq (u_*)^*.$$

Hence $u(x, t) \geq (u_*)^*$ for all $t \geq t_0$ and

$$u(x, t_0) = u_i \quad \text{for} \quad x \leq X_i(M),$$

$$= u_r \quad \text{for} \quad x > X_r(m).$$

Thus for $t \geq t_0$, $u(x, t)$ are restricted in the region $f''(u(x, t)) > 0$. Hence the theorem follows from Liu [9, Theorem 1(ii)]. Q.E.D.

**Remark.** We have a similar theorem for the case $u_r \leq u_i < 0$.

**Theorem 2.6.** If $0 < u_i \leq u_r$, let $p_i(t)$ and $q_i(t)$ be as defined in Theorem 2.5. Define the generalized N-waves as

$$N_2(x, t) = u_i \quad \text{for} \quad x - f'(u_i)t \leq -\sqrt{-2p_i(\infty)f''(u_i)t},$$

$$= u_r \quad \text{for} \quad x - f'(u_r)t \geq \sqrt{2q_i(\infty)f''(u_r)t}, \quad t > 0,$$

$$= h_2 \left( \frac{x}{t} \right) \quad \text{otherwise};$$
then we have

(i) the edges of $N_2$ and $u$ have finite distance for all time, i.e.,

$$|x_i(t) - f'(u_i)t + \sqrt{-2p_1(\infty)f''(u_i)t}| + |x_r(t) - f'(u_r)t - \sqrt{-2q_1(\infty)f''(u_r)t}| = O(S),$$

(ii) $|u(x, t) - N_2(x, t)| \leq A_1^{-1}O(S)^{-1}$ for any $x$ that lies between $\max(x_i(t), f'(u_i)t - \sqrt{2p_1(\infty)f''(u_i)t})$ and $\min(x_i(t), f'(u_i)t + \sqrt{2q_1(\infty)f''(u_i)t}),$

(iii) $|u(x, t) - N_2(x, t)| = O(S)^{-1/2}$ for any $x$ between $x_i(t)$ and $(f'(u_i)t - \sqrt{-2p_1(\infty)f''(u_i)t})$ or between $x_r(t)$ and $(f'(u_r)t + \sqrt{2q_1(\infty)f''(u_r)t}),$

(iv) $u(x, t) = N_2(x, t)$ if $x$ lies outside the regions of (ii) and (iii),

where $x_i(t)$ and $x_r(t)$ are defined in Definition 2.3, $A_1 = \min_{B > 0}(u_i, B)$ is a bound for $u''(x)$.

Remark. We have a similar theorem for the case $u_i \leq u_f < 0$.

Proof. This theorem is an easy consequence of Lemma 2.2, Theorem 2.5, and Theorem 4 of Liu [9]. We omit the proof.

**Theorem 2.7.** If $u_i < u_f^* \leq u_r$, then there exist $x_0, t_0 > 0$, such that

(i) $x_i(t) \geq \sigma(u_i, u_f^*)t$,

(ii) $x_r(t) \leq x_0 + \sigma(u_i, u_f^*)t$,

(iii) $\lim_{t \to \infty} [x_0 + \sigma(u_i, u_f^*)t - x_i(t)] = 0$,

(iv) $u(x, t) \geq ((u_f^*)^*)^*$ for $x > x_i(t), t \geq t_0$.

Remark. We have a similar theorem for the case $u_f > u_f^* > u_r$.

