ON MINIMUM CRITICALLY $n$-EDGE-CONNECTED GRAPHS*

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Abstract. Let $n$ be an integer with $n \geq 2$. A graph $G$ is called critically $n$-edge-connected if the edge-connectivity $\lambda(G) = n$ and for any vertex $v$ of $G$, $\lambda(G - v) = n - 1$. The sizes of critically $n$-edge-connected graphs are important and interesting in applications in communication networks. The maximum graphs with this property have been characterized [2]. In this paper, we first discuss some properties of minimum graphs, then show that the problem of finding a minimum critically $n$-edge-connected spanning subgraph of a given graph $G$ is NP-complete.

Key words. graph theory, edge-connectivity $\lambda(G)$, connectivity $\kappa(G)$, NP-completeness

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1. Introduction. Let $n$ be a fixed integer with $n \geq 2$. A graph $G$ shall be called $n$-edge-connected if the edge-connectivity $\lambda(G) = n$. A graph $G$ is called critically $n$-edge-connected if $G$ is $n$-edge-connected and for any vertex $v$ of $G$, $\lambda(G - v) = n - 1$. A graph $G$ is called $n$-connected if the vertex connectivity, $\kappa(G) = n$. A graph $G$ is called critically $n$-connected if $G$ is $n$-connected and for any vertex $v$ in $G$, $\kappa(G - v) = n - 1$. A graph $G$ is a minimum (maximum) critically $n$-edge-connected graph if no critically $n$-edge-connected graphs with the same number of vertices has fewer (more) edges than $G$.

In a communication network and circuit design, reliability is often determined by the connectivity and edge-connectivity of the corresponding graph. Therefore it is important to investigate, for fixed $n$, critically $n$-connected graphs ([3], [7]), and critically $n$-edge-connected graphs. We characterized the maximum graphs in a subset of critically $n$-edge-connected graphs, for each $n \geq 2$ in [2]. Here we investigate the minimum critically $n$-edge-connected graphs.

We use $\{x\}$ to denote the least integer greater than or equal to $x$, and $[x]$ the greatest integer less than or equal to $x$.

2. An example of a minimum critically $n$-edge-connected graph. For any fixed integers $n$, $m$, $m \geq n + 1$, Harary [5] constructed classes of graphs $H_{n,m}$, that are minimum $n$-connected. These same graphs are minimum critically $n$-edge-connected graph with order $m$. $H_{n,m}$ is constructed as follows:

Case 1. $n$ is even. Let $n = 2r$. Then $H_{2r,m}$ has vertices $0, 1, 2, 3, \ldots, m - 1$ and two vertices $i$ and $j$ are adjacent if $i - r \leq j \leq i + r$ (where addition is taken modulo $m$). $H_{4,8}$ is shown in Fig. 1.

Case 2. $n$ is odd ($n > 1$), $m$ is even. Let $n = 2r + 1$ ($r > 0$). Then $H_{2r+1,m}$ is constructed by first drawing $H_{2r,m}$, and then adding edges joining vertex $i$ to vertex $i + m/2$ for $1 \leq i < m/2$. $H_{5,8}$ is shown in Fig. 2.

Case 3. $n$ is odd ($n > 1$), $m$ is odd. Let $n = 2r + 1$ ($r > 0$). Then $H_{2r+1,m}$ is constructed by first drawing $H_{2r,m}$, and then adding edges $[0, (m - 1)/2]$ and $[i, i + (m + 1)/2]$ for $1 \leq i < (m - 1)/2$. $H_{5,9}$ is shown in Fig. 3.

In Case 1 and Case 2, $\deg_{H_{n,m}}(i) = n$, for all $i \in V(H_{n,m})$ so that $|E(H_{n,m})| = \frac{1}{2} \sum_{i \in V(H_{n,m})} \deg_{H_{n,m}}(i) = \frac{1}{2} n \cdot m$. 

