DEPENDENCE OF TREE COPY FUNCTIONS

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The graph copy function when restricted to the set of trees or of rooted trees is called the tree copy function or the rooted tree copy function, respectively. We show that both the set of tree copy functions and the set of rooted tree copy functions are strongly linearly independent. We also show that the set of all tree copy functions are algebraically independent but the set of all rooted tree copy functions are not. An algebraic base for the algebra generated by rooted tree copy functions is constructed in this paper.

1. Definition and introduction

For any graph $H$, the function $c_H$ from the set of all graphs $\mathcal{G}$ into $\mathbb{R}$ is defined by setting $c_H(G) = |\{W \subseteq V(G) : |G|_W = H\}|$ for every $G \in \mathcal{G}$. In 1932, Whitney [4] proved that the functions $c_H$, $H$ connected, are algebraically independent. From that time on, mathematicians have tried to get similar results by weakening the domain or by working with other graph functions. (See [1, 2, 3].) In this paper, we restrict our discussion to the sets of trees and rooted trees.

A tree is defined to be a connected graph without cycles and with at least one edge. $\mathcal{T}$ denotes the family of all trees. For a fixed tree $T$, $c_T$ is defined to be $c_T = c_{\mathcal{T}[T]}$, the restriction of $c_T$ to the set of all trees. We say that $c_{T_1}, c_{T_2}, \ldots, c_{T_n}$ are linearly independent if $\sum_{i=1}^n d_i c_{T_i} = 0$ for all $T \in \mathcal{T}$ implies $d_i = 0$ for every $i$; $c_{T_1}, c_{T_2}, \ldots, c_{T_n}$ are strongly linearly independent if $\sum_{i=1}^n d_i c_{T_i} = 0$ for all $T \in \mathcal{T}$ implies $d_i = 0$ for every $i$; and $c_{T_1}, c_{T_2}, \ldots, c_{T_n}$ are algebraically independent if for any polynomial $P$ in $n$ variables such that $p(c_{T_1}, c_{T_2}, \ldots, c_{T_n})(T) = 0$ for all $T \in \mathcal{T}$ implies $p = 0$. If $B \subseteq \{c_T \mid T \in \mathcal{T}\}$, we say $B$ is linearly (strongly linearly, algebraically, respectively) independent if any finite elements in $B$ are linearly (strongly linearly, algebraically, respectively) independent. If $B$ is not linearly (strongly linearly, algebraically, respectively) independent, then we say that $B$ is linearly (strongly linearly, algebraically, respectively) dependent. It is easy to see that algebraic independence implies strongly linear independence which in turn implies linear independence. But the converses need not be true.

2. Dependence of tree copy functions

Given a tree $T$, a vertex $x$ is called a brink point if $|\{y \mid \deg_T(y) = 1, x$ is adjacent to $y\}| = \deg_T(x) - 1$. Let $\text{Br}(T) = \{x \mid x$ is a brink point in $T\}$ and call $\text{Br}(T)$ the
brink set of $T$. Note that $\text{Br}(T) = \emptyset$ if and only if $T \cong S_m$ for some $m$. We define the brink degree of $T$, $\deg \text{Br}(T)$, to be $\max \{ \deg_T(x) - 1 \mid x \in \text{Br}(T) \}$ if $\text{Br}(T) \neq \emptyset$ and $\deg \text{Br}(T)$ to be zero if $\text{Br}(T) = \emptyset$. For those $T$ with $\text{Br}(T) \neq \emptyset$, we pick a fixed point $x \in \text{Br}(T)$ with $\deg_T(x) = \deg \text{Br}(T)$ and for those $T$ with $\text{Br}(T) = \emptyset$, we pick a fixed point $x \in V(T)$ such that $\deg_T(x) = m$ where $T \cong S_m$. We call this point a cliff point of $T$.