Proof. (i) is obvious. From (i), $(x_i(t) - \sigma(u_i, u_f^*)t)$ is nondecreasing. But $x_i(t) \leq x_m + \sigma(u_i, u_f^*)t$, where $x_m$ is the $x_\alpha$ when $\alpha = m$ in Lemma 2.3 and $m$ is the number defined in (2.4). Thus $x_i(t) - \sigma(u_i, u_f^*)t$ is nondecreasing and bounded. Hence, $\lim_{t \to \infty}(x_i(t) - \sigma(u_i, u_f^*)t)$ exists; let it be $x_0$. We already proved (ii) and (iii). Now it is easy to see that we can find a time $t_1$ sufficiently large, such that $u(x, t_1) \geq ((u_f^*)^*)^*$ for $x_1(t_1) < x < x_0 + \sigma(u_i, u_f^*)t_1$ and $u(x_0 + \sigma(u_i, u_f^*)t_1 + 0, t_1) = u_f^*$. Furthermore, the line segment $x_0 + \sigma(u_i, u_f^*)t_1, t \geq t_1$, is the characteristic line passing through the point $(x_0 + \sigma(u_i, u_f^*)t_1 + 0, t_1)$ and $t \geq t_m$. Thus we have $u(x_0 + \sigma(u_i, u_f^*)t_1 + 0, t) = u_f^*$ and $u(x, t) \geq u_f^*$ for all $t \geq t_1, x > x_m + \sigma(u_i, u_f^*)t$. Hence we can choose $t_1$ sufficiently large, such that $u(x, t) \geq u_f^*$ for all $t \geq t_1, x > x_0 + \sigma(u_i, u_f^*)t$. Choose $t_0 = t_1$ to complete the proof for (iv).

Q.E.D.
THEOREM 2.8. Under the assumptions of Theorem 2.7, let

\[ q(t) = \max_x \int_x^\infty (u(y, t) - u_r) \, dy; \]

then

(i) \( q'(t) \leq 0, \)

(ii) there exists \( t_0, \) such that \( q(t) = q(t_0) \) for all \( t \geq t_0. \)

Proof: (i) follows from Liu [9, Theorem 1(i)]. To prove (ii), take the \( t_0 \) of Theorem 2.7 as the \( t_0 \) we want. Assume that the maximum point in the definition of \( q(t) \) is taken place at \( x^*(t). \) We want to prove (1) \( u(x^*(t), t) = u_r \) and \( u(x, t) \) is continuous at \( x^*(t), \) and (2) \( q'(t) = 0 \) for all \( t \geq t_0. \) If \( x^*(t) \) is a discontinuity, then since \( x^*(t) > x_r(t), \) we must have \( u(x^*(t) - 0, t) > u(x^*(t) + 0, t). \) But in this case, \( x^*(t) \) is not the maximum point. Hence \( u(x, t) \) must be continuous at \( x^*(t). \) Now if \( u(x^*(t), t) \neq u_r, \) then \( x^*(t) \) cannot be the maximum point too. This proves (1). To prove (2), we know that from (1), \( dx^*(t)/dt \) exists and is equal to \( f'(u_r). \) From the definition of \( q(t), \) we have

\[ q(t) = \int_{x^*(t)}^{x^*(t) - 0} (u(y, t) - u_r) \, dy. \]

Hence

\[
q'(t) = \left[ u(x_r(t) - 0, t) - u_r \right] x_r'(t) + \int_{x^*(t)}^{x_r(t) - 0} u_r \, dy \\
- \left[ u(x_r(t) - 0, t) - u_r \right] x_r'(t) - f(u(x_r(t) - 0)) - f(u_r) \\
= 0 \quad \text{(R–H).}
\]

This proves (ii). Q.E.D.

THEOREM 2.9. Under the assumptions of Theorems 2.7 and 2.8, let \( x_0 \) and \( q(\infty) = q(t_0) \) be the respective constants in Theorems 2.7 and 2.8. Define the following one-sided generalized N-wave

\[
N(x, t) = u_l \quad \text{for} \quad x \leq x_0 + \sigma(u_l, u_r^*) t, \\
= u_r \quad \text{for} \quad x > f'(u_r) t + \sqrt{2q(\infty)f''(u_r)} t + x_0, \\
= h_2 \left( \frac{x - x_0}{t} \right) \quad \text{otherwise.}
\]
Then we have

(i) there exists $t_0 > 0$, such that $x_i(t) = x_0 + \sigma(u_i, u_i^*) t$ for all $t \geq t_0$,