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\[ H_{4,8} \]

**Fig. 1**

\[ H_{5,8} \]

**Fig. 2**

\[ H_{5,9} \]

**Fig. 3**
In Case 3, \( \text{deg}_{H_{n,m}}(i) = n \) for \( i = 1, 2, \ldots, m - 1 \), and \( \text{deg}_{H_{n,m}}(0) = n + 1 \). So that
\[
\sum_{i \in V(H_{n,m})} \text{deg}_{H_{n,m}}(i) = nm + 1 = 2 \cdot |E|. \quad \text{So} \quad |E(H_{n,m})| = \frac{(nm + 1)/2}.
\]
Therefore for any fixed integers \( n, m, m \geq n + 1 \), \( |E(H_{n,m})| = \{nm/2} \).
Now we show that \( H_{n,m} \) is a minimum critically \( n \)-edge-connected graph. \( \delta(G) \) is the least degree over all vertices of \( G \).

**Theorem 1.** The graph \( H_{n,m} \) is \( n \)-connected [5].
From the construction of \( H_{n,m} \), it is clear that \( \delta(H_{n,m}) = n \), and since
\[
n \leq \kappa(H_{n,m}) \leq \lambda(H_{n,m}) \leq \delta(H_{n,m}) = n,
\]
we have \( \lambda(H_{n,m}) = \delta(H_{n,m}) = n \). Therefore, we have the following theorem.

**Theorem 2.** The graph \( H_{n,m} \) is \( n \)-edge-connected.

For vertices \( j \) and \( k \) in a graph \( G \), a \( (j, k) \)-cutset of \( G \) is a vertex cutset \( T \) such that
\[
\delta(G - T) = n - 1 \quad \text{and} \quad |T| = n - 1.
\]

**Theorem 3.** The graph \( H_{n,m} - \{i\} \) is \( (n - 1) \)-connected, for any vertex \( i \) in \( H_{n,m} \).
Proof. Let \( n = 2r \) if \( n \) is even, \( 2r + 1 \) if \( n \) is odd. The minimum degree,
\[
\delta(H_{n,m} - \{i\}) = n - 1 \quad \text{so there exists a vertex cutset of size } n - 1. \quad \text{We will show that}
\]
there is no vertex cutset with fewer than \( n - 1 \) vertices. Suppose there exists a vertex cutset \( T \) such that \( 2 \leq |T| < n - 1 \). Let \( j \) and \( k \) be vertices belonging to different components of \( (H_{n,m} - \{i\}) - T \) such that if \( i \) is between \( j \) and \( k \) then \( 0 \leq k < i < j \), and if \( i \) is not between \( j \) and \( k \) then \( j < k \). Define two vertex sets \( A \) and \( B \) in \( H_{n,m} - \{i\} \) (addition is modulo \( m \)):
\[
A = \{j, j+1, j+2, \ldots, k-1, k\},
\]
\[
B = \{k, k+1, k+2, \ldots, i-1, i+1, \ldots, j-1, j\}.
\]
Note that \( A \cup B = V(H_{n,m} - \{i\}) \) and \( A \cap B = \{j, k\} \). Since \( |T| < n - 1 \), \( |T| < 2r \).

Therefore not both \( T \cap A \) and \( T \cap B \) can have \( r \) or more elements.

**Case 1.** \( |T \cap A| < r \). \( A - T = A - (A \cap T) \) so no more than \( r - 1 \) consecutive elements are removed from \( A \) by \( T \). Hence \( A - T \) has a sequence of distinct vertices starting with \( j \) and ending with \( k \) with no difference greater than \( r \) between any pair of consecutive vertices. This sequence is a \( (j, k) \)-path in \( (H_{n,m} - \{i\}) - T \), a contradiction to \( T \) being a \( (j, k) \)-cutset.

**Case 2.** \( |T \cap B| < r \).

**Subcase (i).** \( |T \cap B| < r - 1 \). As in Case 1, no more than \( r - 2 \) consecutive elements are removed from \( B \) by \( T \). Hence \( B - T \) has a sequence of distinct vertices starting with \( k \) and ending with \( j \), and the difference between any two consecutive vertices is at most \( (r - 1) + 1 = r \). (There is an additional 1 in the gap between \( i - 1 \) and \( i + 1 \).) This sequence is a \( (k, j) \)-path of \( (H_{n,m} - \{i\}) - T \), a contradiction to \( T \) being a \( (j, k) \)-cutset.