**Lemma 2.1.** Let $B = \{ T_1, T_2, \ldots, T_n \}$ be a finite set of distinct trees. Then there exists a tree $T_i$ in $B$ such that for any $\varepsilon > 0$ and $K > 0$, there exists a tree $T' = T'(\varepsilon, K)$ satisfying $c_T'(T') > K$ and $c_T'(T') < \varepsilon c_T'(T')$ for all $T_j \neq T_i$.

**Proof.** Consider $C = \{ T_i \mid \text{Br}(T_i) = \emptyset \}$ a subset of $B$. If $C \neq \emptyset$, say $C = \{ T_1, T_2, \ldots, T_m \}$, where $T_i \cong S_m$, then take $T_m$ to be the tree in $C$ with the largest $m_i$. Let $B(m)$ be $S_m$. We have $c_T'(B(m)) = (m)_i$ if $T_i \in C$ and $c_T'(B(m)) = 0$ if $T_i \notin C$. By choosing $m$ sufficiently large, we have $c_T'(B(m)) > K$ and $c_T'(B(m)) < \varepsilon c_T'(B(m))$ for $T_j \neq T_m$.

If $C = \emptyset$, take $D$ to the subset of trees in $B$ with the largest brink degree and take $T_m$ to be a tree in $D$ with the least number of edges. Construct $B(m)$ by adding $m - \deg \text{Br}(T_m)$ pendant edges to the cliff point of $T_m$. For those tree $T_i$ not in $D$, $c_T'(B(m)) = P_i(m)$ where $P_i$ is a polynomial with degree at most $\deg \text{Br}(T_i)$ with $\deg \text{Br}(T_i) < \deg \text{Br}(T_m)$, and $c_T'(B(m)) = P_i(m)$, where $P_i$ is a polynomial with degree $\deg \text{Br}(T_i)$. However, for those trees $T_j$ in $D$ and $T_j \neq T_m$, we have $c_T'(B(m)) = 0$. Therefore we can choose sufficiently large $m$ such that $c_T'(B(m)) > K$ and $c_T'(B(m)) < \varepsilon c_T'(B(m))$ for $T_j \neq T_m$. □

**Lemma 2.2.** The set of all tree copy functions is strongly linearly independent and hence it is linearly independent.

**Proof.** If it is not strongly linearly independent, we can find $T_1, T_2, \ldots, T_n$ such that there exist $d_0, d_1, \ldots, d_n$ with $d_j \neq 0$ for $j \neq 0$ which satisfy $\sum_{i=0}^n d_i c_T'(T) = d_0$ for all $T \in \mathcal{T}$. By Lemma 2.1, we can find $T_s$ such that for $0 < \varepsilon < \min \{|d_i|/|d_j|\mid j \neq s\}$ and $K > n |d_j|/d_s|$, there exists a tree $T'$ which satisfies $c_{T_s}'(T') > K$ and $c_{T_s}'(T') < \varepsilon c_{T_s}'(T')$ for $T_j \neq T_s$. Since $\sum_{i=0}^n d_i c_{T_s}'(T') = d_0$, we have $d_s c_{T_s}'(T') = d_0 - \sum_{i \neq 0, s} d_i c_{T_s}'(T')$. Therefore

$$|c_{T_s}'(T')| = \left| \frac{d_0}{d_s} - \sum_{i \neq 0, s} \frac{d_i}{d_s} c_{T_s}'(T') \right| < \left| \frac{d_0}{d_s} \right| + \sum_{i \neq 0, s} \left| \frac{d_i}{d_s} \right| c_{T_s}'(T')$$

$$< \frac{c_{T_s}'(T')}{n} + (n-1) \frac{c_{T_s}'(T')}{n} = c_{T_s}'(T').$$

We get a contradiction. Therefore the set of all tree copy functions is strongly linearly independent. □
Definition. A tree \( T \) is accessible from \( T_1, T_2, \ldots, T_n \) if
(1) there exist subtrees \( A_0, A_1, \ldots, A_n \) of \( T \) such that \( T_i = A_i \) for \( i \geq 1 \),
(2) all the pendant of \( A_0 \) is identified with some node of \( A_i, i \geq 1 \) and
(3) the subtree of \( T \) generated by \( A_0, A_1, \ldots, A_n \) is \( T \) itself.