(ii) $|u(x, t) - N(x, t)| \leq A^{-1} O(S) t^{-1}$ for $x$ between $x_0 + \sigma(u_i, u_i^*) t$ and $\min(x_i(t), \sigma(u_i, \sigma^*)) t + \sigma(u_i, f''(u_i) t + x_0)$,

(iii) $|x_i(t) - f''(u_i) t - \sqrt{2q(\infty)} f''(u_i) t| = O(S)$,

(iv) $|u(x, t) - N(x, t)| \leq O(S) t^{-1/2}$ for $x$ between $x_i(t)$ and $f''(u_i) t + \sqrt{2q(\infty)} f''(u_i) t + x_0$, where $A = \min_{i \leq k \leq \infty} \sigma(u_i, \sigma^*)$.

Remark. We have a similar theorem for the case $u_i \leq u_i^* < u_i$.

Proof. Let $-X(t) = x_i(t) - x_0 - f''(u_i^*) t$, then

$$-X'(t) = x_i'(t) - f''(u_i^*) = \sigma(u_i, u(x_i(t) + 0, t)) - \sigma(u_i, u_i^*)$$

From Theorem 2.7, $X(t) \to 0$, $X'(t) \to 0$ as $t \to \infty$. Hence we can expand $\sigma(u_i, u(x_i(t) + 0, t))$ to obtain

$$\sigma(u_i, u(x_i(t) + 0, t)) \equiv \sigma(u_i, u_i^*) - \frac{f''(u_i^*)}{2(u_i - u_i^*)} (u - u_i^*)^2.$$ 

But

$$u_i^* = h_2(f''(u_i^*)) = h_2 \left( \frac{x_0 + f''(u_i^*) t - x_0}{t} \right),$$

hence for $t$ large

$$u(x_i(t) + 0, t) \approx h_2 \left( \frac{x_i(t) - x_0}{t} \right)$$

$$\approx h_2 \left( \frac{-X(t) + f''(u_i^*) t}{t} \right) = h_2 \left( \frac{f''(u_i^*)}{t} - \frac{X(t)}{t} \right)$$

$$= u_i^* - h_2(f''(u_i^*)) \frac{X(t)}{t}.$$

Hence

$$X'(t) \equiv -A \frac{X^2(t)}{t^2} \quad \text{with} \quad A = \frac{f''(u_i^*)}{2(u_i^* - u_i)} \left( h_2(f''(u_i^*)) \right)^2,$$

or

$$X(t) \equiv \frac{t}{ct - A} \quad \text{and} \quad X(t) \to \frac{1}{c} \quad \text{as} \quad t \to \infty.$$
But as we know $X(t) \to 0$ as $t \to \infty$. Hence $c = \infty$, which means that for some finite $t_0$, $X(t_0) = 0$. This proves (i). Parts (ii), (iii) and (iv) are an easy consequence of Theorem 2.7, Theorem 2.8 and Liu [9, Theorem 4].

**Lemma 2.10.** If $u_i = u_e = 0$ and $f(u) \equiv f'(0)u + Au^{2n+1}$, $n \geq 1$, $A > 0$, for $|u|$ small, then we have

$$|u_\alpha(x, t)| = O(S) t^{-\frac{1}{(2n+1)}}.$$

**Proof.** If $\alpha < 0$, then the solution $u_\alpha(x, t)$ is

$$u_\alpha(x, t) = 0 \quad \text{for} \quad x \leq x_1(t),$$

$$= h_1 \left( \frac{x + S}{t} \right) \quad \text{for} \quad x_1(t) < x \leq x_2(t), \quad 0 < t \leq t_1,$$

$$= \alpha \quad \text{for} \quad x_2(t) < x \leq x_3(t),$$

$$= 0 \quad \text{for} \quad x_3(t) < x,$$

$$u_\alpha(x, t) = 0 \quad \text{for} \quad x \leq x_1(t),$$

$$= h_1 \left( \frac{x + S}{t} \right) \quad \text{for} \quad x_1(t) < x \leq x_4(t), \quad t_1 < t,$$