**Subcase (ii).** \( |T \cap B| = r - 1 \). Since \( j \) and \( k \) are not in \( T \),
\[
|T \cap A| = |T| - |T \cap B| < n - 1 - (r - 1) = n - r < r + 1.
\]
If \( |T \cap A| < r \) then Case 1 applies. Therefore \( |T \cap A| = r \), \( |A| + |B| = (m + 2) - 1 = m + 1 \). Therefore not both of \( |A| \) and \( |B| \) can be greater than \( \{(m + 1)/2} \), but at least one is greater than or equal to \( \{(m + 1)/2} \).

Suppose \( |A| \geq \{(m + 1)/2} \). If there exists a sequence of vertices in \( A - T \) beginning with \( j \) and ending with \( k \) such that no pair of consecutive terms has a difference \( \geq r + 1 \), then this sequence is a \( (j, k) \)-path in \( (H_{n,m} - \{i\}) - T \), a contradiction to \( T \) being
a \((j, k)\)-cutset. Thus we may assume that every sequence of vertices in \(A - T\) beginning with \(j\) and ending with \(k\) has a pair of consecutive terms with difference \(\geq r + 1\). In fact, since \(|T \cap A| = r\), this difference is exactly \(r + 1\), and there is only one such consecutive pair with difference \(r + 1\). All other consecutive pairs have a difference of 1. Call the pair of vertices with difference \(r + 1\), \(s\) and \(s + r + 1\), in the sequence \(A - T\). Thus we can write \(A - T\) as \(\{j, j + 1, j + 2, \ldots, s - 1, s, s + r + 1, \ldots, k - 1, k\}\). (Note that \(j\) can be \(s\).) Split \(A - T\) into two parts:

\[A_1 = \{j, j + 1, \ldots, s - 1, s\} \quad \text{and} \quad A_2 = \{s + r + 1, s + r + 2, \ldots, k - 1, k\} \]

The difference in consecutive terms in each \(A_i\) is 1, so there is an edge in \((H_{n,m} - \{i\}) - T\) between them. But \(m \geq n + 1 \geq 2r + 1\) implies \(m/2 \geq r + \frac{1}{2} > r\) if \(m\) is even, and \((m + 1)/2 \geq r + 1 > r\) if \(m\) is odd. Thus there are some \(a_1 \in A\) and \(a_2 \in A_2\) such that \(a_2 = a_1 + [(m + 1)/2]\). The sequence \(\{j, j + 1, \ldots, a_1, a_2, \ldots, k - 1, k\}\) is a \((j, k)\)-path in \((H_{n,m} - \{i\}) - T\), a contradiction to \(T\) being a \((j, k)\)-cutset.

If \(|B| \geq \{(m + 1)/2\}\) then the same argument applies since \(n - 1 > |T| \geq 2\) implies \(n \geq 4\), hence \(r \geq 2\), so there is an edge between \(i - 1\) and \(i + 1\) in \(H_{n,m} - \{i\}\).

All that remains is to show that no vertex cutset of only one vertex exists for \(H_{n,m} - \{i\}\). Suppose \(T = \{p\}\) is a vertex cutset of \(H_{n,m} - \{i\}\). Since \(|T| < n - 1\), \(n \geq 3\).

Case 1. If \(p = i - 1\) (equivalently \(i = p + 1\)), then \(i + 1, i + 2, \ldots, m - 1, 0, \ldots, i - 2\) is a path containing all the vertices of \(H_{n,m} - \{i, p\}\), a contradiction to \(T\) being a cutset of \(H_{n,m} - \{i\}\).

Case 2. \(p \neq i - 1\) and \(p \neq i + 1\). Without loss of generality assume \(i < p < m - 1\). Now \(P_1 = p + 1, p + 2, \ldots, m - 1, 0, \ldots, i - 1\) is a path and \(P_2 = i + 1, i + 2, \ldots, p - 1\) is a path in \((H_{n,m} - \{i\}) - \{p\}\). If \(n\) is even then \(r \geq 2\) and

\[\{i - 1, i + 1\} \in E(H_{n,m})\]

so there is only one component of \((H_{n,m} - \{i\}) - \{p\}\). If \(n\) is odd then there exists an edge between some \(x\) in \(P_1\) and \(x + [(m + 1)/2]\) in \(P_2\), again contradicting \(T = \{p\}\) being a cutset of \(H_{n,m} - \{i\}\). Therefore, there exists no cutset with only one vertex, and the theorem is proved.