Let \( A(T_1, T_2, \ldots, T_n) \) denote the set of all trees accessible from
\( T_1, T_2, \ldots, T_n \); and for \( T \in A(T_1, T_2, \ldots, T_n) \), we let \( \psi_{T_1, T_2, \ldots, T_n}(T) \) be the number
of all possible ways to select \( A_0, A_1, \ldots, A_n \) such that
(1) \( T_i = A_i \) for \( i \geq 1 \);
(2) all pendant of \( A_0 \) is identified with some node of \( A_i \) where \( i > 1 \);
(3) the subtree of \( T \) generated by \( A_0, A_1, \ldots, A_n \) is \( T \) itself; and
(4) \( A_0 \) is minimum with the properties (1), (2) and (3).

Lemma 2.3.
\( T = \psi_{T_1, T_2, \ldots, T_n}(T) \)

Proof. For any \( B \in \mathcal{G} \), we have \( \psi_{T_1, T_2, \ldots, T_n}(B) = \left\{ (A_0, A_1, \ldots, A_n) \mid A_i \text{ is a}
\text{subtree of } B \text{ and } A_i \equiv T_i \text{ for all } i \right\} \).

For each \( (A_1, A_2, \ldots, A_n) \) with \( A_i \) a subtree of \( B \) and \( A_i \equiv T_i \) for all \( i \), let \( T \) be
a minimum subtree of \( B \) containing \( A_i \) for all \( i \). Then \( T \in A(T_1, T_2, \ldots, T_n) \).
By summing over all \( T \in A(T_1, T_2, \ldots, T_n) \), we have
\[
\psi_{T_1, T_2, \ldots, T_n}(B) = \sum_{T \in A(T_1, T_2, \ldots, T_n)} \psi_{T_1, T_2, \ldots, T_n}(T) \psi_{T_1, T_2, \ldots, T_n}(B).
\]

Example 1. Let \( T_1 = T_2 = P_i \). Then \( A(T_1, T_2) = A(2P_i) = \{P_i \mid i > 1\} \), where \( P_i \) is
the path graph of length \( i \). Therefore we have
\[
\psi_{P_i} = \psi_{P_i} + 2 \sum_{i=2}^{\infty} \psi_{P_i}.
\]

Theorem 2.1. \( \{c_T \mid T \in \mathcal{G}\} \) is algebraically independent.

Proof. If this theorem is not true, there is a polynomial which is zero for all
trees but not identically zero. Let \( P \) be such a polynomial containing \( q \) variables
\( c_{T_1}, c_{T_2}, \ldots, c_{T_n} \) where \( T_i \) are arranged in descending order on the number of
edges. Then \( P \) can be written as
\[
P(c_{T_1}, c_{T_2}, \ldots, c_{T_n}) = \sum_{i=1}^{n} a_{i} c_{T_1}^{m_{1}} c_{T_2}^{m_{2}} \cdots c_{T_n}^{m_{n}},
\]
where \( \sum_{i=1}^{q} c_{i_{1}}, c_{i_{2}}, \ldots, c_{i_{q}} \) is greater than \( \sum_{j=1}^{q} a_{j} \) lex-
igraphically if and only if \( i < k \). By Lemma 2.3 \( P \) can be written as
\[
P(c_{T_1}, c_{T_2}, \ldots, c_{T_n})
= \sum_{T \in A} \psi_{T_1, T_2, \ldots, T_n}(T) c_{T_1} \cdots c_{T_n}
= \sum_{T \in A} a_{T} c_{T}.
\]
where $T \in A$ if and only if

$$\sum_{i=1}^{n} \sum_{T \in A(\alpha_{1}, \alpha_{2}, \ldots, T_{q})} a_{i} \psi_{\alpha_{1}, \alpha_{2}, \ldots, T_{q}}(T)$$

is not zero.