$$= 0 \quad \text{for} \quad x_4(t) < x,$$

where

$$x_1(t) = -S + f'(0)t,$$

$$x_2(t) = -S + f'(\alpha)t,$$

$$x_3(t) = S + \sigma(\alpha, 0)t,$$

$x_4(t)$ satisfies

$$x'_4(t) = \sigma \left( h_1 \left( \frac{x_4(t) + S}{t} \right), 0 \right) - f \left( h_1 \left( \frac{x_4(t) + S}{t} \right) \right) \left/ h_1 \left( \frac{x_4(t) + S}{t} \right) \right.,$$

with

$$x_4(t_1) = x_2(t_1) = x_3(t_1), \quad t \geq t_1.$$

From the entropy condition,

$$x'_4(t) < f' \left( h_1 \left( \frac{x_4(t) + S}{t} \right) \right) = \frac{x_4(t) + S}{t}. $$
hence
\[
\frac{d}{dt} \left( \frac{x_4(t) + S}{t} \right) = \frac{x_4(t)}{t} - \frac{x_4(t) + S}{t^2} < 0.
\]

But it is obvious that
\[
\frac{x_4(t) + S}{t} \geq f'(0);
\]
hence \((x_4(t) + S)/t \to f''(0)\) as \(t \to \infty\). And thus \(h_1((x_4(t) + S)/t) \to 0\) as \(t \to \infty\). Now for \(t\) large, we can have
\[
X(t) = x_4(t) + S,
\]

But on the other hand, from the definition of \(h_1\),
\[
(2n + 1) A h_1^2 \left( \frac{x_4(t) + S}{t} \right) + f''(0) \equiv f'' \left( h_1 \left( \frac{x_4(t) + S}{t} \right) \right) = \frac{x_4(t) + S}{t},
\]

Letting \(X(t) = x_4(t) + S\), we get
\[
X'(t) \equiv f''(0) + \frac{1}{2n + 1} \left( \frac{X}{t} - f''(0) \right)
\quad \text{for } t \text{ large.}
\]

Hence
\[
X(t) \equiv f''(0)t + O(S) t^{1/(2n + 1)}
\]
and
\[
h_1 \left( \frac{X(t)}{t} \right) \equiv \left[ \frac{1}{(2n + 1) A} \left( \frac{X(t)}{t} - f''(0) \right) \right]^{1/2n} = O(S) t^{-1/(2n + 1)}.
\]

Similarly we can consider the case \(a > 0\). This completes the proof. Q.E.D.

**Theorem 2.11.** If \(u_i = u_r = 0\) and \(f(u) \equiv f'(0)u + Au^{2n+1}, n \geq 1, A > 0,\) for \(|u|\) small, then we have
\[
|u(x, t)| = O(S) t^{-1/(2n + 1)}.
\]

**Proof:** This theorem is a consequence of Lemma 2.10 and the Ordering Principle.
**Theorem 2.12.** If $u_1 = 0 < u_r$ and $f(u) \equiv f'(0)u + Au^{2^n+1}$, $n \geq 1$, $A > 0$, for $|u|$ small, define

$$p(t) = \min_x \int_0^x u(y, t) \, dy,$$

$$q(t) = \max_x \int_0^x (u(y, t) - u_r) \, dy,$$

then

(i) if $p(0) = 0$, then $p(t) = 0$ for all $t$,

(ii) if $p(0) < 0$, then $p'(t) > 0$ for all $t$ and $p(t) \to 0$ as $t \to \infty$,

(iii) if $p(t) = \min_x \int_0^x u(y, t) \, dy = \int_0^{x(t)} u(y, t) \, dy$, then

$$x_p(t) \approx f'(0) t + O(S) t^{2^n} - S \quad \text{for } t \text{ large,}$$

where $2n\xi^{2^n} - \xi^{2^n-1} - \cdots - \xi - 1 = 0$, $-1 < \xi < 0$,

(iv) $q'(t) < 0$ for all $t \geq 0$,

(v) there exists $t_0 > 0$, such that for all $t \geq t_0$, $q(t) = q(t_0)$.