QED

Since \(n - 1 \leq \kappa(H_{n,m} - \{i\}) \leq \lambda(H_{n,m} - \{i\}) \leq \delta(H_{n,m} - \{i\}) = n - 1\), we have \(\lambda(H_{n,m} - \{i\}) = \delta(H_{n,m} - \{i\}) = n - 1\). Therefore, we have the following theorem.

**Theorem 4.** The graph \(H_{n,m} - \{i\}\) is \((n - 1)\)-edge-connected, for any vertex \(i\) in \(H_{n,m}\).

Now we can show the main theorem of this section.

**Theorem 5.** *For any given positive integers \(m, n\), \(m \geq n + 1\), there exists a minimum critically \(n\)-edge-connected graph with order \(m\).*

**Proof.** By Theorem 2 and Theorem 4, \(H_{n,m}\) is critically \(n\)-edge-connected.

\[|E(H_{n,m})| = \{mn/2\} \quad \text{and} \quad |V(H_{n,m})| = m.\]

Let \(G = (V, E)\) be a critically \(n\)-edge-connected graph with \(|V| = m\). Thus \(\lambda(G) = n\), and for any vertex \(v\) in \(G\), \(\lambda(G) \leq \delta(G) \leq \deg_G v\). Hence

\[2 \cdot |E| = \sum_{v \in V(G)} \deg_G v \geq m \cdot \delta(G) = m \cdot n.\]

So \(|E| \geq mn/2\). \(|E|\) is an integer, hence \(|E| \geq \{mn/2\} = |E(H_{n,m})|\). So no critically \(n\)-edge-connected graph with \(m\) vertices has fewer edges than \(H_{n,m}\). Therefore \(H_{n,m}\) is a minimum critically \(n\)-edge-connected graph with order \(m\). QED
3. Characterizations of minimum critically \( n \)-edge-connected graphs. In addition to \( H_{n,m} \), there are other minimum critically \( n \)-edge-connected graphs. First we discuss some properties of minimum critically \( n \)-edge-connected graphs.

From the discussion of the graph \( H_{n,m} \), it is easy to obtain the following lemma.

**Lemma 6.** If \( G \) is a minimum critically \( n \)-edge-connected graph with order \( m \), then \( |E(G)| \geq \lceil mn/2 \rceil \).

A graph \( G \) is called almost regular of degree \( n \) if there is at most one vertex of degree \( n+1 \) and all other vertices have degree \( n \). Clearly, an \( n \)-regular graph is almost regular of degree \( n \).

**Theorem 7.** If \( G = (V, E) \) is a minimum critically \( n \)-edge-connected graph, then \( G \) is almost regular of degree \( n \). The proof follows from Lemma 6.

The converse of Theorem 7 is not true. \( G \), as shown in Fig. 4, is almost regular of degree 5, but \( G \) is not critically 5-edge-connected, since \( \lambda(G) = 5 \), and \( \lambda(G - a_{10}) = 3 \neq 5 - 1 \).

If \( G \) is \( n \)-edge-connected, then the order of \( G \), \( m \), is such that \( m \geq n + 1 \). For \( n + 1 \leq m \leq 2n \), we have a characterization of minimum critically \( n \)-edge-connected graphs.

**Theorem 8.** Let the order of \( G \) be \( m \). For any \( n \) such that \( n + 1 \leq m \leq 2n \), \( G = (V, E) \) is a minimum critically \( n \)-edge-connected graph if and only if \( G \) is almost regular of degree \( n \).

To prove Theorem 8, we will use the following lemma.

**Lemma 9.** If \( G \) has \( m \) vertices and \( \delta(G) \geq \lceil m/2 \rceil \), then \( \lambda(G) = \delta(G) \) [1].

**Proof of Theorem 8.** By Theorem 7, if \( G \) is a minimum critically \( n \)-edge-connected graph, then \( G \) is almost regular of degree \( n \).