The cardinality of $A$ can be zero, a positive integer or infinite. First, we claim that $|A| \neq 0$. Consider the first term of $P$, $a_{1}c_{1}^{\alpha_{1}}c_{2}^{\alpha_{2}} \cdots c_{q}^{\alpha_{q}}$. Let $k = \max\{d(T) \mid i = 1, 2, \ldots, q\}$, where $d(T)$ is the length of the longest path in $T$. Form a tree $B$ as follows: For each of the $m$ given $P_{k}$ where $m = \sum_{i=1}^{q} \alpha_{i}$, pick up an endpoint and identity all these $m$ points to get a star-like graph, $P_{m,k}$. We identify each cliff point of trees in $\alpha_{1,1}T_{1}, \alpha_{1,2}T_{2}, \ldots, \alpha_{1,q}T_{q}$ with one and only one pendant vertex of $P_{m,k}$. $B$ is the resultant graph. It is easy to check that $B$ is an element of $A(\alpha_{1,1}T_{1}, \alpha_{1,2}T_{2}, \ldots, \alpha_{1,q}T_{q})$ but not of $A(\alpha_{1,1}T_{1}, \alpha_{i,2}T_{2}, \ldots, \alpha_{i,q}T_{q})$ where $i \neq 2$. Therefore the term $a_{B}c_{B}^{\alpha_{B}}$ cannot be cancelled out in ($\ast$). Hence $|A| \neq 0$. Second, we assume that $|A|$ is a positive integer. Then we have $0 = P(c_{1}, c_{2}, \ldots, c_{q}) = \sum_{T \in A} a_{T}c_{T}^{\gamma}$ which contradicts Lemma 2.2. Finally, assume that $|A|$ is infinite. Choose $B$ in $A$ with the least number of edges. We have $\sum_{T \in A} a_{T}c_{T}^{\gamma}(B) = a_{B}c_{B}^{\alpha_{B}}(B) \neq 0$, a contradiction. From the discussions above, we conclude that \{c_{T}^{\gamma} \mid T \in T\} is algebraically independent. $\square$

**Remark.** We must exclude $K_{1}$ as a tree for otherwise we have $c_{K_{1}} - c_{K_{1}} = 1$.

### 3. Dependence of rooted tree copy functions

A tree with one point, its root, distinguished from the other points is called a rooted tree. Let $T_{1}$ be a rooted tree with root $x_{1}$, and $T_{2}$ be a rooted tree with root $y_{1}$. $c_{T_{1}}^{\gamma}(T_{2})$ is the number of rooted subtrees in $T_{2}$ which is isomorphic to $T_{1}$. Let $T_{1}$, $T_{2}$ be the rooted trees in Fig. 1, with roots $x_{1}$, $y_{1}$, respectively. Then $c_{T_{1}}^{\gamma}(T_{2}) = 1$. Note that the subtree generated by $y_{2}$, $y_{3}$ is not a rooted subtree of $T_{2}$.

The term of linearly (strongly linearly, algebraically) dependence for rooted tree copy function is similarly defined to that for tree copy function. In this section we are going to discuss the dependence of rooted tree copy functions. Our result

![Fig. 1.](image)
Dependence of tree copy functions

is that they are strongly linearly independent but not algebraically independent. One trivial example is

$$2C^n_n = C^n_{n+2} - C^n_{n+1}$$

Since every rooted tree $T$ is planar, we can map it on the plane in such a way that the root $x$ of $T$ is always at bottom and those vertices with equal distance to the root $x$ are on the same level. The height of a rooted tree is the maximum distance from the root $x$ to any particular point. Therefore for every rooted tree $T$, we can associate with it at least one pictured-tree on paper, say $P_T$. We may represent $P_T$ by a unique sentence, the first word of which reflects the number of branches $P_T$ has, and the $(i+1)$th word reflects the number of sub-branches for that of the $i$th step, recorded from left to right. We denote the $i$th word by $W_i(P_T)$; the $j$th digit in $W_i(P_T)$ by $d^i_j(P_T)$. The sentence representing $P_T$ is called a representation of $T$.