**Proof:** From Liu [9, Theorem 1(i)], $p'(t) \geq 0$ and $p(t) \leq 0$; thus if $p(0) = 0$, then $p(t) = 0$ for all $t$ which proves (i). If $p(0) < 0$, then we want to prove that the maximum point $x_p(t)$ as defined in (iii) is a shock curve with $u(x_p(t) - 0, t) < 0$ and $0 < u(x_p(t) + 0, t) \leq u(x_p(t) - 0, t)^*$. If $u(x, t)$ is continuous at $x_p(t)$, then $u(x_p(t), t) = 0$ and $u(x_p(t) - \varepsilon, t) < 0$. $u(x_p(t) + \varepsilon, t) > 0$ for sufficiently small $\varepsilon > 0$. But this is impossible, because the characteristics from the immediate left-hand side of $x_p(t)$ will intersect the characteristics from $x_p(t)$ immediately. Thus $x_p(t)$ must be a shock curve. It is then obvious that $u(x_p(t) - 0, t) < 0$ and $0 < u(x_p(t) + 0, t) \leq u(x_p(t) - 0, t)^*$. Thus

$$p'(t) = u(x_p(t) - 0, t) x_p(t) - \left[ f(u(x_p(t) - 0, t)) - f(0) \right]$$

$$= u(x_p(t) - 0, t) \left[ \frac{f(u(x_p(t) - 0, t)) - f(u(x_p(t) + 0, t))}{u(x_p(t) - 0, t) - u(x_p(t) + 0, t)} \right] > 0;$$

this proves (ii). Now we would like to estimate the order of $x_p(t)$. From (R–H) and the above arguments, we have

$$x_p(t) = \frac{f(u(x_p(t) - 0, t)) - f(u(x_p(t) + 0, t))}{u(x_p(t) - 0, t) - u(x_p(t) + 0, t)} < f'(u(x_p(t) - 0, t)).$$
For $t$ large, $u(x_p(t) - 0, t) \equiv h_1((x_p(t) + S)/t)$; hence
\[
\frac{d}{dt} \left( \frac{x_p(t) + S}{t} \right) < 0 \quad \text{and} \quad \frac{x_p(t) + S}{t} \to f'(0) \quad \text{as} \quad t \to \infty.
\]
Thus for $t$ large, we have $u(x_p(t) + 0, t) = u(x_p(t) - 0, t)^*$ and
\[
x_p'(t) \approx \frac{f(h_1((x_p(t) + S)/t)) - f(h_1((x_p(t) + S)/t)^*)}{h_1((x_p(t) + S)/t) - h_1((x_p(t) + S)/t)^*}
= f'(0) + A \left[ h_1^{2n} + h_1^{2n-1} h_1^* + \cdots + h_1^{2n} \right]
= f'(0) + A h_1^{2n} \left( \frac{x_p(t) + S}{t} \right) \cdot [1 + \xi + \xi^2 + \cdots + \xi^{2n}],
\]
where $\xi$ satisfies
\[
2n\xi^{2n} - \xi^{2n-1} - \xi^{2n-2} - \cdots - \xi - 1 = 0, \quad -1 < \xi < 0.
\]
Thus
\[
x_p'(t) \approx f'(0) + A(2n + 1) h_1^{2n} \left( \frac{x_p(t) + S}{t} \right) \cdot \xi^{2n}
\approx f'(0) + \xi^{2n} \left( \frac{x_p(t) + S}{t} \right) - f'(0) .
\]
Hence
\[
x_p(t) + S \approx f'(0) t + O(S) t^{2n}.
\]
This proves (iii); (iv) follows from Liu [9, Theorem 1(i)]. To prove (v), first, we use the solution $u_{m}(x, t)$ to prove that $u(x, t) \geq u_{m}(x, t) > (u_r)^*$ for all $t \geq t_1$, where $m$ is the infimum of $u(x, 0)$ and $t_1$ is some constant greater than zero. This is by direct construction of the solution $u_{m}(x, t)$. We do not want to repeat it here. Assume that the maximum point in the definition of $q(t)$ is taken place at $X(t)$. Since $u(x, t) > (u_r)^*$, for all $t \geq t_1$, $X(t)$ cannot be a shock curve. Hence $u(x, t)$ is continuous at the point $X(t)$ and $u(X(t), t) = u_r$. Direct calculation of $q'(t)$ will prove (v). Q.E.D.