Conversely, if \( G \) is almost regular of degree \( n \), then \( \delta(G) = n \geq m/2 \geq \lceil m/2 \rceil \). By Lemma 9, we have \( \lambda(G) = \delta(G) = n \). For any vertex \( u \in V(G) \), \( \delta(G - u) = n - 1 \geq m/2 - 1 \). Since \( n - 1 \) is an integer, \( n - 1 \geq \lfloor m/2 \rfloor \).

**Case 1.** \( m \) is odd.

\[
\delta(G-u) = n-1 \geq \left\lfloor \frac{m}{2} - 1 \right\rfloor, \quad n-1 \geq \frac{m+1}{2} - 1 = \frac{m-1}{2} = \left\lfloor \frac{m-1}{2} \right\rfloor.
\]

**Case 2.** \( m \) is even.

\[
\delta(G-u) = n-1 \geq \left\lfloor \frac{m}{2} - 1 \right\rfloor = \frac{m-2}{2} = \left\lfloor \frac{m-1}{2} \right\rfloor.
\]
By Lemma 9, we have \( \lambda(G - u) = \delta(G - u) = n - 1 \).

\[
\frac{|E(G)|}{2} = \sum_{v \in V(G)} \deg_G v
\]

\[
= \begin{cases} 
\frac{mn}{2}, & \text{or} \\
\frac{1}{2}((m-1)n+n+1) = \frac{1}{2}(mn+1) 
\end{cases}
\]

Therefore, \( G \) is a minimum critically \( n \)-edge-connected graph. QED

The reader should note that \( G \) need not be \( n \)-connected in Theorem 8.

In general, the converse of Theorem 7 is not true, but if the vertex connectivity \( \kappa(G) = n \), then we can give a characterization of minimum critically \( n \)-edge-connected graphs.

**Theorem 10.** Let \( \kappa(G) = n \). \( G = (V, E) \) is a minimum critically \( n \)-edge-connected graph if and only if \( G \) is almost regular of degree \( n \).

**Proof.** Let the order of \( G \) be \( m \). By Theorem 7, we obtain the “only if part.”

Conversely, if \( G \) is almost regular of degree \( n \), then \( \delta(G) = n \). Since \( n = \kappa(G) \leq \lambda(G) \leq \delta(G) = n \), we have \( \lambda(G) = n \).

For any vertex \( u \) in \( G \), \( \kappa(G - u) \leq \lambda(G - u) \leq \delta(G - u) = n - 1 \). Suppose that \( \lambda(G - u) < \delta(G - u) \), for some vertex \( u \) in \( G \), then \( \kappa(G - u) \leq \lambda(G - u) < n - 1 \). Thus, the connectivity \( \kappa(G) < n \), a contradiction. So for any vertex \( u \) in \( G \), we have \( \lambda(G - u) = \delta(G - u) = n - 1 \).

\( G \) is almost regular of degree \( n \), so by the proof of Theorem 8, \( |E(G)| = \{mn/2\} \).

Therefore, \( G \) is a minimum critically \( n \)-edge-connected graph. QED

The condition \( \kappa(G) = n \) in Theorem 10 is necessary, since we can find a graph \( G \), the one shown in Fig. 4, which is almost regular of degree \( n \) with \( \kappa(G) < n \), \( G \) is a minimum \( n \)-edge-connected graph, but \( G \) is not critical with respect to \( \lambda(G) \). Here \( \kappa(G) = 4 \), since \( \{a_2, a_9, a_{10}, a_{11}\} \) is a vertex cutset.

For \( m \geq 2n + 1 \), we can give some characterizations of minimum critically \( n \)-edge-connected graphs.

**Theorem 11.** For any given positive integers \( m, n, m \geq 2n + 1 \), and \( |V(G)| = m \), \( G = (V, E) \) is a minimum critically \( n \)-edge-connected graph if and only if \( G \) is almost regular of degree \( n \), and for each vertex \( u \) in a vertex cutset \( T \) with \( |T| \leq n - 1 \), \( \lambda(G - u) \geq n - 1 \).

**Proof.** By Theorem 7, if \( G \) is a minimum critically \( n \)-edge-connected graph, then \( G \) is almost regular of degree \( n \). Since \( G \) is critical with respect to \( \lambda(G) \), for each vertex \( u \) in \( G \), \( \lambda(G - u) = n - 1 \). So “the only if part” is complete.