**Example 2.**

$$\begin{array}{c}
\text{d}^3_3(P_T) \\
(3;3,1,2,0;3,0,1,0,0,0,0,2) \\
W_3(P_T)
\end{array}$$

It is easy to see that $\sum d^i_j -$ the number of digits in $W_{i+1}$ and $\sum_{i,j} d^i_j =$ the number of edges in $T$. Since every rooted tree $T$ may have many $P_T$'s associated with it, the representation of $T$ is not uniquely defined. To avoid this complexity, we may define the principal representation, $PR(T)$, of $T$ to be the largest representations for $T$. Then $PR(T)$ is the representation for the most left-tilted pictured-tree $P_T$ which $T$ may form. In the following discussions, the representation of $T$ refers to the principal representation of $T$; and the order refers to the lexicographical order. We have the following lemma whose proof is similar to that of the strongly linear independence for tree copy functions.

**Lemma 3.1.** Let $B_0 = \{T_1, T_2, \ldots, T_n\}$ be a family of distinct rooted trees. Then there exists a rooted tree in $B_0$, say $T_*$, such that for any $\varepsilon > 0$ and $K > 0$, there exists a rooted tree $T$ which satisfies $c_T^*(T) > K$ and $c_T^*(T) < ec_T(T)$ for every $T, T_*$.

**Proof.** We need an algorithm to find $T_*$ and to construct $T$. We define a sequence of sets by $B_i = \{T \in B_{i-1} \mid W_i(T)$ is the least with respect to the lexicographic order). After finite steps, we have an $\alpha$ such that $B_\alpha$ contains only one element, say $T_\alpha$. In fact, $T_\alpha$ is the least element in $B_0$. Let $\alpha$ be the first index such that $B_\alpha = \{T_\alpha\}$. If the height of $T_\alpha$ is greater than $\alpha$, take $T_\alpha$ to be $T_\alpha$. Let $d^\alpha_1$ be the first nonzero digit of the last word in $PR(T_\alpha)$. Construct $T(m)$ to be the rooted tree whose representation differs from that of $T_\alpha$ by changing $d^\alpha_1$ to $m$. If the height of
Tk equals \( \alpha \), consider \( A = \{ T \in B_{\omega-1} \mid \text{height of } T \text{ is } \alpha \} \). Since \( A \neq \emptyset \), we can choose \( T_k \) to be the largest element in \( A \). Let \( i(1) < i(2) < \cdots < i(p) \) be the indices such that \( d^{(i)}_{\alpha} \neq 0 \). Construct \( T(m) \) to be the rooted tree whose representation differs from that of \( T_k \) by a replacement of \( d^{(i)}_{\alpha} \) by \( m(p-i+1) \), where \( m(1) = m \), \( m(i+1) = 2m(i) \). Now we can show that \( T(m) \) and \( T_k \) satisfy our requirement for large \( m \).

For the first case, that is, the height of \( T_k \) is greater than \( \alpha \), we have \( c_{\alpha}^T(T(m)) = 0 \) for all \( i \neq s \). Therefore a sufficiently large \( m \) may be chosen to do the job. For the second case, that is, the height of \( T_k \) equals \( \alpha \), we have \( c_{\alpha}^T(T(m)) = 0 \) for those rooted trees not in \( A \); and

\[
c_{\alpha}^T(T(m)) = (a_i + o(1)) \prod_{i=1}^{p} \left( \frac{m(p-i+1)}{d^{(i)}_{\alpha}(T_i)} \right),
\]

where \( a_i \) is a positive constant. For those rooted trees \( T_i \) in \( A \) with \( T_i \neq T_k \), \( c_{\alpha}^T(T(m)) \) is at most

\[
(a_i + o(1)) \prod_{i=1}^{p} \left( \frac{m(p-i+1)}{d^{(i)}_{\alpha}(T_i)} \right),
\]

where \( a_i \) is a positive constant. Since \( W_{\alpha}(T_i) < W_{\alpha}(T_k) \), we can choose a sufficiently large \( m \) such that \( c_{\alpha}^T(T(m)) > K \) and \( c_{\alpha}^T(T(m)) < \epsilon c_{\alpha}(T(m)) \). \( \square \)

**Theorem 3.1.** \( \{ c_{\alpha}^T \mid T \text{ is a rooted tree} \} \) is strongly linearly independent.

**Proof.** Similar to the proof of Lemma 2.2. \( \square \)

As we mentioned above, \( \{ c_{\alpha}^T \mid T \text{ is a rooted tree} \} \) is algebraically dependent. We would like to find an algebraic base for it. For this purpose, we consider the set of stem trees.