**Theorem 2.13.** Under the same assumptions of Theorem 2.12, define the one-sided $\varepsilon$--N-wave $N_{\varepsilon}(x, t)$ as
\[
N_{\varepsilon}(x, t) = u_r, \quad \text{for} \quad x > f'(u_r) t + \sqrt{2q(\infty)f''(u_r)} t,
= h_2 \left( \frac{x}{t} \right), \quad \text{for} \quad f'(\varepsilon) t < x \leq f'(u_r) t + \sqrt{2q(\infty)f''(u_r)} t.
\]
then

(i) \[ |x(t) - f'(u) t - \sqrt{2q(\infty)f''(u)} t| = O(S) \text{ for all } t, \]

(ii) \[ |N(x, t) - u(x, t)| \leq A^{-1}(\varepsilon) O(S) t^{-1} \text{ for } x \text{ between } f'(\varepsilon) t \text{ and } \min(x, f'(\varepsilon) t + \sqrt{2q(\infty)f''(u)} t), \]

(iii) \[ |N(x, t) - u(x, t)| \leq O(S) t^{-1/2} \text{ for } x \text{ between } x(x, t) \text{ and } f'(u) t + \sqrt{2q(\infty)f''(u)} t, \]

where \( \varepsilon \) is a small fixed number with \( 0 < \varepsilon < u \), and \( q(\infty) \) is the constant \( q(t, s) \) in Theorem 2.12(v) and \( A(\varepsilon) = \min_{a < \varepsilon} f''(u) \).

Remark. We can have a similar theorem for the case \( u_1 = 0 > u_r \).

Proof. From Theorem 2.12(iii), we find a time \( t_1 > 0 \), such that \( f'(\varepsilon) t > x_p(t) \) for \( t \geq t_1 \). Thus for \( t \geq t_1 \), \( u(x, t) \geq \varepsilon \) for all \( x \geq f'(\varepsilon) t \). Then (i), (ii) and (iii) follows from Theorem 4 of Liu [9]. Q.E.D.

3. THE CASE WHEN \( f'' \) VANISHES AT \( N \) POINTS AND CHANGES SIGN AT THESE POINTS

Without loss of generality, we assume that \( f'' \) vanishes at \( a_1, a_2, \ldots, a_N \), where \( a_1 < a_2 < \cdots < a_N \), and \( f''(u) < 0 \) for \( u < a_1 \), \( f''(u) > 0 \) for \( a_1 < u < a_2, \ldots, \) etc. We also adopt the definitions of \( u_a(x, t), M, m, x_i(t) \) and \( x_r(t) \) of Section 2. For convenience, we put \( a_0 = -\infty \) and \( a_{N+1} = +\infty \). In this section, we use \( u(x, t) \) to denote the solution of (1.1), (1.2) with initial condition (1.3), where \( f \) is under the assumption of this section.

We may need direct construction of solution \( u(x, t) \) in the proof of the following lemmas and theorems. We will give only some indications and omit the details. These constructions are similar to the constructions in Section 2.

**Lemma 3.1.** If \( u_i \in (a_{i-1}, a_i), u_r \in (a_{j-1}, a_j) \), where \( 1 \leq i \leq j \leq N + 1 \), then there exists \( t_a \geq 0 \), such that for all \( t \geq t_a \), \( u_a(x, t) \in (A_1, A_r) \), where \( A_1, A_r \) are two fixed constants with \( A_i \in (a_i, u_i) \) and \( A_r \in (u_r, a_j) \).