Conversely, if \( G \) is almost regular of degree \( n \), then \( \delta(G) = n \). Since \( \lambda(G - u) \geq n - 1 \) for some vertex \( u \) in \( G \), and \( \delta(G) = n \), we have \( \lambda(G) \geq n - 1 \).

Suppose \( \lambda(G) = n - 1 \). Let \( E_1 \) be a minimum edge-cutset and \( G_1, G_2 \) be two components of \( G - E_1 \). \( \delta(G) = n \) and \( |E_1| = n - 1 \), so \( |V(G_1)| \geq 2 \) and \( |V(G_2)| \geq 2 \). Since \( m \geq 2n + 1 \), without loss of generality, we may let \( |V(G_1)| \geq n + 1 \). Let \( A \) be the set of vertices in \( G_1 \), which are incident with \( E_1 \), \( |A| \leq n - 1 \), since \( |E_1| = n - 1 \). So \( A \) is a vertex cutset with \( |A| \leq n - 1 \), and for any vertex \( u \) in \( A \), \( \lambda(G - u) \leq n - 2 \), a contradiction.
Therefore \( \lambda(G) > n - 1 \). \( n - 1 < \lambda(G) \leq \delta(G) = n \), so \( \lambda(G) = n \). Therefore \( G \) is \( n \)-edge-connected. We show next that \( G \) is critically \( n \)-edge-connected.

For each vertex \( u \) in \( G \), we consider the following two cases for a cutset containing it.

**Case 1.** \( u \) is in a vertex cutset \( T \) with \( |T| \leq n - 1 \), then \( \lambda(G - u) \geq n - 1 \). Since \( \lambda(G - u) \leq \delta(G - u) = n - 1 \), we have \( \lambda(G - u) = n - 1 \).

**Case 2.** Every vertex cutset containing \( u \) has at least \( n \) vertices. Suppose \( \lambda(G - u) < n - 1 \). Let \( \mathcal{E} \) be a minimum edge-cutset of \( G - u \), and \( H_1, H_2 \) be two components of \( (G - u) - \mathcal{E} \). \( |V(H_1)| + |V(H_2)| = m - 1 \leq (2n + 1) - 1 = 2n \). Without loss of generality, let \( |V(H_1)| \geq n \). Since \( |\mathcal{E}| < n - 1 \), \( u \) must be adjacent to some vertices in \( H_1 \) and some vertices in \( H_2 \), as shown in Fig. 5.

Let \( T_1 \) be the set of vertices in \( H_1 \) which are incident with \( \mathcal{E} \). \( |T_1| < n - 1 \), since \( |\mathcal{E}| < n - 1 \). \( |V(H_1) - T_1| > 1 \). Thus \( T_1 \cup \{u\} \) is a vertex cutset of \( G \) and \( |T_1 \cup \{u\}| \leq n - 1 \), a contradiction to the assumption of this case. So \( \lambda(G - u) \geq n - 1 \). Since \( \lambda(G - u) \leq \delta(G - u) = n - 1 \), we have \( \lambda(G - u) = n - 1 \). Therefore \( G \) is critical with respect to \( \lambda(G) \).

\( G \) is almost regular of degree \( n \), by the proof of Theorem 8, \( |E(G)| = \frac{mn}{2} \), where \( m \) is the order of \( G \). Therefore, \( G \) is a minimum critically \( n \)-edge-connected graph with order \( m \). QED

A vertex \( u \) of a graph \( G \) is called **critical** if \( u \) is contained in a minimum vertex cutset. Thus, we have the following lemma.

**Lemma 12.** A vertex \( u \) in graph \( G \) is critical if and only if \( \kappa(G - u) = \kappa(G) - 1 \).

**Corollary 13.** For any given positive integers \( m, n \), such that \( m \geq 2n + 1 \), \( |V(G)| = m \), and \( \kappa(G) \geq n - 1 \), \( G = (V, E) \) is a minimum critically \( n \)-edge-connected graph if and only if \( G \) is almost regular of degree \( n \), and for any critical vertex \( u \), \( \lambda(G - u) \equiv n - 1 \).

Next, we give some examples to illustrate Theorem 11 and Corollary 13.