A rooted tree \( T \) with root \( x \) is called a stem tree if \( \deg_T(x) = 1 \). Given an arbitrary rooted tree \( T \), we can decompose it into branches of stem trees \( B_1, B_2, \ldots, B_n \). Then we write \( T = B_1 \oplus B_2 \oplus \cdots \oplus B_n \). We let \( iT \) denote \( T \oplus T \oplus \cdots \oplus T \) (\( i \) times), and \( ST = \{ T \mid T \text{ is a stem tree} \} \). (See Fig. 2).

![Fig. 2](image-url)

Fig. 2. \( T_1 \) and \( T_2 \) are both stem trees but \( T_3 \) is not. Moreover \( T_3 = T_1 \oplus 2T_2 \).
Lemma 3.2. \( \{c_T \mid T \in ST\} \) is algebraically independent.

**Proof.** We follow Whitney's idea. If the assertion is not true, there is a non-zero polynomial which is zero for all graphs. Let \( P \) be such a polynomial containing the least possible number, say \( q(>0) \), of variables. Let us call the variables \( c_T^v, c_T^{u_1}, \ldots, c_T^n \) where \( T_i \) is one of the stem trees with the least number of vertices among \( \{T_i\}_{i=1}^n \). Arrange the polynomial in descending power of \( c_T^v \) and assume that it is of degree \( \alpha \) in this variable:

\[
P(c_T^v, c_T^{u_1}, \ldots, c_T^n) = P_0(c_T^{u_1}, \ldots, c_T^n) c_T^{\alpha\alpha - 1} + \cdots + P_n(c_T^{u_1}, \ldots, c_T^n) (P_0 \neq 0).
\]

Take any rooted tree \( B_o \) and form \( B_1, B_2, \ldots, B_n \) by letting \( B_i = B_o \oplus iT_i \) for \( i = 1, 2, \ldots, n \). We get \( c_{T_i}^v(B_i) = c_{T_i}^v(B_{i-1}) + 1 \) and \( c_{T_i}^v(B_i) = c_{T_i}^v(B_{i-1}) \) for \( i = 1, 2, \ldots, n, 2 \leq j \leq q \). \( P \) vanishes for these \( \alpha + 1 \) distinct values of \( c_T^v \) and the coefficients \( P_0, P_1, \ldots, P_n \) are constants for these rooted trees. They vanish, in particular, for the rooted tree \( B_0 \). Since \( B_0 \) is arbitrary, we get \( P_0(c_T^{u_1}, \ldots, c_T^n) = 0 \) for all rooted trees, which contradict the choice of \( P \). Therefore \( \{c_T^v \mid T \in ST\} \) is algebraically independent. \( \square \)

Our goal is to prove that \( \{c_T^v \mid T \in ST\} \) is actually an algebraic base for the algebra generated by \( \{c_T^v \mid T \) is a rooted tree\}. All we need is to prove that for any given rooted tree \( T, c_T^v \) can be expressed algebraically by elements of \( \{c_T^v \mid T \in ST\}. \)

Let \( T_1, T_2, \ldots, T_n \) be stem trees. We say a stem tree \( T \) is constructible by \( T_1, T_2, \ldots, T_n \) if there exist rooted subtrees \( A_1, A_2, \ldots, A_n \) of \( T \) such that \( T_i \equiv A_i \) for every \( i \) and the rooted subtree of \( T \) generated by the union of \( A_1, A_2, \ldots, A_n \) is \( T \) itself. Let \( I(T_1, T_2, \ldots, T_n) \) denote the set of all constructible stem trees of \( T_1, T_2, \ldots, T_n \); and for \( T \in I(T_1, T_2, \ldots, T_n) \), let \( \Psi_{T_1,T_2,...,T_n}(T) \) be the number of all possible ways to select rooted subtrees \( A_1, A_2, \ldots, A_n \) of \( T \) such that \( T_i \equiv A_i \) for every \( i \) and their union generates \( T \).