**Lemma 3.2.** Under the assumptions of Lemma 3.1, there exists \( t_0 \geq 0 \), such that for all \( t \geq t_0 \), \( u(x, t) \in (A_1, A_r) \), where \( A_1, A_r \) are as in Lemma 3.1.

Proof of Lemmas 3.1 and 3.2. Using the Ordering Principle, we can easily establish Lemma 3.2 from Lemma 3.1 if \( u(x, 0) \in (a, a_j) \) for all \( x \). To prove Lemma 3.1, we use induction. If \( a \in (a_{i-1}, a_i) \), then Lemma 3.1 is obviously true. Now assume that when \( a \in (a_{i-k}, a_{i-k+1}) \), Lemma 3.1 is true, and hence Lemma 3.2 is also true when \( u(x, 0) \in (a, a_j) \). We would like
to establish that when $a \in (a_{i-k-1}, a_{i-k})$, Lemma 3.1 is true. The solutions for the Riemann problems $(u_1, a)$ and $(a, u_r)$ are combinations of shock waves and rarefaction waves. Let us denote these simple wave resolutions of $(u_1, a)$ and $(a, u_r)$ by $(u_1, v_1), (v_1, v_2), \ldots, (v_m, a)$ and $(a, w_1), (w_1, w_2), \ldots, (w_n, u_r)$. It is easy to see that at least one of the simple waves $(v_m, a)$ and $(a, w_1)$ must be a shock wave. It is this simple consequence of entropy condition (E) that causes the cancellation of waves. Now it is easy to see that shock wave $(v_m, a)$ or $(a, w_1)$ will kill the $a$-states in a finite time. After that, the remaining rarefaction wave, $(v_m, a)$ or $(a, w_1)$, will be killed by a combination of type I and type II shocks in a finite time. Thus there exists $t_1 > 0$, such that for $t \geq t_1$, $u_a(x, t) \in (a_{i-k}, a_j)$. Using induction hypotheses, we prove that when $a \in (a_{i-k-1}, a_{i-k})$, Lemma 3.1 is true. Similarly, we can consider the case $a > a_j$. This completes the proof of Lemmas 3.1 and 3.2.

Q.E.D.

Remark. We have two similar lemmas when $1 \leq j \leq i \leq N + 1$.

**Definition.** If the solution of the Riemann problem $(u_i, u_r)$ consists of a simple shock wave with $f'(u_i) > \sigma(u_i, u_r) > f'(u_r)$ and $\sigma(u_i, u) > \sigma(u_i, u_r) > \sigma(u, u_r)$ for all $u$ between $u_i$ and $u_r$, then we call $(u_i, u_r)$ a strict shock.

**Theorem 3.3.** If $(u_i, u_r)$ is a strict shock, then there exists $x_0$ and $t_0$, $t_0 > 0$, such that for all $t \geq t_0$,

\[ u(x, t) = \begin{cases} u_i & \text{for } x \leq x_0 + \sigma(u_i, u_r)t, \\ u_r & \text{for } x > x_0 + (u_i, u_r)t, \end{cases} \]

where

\[ x_0 = \frac{1}{u_r - u_i} \int_{-N}^{N} \left( \frac{u_r + u_i}{2} - u^0(x) \right) dx, \quad N \geq S. \]

**Proof.** From Lemma 3.2, if $u_i \in [a_{i-1}, a_i], u_r \in [a_{j-1}, a_j]$ and $i \geq j$ (note that the case $i \leq j$ can be similarly considered), then we can choose $A_i \in (a_{i-1}, u_i), A_r \in (u_r, a_j)$ such that $(A_i, u_r)$ and $(u_i, A_r)$ are all strict shocks. It is easy to construct the solutions $u_{A_i}(x, t)$ and $u_{A_r}(x, t)$ directly and find $t_{A_i}, t_{A_r}, x_{A_i}, x_{A_r}$ such that

\[ \begin{align*} u_{A_i}(x, t) &= u_i & \text{for } x \leq x_{A_i} + \sigma(u_i, u_r)t, & t \geq t_{A_i}, \\
&= u_r & \text{for } x > x_{A_i} + \sigma(u_i, u_r)t, \\
\end{align*} \]