**Example 1.** \( G \) is shown in Fig. 6.

\( G \) is almost regular of degree \( 5 \), \( \kappa(G) = 3 \). For any vertex \( u \) in a vertex cutset \( T \) with \( |T| \leq 4 \), \( \lambda(G - u) \geq 4 \). By Theorem 11, \( G \) is a minimum critically 5-edge-connected graph.

**Example 2.** \( G \) is shown in Fig. 7.

\( G \) is almost regular of degree \( 5 \), \( \kappa(G) = 4 \), and for any critical vertex \( u \), \( \lambda(G - u) \equiv 4 \). By Corollary 13, \( G \) is a minimum critically 5-edge-connected graph.

**Example 3.** \( G \) is shown in Fig. 8.
$G$ is almost regular of degree 5, $\kappa(G) = 3$, $u$ is in a vertex cutset $S$, $|S| = 4$, $\lambda(G - u) = 3 < 5 - 1$. By Theorem 11 $G$ is not a minimum critically 5-edge-connected graph. In fact, $G$ is not critical with respect to $\lambda(G)$.

Example 4. $G$ is shown in Fig. 9.

$G$ is almost regular of degree 5, $\kappa(G) = 4$, $u$ is a critical vertex, but $\lambda(G - u) = 3 < 5 - 1$. By Corollary 13, $G$ is not a minimum critically 5-edge-connected graph. In fact, $G$ is not critical with respect to $\lambda(G)$.

**Corollary 14.** For positive integers $m, n, m \geq 2n + 1$, at least one of $n$ or $m$ is even, and $|V(G)| = m$, $G = (V, E)$ is a minimum critically $n$-edge-connected graph if and only if $G$ is regular of degree $n$, and for any vertex $u$ in a vertex cutset $T$ with $|T| \leq n - 1$, $\lambda(G - u) \geq n - 1$.

**Proof.** By Theorem 11, $G$ is a minimum critically $n$-edge-connected graph if and only if $G$ is almost regular of degree $n$, and for any vertex $u$ in a vertex cutset $T$ with $|T| \leq n - 1$, $\lambda(G - u) \geq n - 1$.

Now, suppose that $G$ is not regular of degree $n$, but $G$ is almost regular of degree $n$.

Then $\sum_{v \in V(G)} \deg_G v = n(m - 1) + (n + 1) = nm + 1$ is odd, since $nm$ is even. But $\sum_{v \in V(G)} \deg_G v = 2 \cdot |E|$, so we obtain a contradiction. Conversely, if $G$ is regular of degree $n$, then $G$ is almost regular of degree $n$. QED

4. **NP-completeness.** A problem is in the class NP if some nondeterministic machine could, in every instance, find the answer in a number of steps which is bounded by some fixed polynomial in the length of the input data. A problem is NP-complete if it is in NP, and the existence of a deterministic polynomial algorithm, for it would imply the
existence of a deterministic polynomial algorithm for all NP problems. The proof technique for NP-completeness in this section uses the restriction technique. An NP-completeness proof by restriction for a given problem $Q \in \text{NP}$ consists simply of showing that $Q$ contains a known NP-complete problem $R$ as a special case.

The main problem in this section is as follows:

**Problem $n$-EDGE.**

**Instance:** $G = (V, E)$, a positive integer $n$, $1 < n \leq |V| - 1$.

**Question:** Is there a minimum critically $n$-edge-connected subgraph $G' = (V, E')$ of $G$?

We shall show that Problem $n$-EDGE is NP-complete. To do this, we will use the NP-complete problem, the Hamiltonian Circuit Problem (HC).

**Problem HC.**

**Instance:** Graph $G = (V, E)$.

**Question:** Does $G$ contain a Hamiltonian circuit?

**Lemma 15.** $G' = (V, E')$ is a connected spanning subgraph of $G = (V, E)$ and $G'$ is almost regular of degree 2 if and only if $G'$ is a Hamiltonian circuit of $G$.

Lemma 15 is proved by using the facts that the number of vertices of odd degree for any graph is even, a connected graph with no vertices of odd degree is Eulerian, and an Eulerian circuit in a 2-regular graph must be a Hamiltonian circuit.

There are many polynomial time algorithms for computing the number of components of a graph $G = (V, E)$ including the one given in [8].