For simplicity, we work on 2-branch rooted trees.

Lemma 3.3. Let \( T = T_1 \ominus T_2 \), where \( T_1, T_2 \) are stem trees. Then \( c_T^v \) can be algebraically expressed in terms of \( c_{T_1}^v, c_{T_2}^v \) and those \( c_T^v \) with \( T'' \in I(T_1, T_2) \). To be more precise,

\[
kc_T^v - c_{T_1}^v c_{T_2}^v - \sum_{T'' \in I(T_1, T_2)} \Psi_{T_1,T_2}(T'') c_T^v,
\]

where \( k = 1 \) if \( T_1 \neq T_2 \) and \( k = 2 \) otherwise.

**Proof.** For any \( n \)-branch rooted tree \( T' \) with branches \( B_1, B_2, \ldots, B_n \), we have \( k c_{T'}^v(T') = |\{ (A, B) \mid A, B \) are rooted subtrees of \( T' \) such that \( A = T_1 \) and \( B = T_2 \); \( A, B \) are not in the same branch \}| and \( c_{T_1}^v c_{T_2}^v(T') = |\{ (A, B) \mid A, B \) are rooted
subtrees of \( T' \) such that \( A = T_1, B = T_2 \). Thus
\[
(c^e_{T_1} c^e_{T_2} - kc^v_2)(T') = \left| \{(A, B) \mid A, B \text{ are in the same branch of } T' \text{ and } A = T_1, B = T_2 \} \right|
\]
\[
= \sum_{i=1}^{n} \left| \{(A, B) \mid A = T_1, B = T_2, A, B \text{ both in } B_i \} \right|
\]

For \( T'' \in I(T_1, T_2) \), we get
\[
\sum_{T'' \in I(T_1, T_2)} \Psi_{T_1, T_2}(T'')c^v_T(T') = \sum_{i=1}^{n} \sum_{T'' \in I(T_1, T_2)} \Psi_{T_1, T_2}(T'') c^v_T(R_i).
\]
For each \( B_i \), we define an equivalence relation \( \sim_i \) on the set \( \{(A, B) \mid A = T_1, B = T_2, A, B \text{ in } B_i \} \): \( (A, B) \sim_i (A', B') \) if the union of \( A \) and \( B \) is equal to the union of \( A' \) and \( B' \). Consider the equivalent class of \( \sim_i \), we get
\[
(c^e_{T_1} c^e_{T_2} - kc^v_2)(R_i) = \sum_{T'' \in I(T_1, T_2)} \Psi_{T_1, T_2}(T'') c^v_T(R_i).
\]
Summing over all \( i \), we have
\[
(c^e_{T_1} c^e_{T_2} - kc^v_2)(T') = \sum_{T'' \in I(T_1, T_2)} \Psi_{T_1, T_2}(T'') c^v_T(T'). \tag*{\Box}
\]

Corollary. Let \( T = T_1 \oplus T_2 \oplus \cdots \oplus T_n \), where each \( T_i \) is a stem tree. Then
\[
kc^v_2 - c^v_{T_1} c^v_{T_2} \cdots c^v_{T_n} = \sum_{\text{finite}} c^v_{T_1} c^v_{T_2} \cdots c^v_{T_n}
\]
where \( T_i \) is a stem tree and \( k \) depend on the repetition type of \( T_1, T_2, \ldots, T_n \).

Combining Lemma 3.2 and the above corollary, we have the following theorem.

Theorem 3.2. \( \{c^v_T \mid T \in ST\} \) is a base for the algebra generated by \( \{c^v_T \mid T \text{ is a rooted tree} \} \).

References