\[ \begin{align*} u_{A_r}(x, t) &= u_i & \text{for } x \leq x_{A_r} + \sigma(u_i, u_r)t, & t \geq t_{A_r}, \\
&= u_r & \text{for } x > x_{A_r} + \sigma(u_i, u_r)t, \end{align*} \]
Thus for \( t \) sufficiently large, say \( t \geq t_0 \), we have

\[
\begin{align*}
    u(x, t) &= u_l \quad \text{for} \quad x \leq x_{A_l} + \sigma(u_l, u_r)t, \\
    &= u_r \quad \text{for} \quad x > x_{A_l} + \sigma(u_l, u_r)t,
\end{align*}
\]

and \( u_l = u_l(x, t) \leq u(x, t) \leq u_r(x, t) = u_r \) for \( x_{A_l} + \sigma(u_l, u_r)t \leq x \leq x_{A_l} + \sigma(u_l, u_r)t \). Using the strict shock properties of \((u_l, u_r)\), we can prove this theorem by using the same arguments as in the proof of Theorem 2.4.

Q.E.D.

**Definition.** If the solution of the Riemann problem \((u_l, u_r)\) is a simple rarefaction wave and if \( u_i \neq a_i \neq u_i \) for all \( i \), then we call \((u_l, u_r)\) a strict rarefaction wave.

**Remark.** From this definition, a strict rarefaction wave can have two possibilities only. Either \( a_{i-1} < u_i < a_i < u_r \) and \( f''(u) > 0 \) for all \( u \in (a_{i-1}, a_i) \) or \( a_{j-1} < u_i < u_r < a_j \) and \( f''(u) < 0 \) for all \( u \in (a_{j-1}, a_j) \).

**Theorem 3.4.** If \((u_l, u_r)\) is a strict rarefaction wave, with the proper definitions of \( p(t) \) and \( q(t) \) in Theorem 2.5 and the definition of \( N(x, t) \) in Theorem 2.6, where we have to replace \( h_i \) by some proper \( h_i \) and \( h_i \) is the inverse function of \( f'(u) \) restricted in \((a_{i-1}, a_i)\), then the proper statements of Theorems 2.5 and 2.6 hold.

**Proof.** In view of the Ordering Principle and Lemmas 3.1 and 3.2, we can push the solution \( u(x, t) \) at a finite time into the interval \((a_{i-1}, a_i)\) which contains \( u_l \) and \( u_r \). Then the whole story of Liu [9] goes and the theorem is proved. Q.E.D.

For nonstrict shocks and nonstrict rarefaction waves, they can be treated as in Theorems 2.9, 2.11, 2.12, and 2.13. We do not treat them here. Similarly we can treat the case of the combination of shocks and rarefaction waves. For example, if the resolutions of \((u_l, u_r)\) to simple waves are \((u_1, v_1), (v_1, v_2), (v_2, u_2)\), where \((u_1, v_1)\) is a shock with \( f'(u_1) > \sigma(u_1, v_1) = f'(v_1) \), \((v_1, v_2)\) is a strict rarefaction wave, \((v_2, u_2)\) is a shock with \( f'(v_2) = \sigma(v_2, u_2) > f'(u_2) \), then we can prove that after a finite time, \( x_i(t) = X_i + \sigma(u_i, v_i)t \), \( x_r(t) = X_r + \sigma(v_2, u_2)t \) and between these two shocks is rarefaction wave \( (v_1, v_2) \). The proof is similar to the proof of Theorem 2.9. For \( u_i \neq u_r = a_i \), the treatment is almost identical to the treatment of Theorem 2.11. Although we did not consider the case \( f''(a_i) = 0 \) and \( f'' \) does not change sign at \( a_i \), it is obvious that we can apply our technique to this case as well.
ACKNOWLEDGMENT

I would like to express my thanks to Tai-Ping Liu for introducing me to this area of research and useful suggestions and discussions. I am solely responsible for all possible errors in this paper.

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