Now we consider Problem AR$n$.

**Problem AR$n$.**

**Instance:** $G = (V, E)$, a positive integer $n$, $1 < n \leq |V| - 1$.

**Question:** Is there a spanning connected subgraph $G' = (V, E')$, such that $G'$ is almost regular of degree $n$?

**Theorem 16.** Problem AR$n$ is NP-complete.

**Proof.** First, we prove that Problem AR$n$ is in NP: Given a yes solution (called certificate) to Problem AR$n$, we give a polynomial checking algorithm:

**Certificate:** a subgraph $G'$ of $G$.

**Certificate-Checking Algorithm (Procedure I):**

Begin

1. If $V(G') \neq V(G)$
   
   Then return "No"
   
   Else

2. If $c(G')$ (the number of components of $G'$) $\geq 2$
   
   Then return "No"
   
   Else

End
3. Sort degrees of vertices in $G'$, such that $d_1 \leq d_2 \leq d_3 \leq \cdots \leq d_m$.
4. If $(d_1 = d_2 = d_3 = \cdots = d_{m-1} = n)$ and $(d_m = n \text{ or } d_m = n + 1)$
   Then return "Yes"
   Else return "No";

Step 2 is a polynomial procedure. Step 3 is a sorting procedure, so it also runs in polynomial time. Therefore, the certificate-checking algorithm runs in polynomial time, Problem AR$n$ is in NP.

Let $n = 2$. Problem AR$n$ is reduced to Problem HC by Lemma 15. So a specified type of instance of Problem AR$n$ is NP-complete. By the "restriction technique," Problem AR$n$ is NP-complete. QED

Problem $n$-EDGE-T.
Instance: $G = (V, E)$, a positive integer $n$, $1 < |V|/2 \leq n \leq |V| - 1$.
Question: Is there a minimum critically $n$-edge-connected subgraph $G' = (V, E')$ of $G$?

THEOREM 17. Problem $n$-EDGE-T is NP-complete.
Proof. By Theorem 8, Problem $n$-EDGE-T is the same as Problem AR$n$. So Problem $n$-EDGE-T is NP-complete. QED

Problem MENS (Minimum $n$-edge-connected subgraph).
Instance: $G = (V, E)$ and positive integers $n \leq |V|$ and $b \leq |E|$.
Question: Is there a subset $E' \subseteq E$ with $|E'| \leq b$ such that $G' = (V, E')$ is $n$-edge-connected?

COROLLARY 18. Problem MENS is NP-complete [4].
Therefore, if $G' = (V, E')$ is a certificate, then there is a polynomial time certificate-checking algorithm for Problem MENS, we call it "Procedure II."

THEOREM 19. Problem $n$-EDGE is NP-complete.
Proof. First, we show that Problem $n$-EDGE is in NP.

CERTIFICATE-CHECKING ALGORITHM:
Begin
1. If $G'$ is not a spanning connected subgraph of $G$ or $G'$ is not almost regular of degree $n$—(Call Procedure I)
   Then return "No"
   Else
2. If $G'$ is not $n$-edge-connected—(Call Procedure II)
   Then return "No"
   Else
3. For $i := 1$ to $|V|$.
   Construct $H' = G' - v_i$, $H = G - v_i$;
   If $H'$ is not $(n - 1)$-edge-connected—(Call Procedure II (Instance: $H, n - 1$))
   Then return "No" and go to 5.
   Else go to loop 3;
4. Return "Yes";
5. End.
In step 1, Procedure I runs in polynomial time $P_1$.
In step 2, Procedure II runs in polynomial time $P_2$.
In step 3, the number of computation steps is $O(P^2_2 |V|)$.
Therefore, the certificate-checking algorithm runs in polynomial time, Problem $n$-EDGE is NP.
If we use instance $n, |V|/2 \leq n \leq |V| - 1$, Theorem 17 and the “restriction technique,”
Problem $n$-EDGE is NP-complete. QED
We have shown that the problem of finding a minimum critically $n$-edge-connected
spanning subgraph of $G$ is NP-complete. If we place any restrictions on graph $G$ other
than the ones imposed in Theorems 8, 10, 11 and Corollary 13 does the problem become
easier?
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