ABSTRACT

We show that any complex singular square matrix $T$ is a product of two nilpotent matrices $A$ and $B$ with $\text{rank } A = \text{rank } B = \text{rank } T$ except when $T$ is a $2 \times 2$ nilpotent matrix of rank one.

An $n \times n$ complex matrix $T$ is nilpotent if $T^n = 0$. It is easily seen that a product of finitely many nilpotent matrices must be singular. The purpose of this note is to prove the converse.

**Theorem.** Any complex singular square matrix $T$ which is not $2 \times 2$ nilpotent is a product of two nilpotent matrices with ranks both equal to $\text{rank } T$.

Fong and Sourour [3] considered the product of two quasinilpotent operators on a complex separable Hilbert space. They proved that on an infinite-dimensional space every compact operator is a product of two compact quasinilpotent operators. The preceding theorem gives the finite-dimensional analogue.

We remark that singular square matrices can also be expressed as products of idempotent matrices (cf. [2], [4], and [1]).

To prove our theorem, we start with the following lemma, which was observed in [3].

* Research supported in part by a project grant from the National Science Council of the Republic of China.
Lemma 1. The $2 \times 2$ matrix

\[
\begin{bmatrix}
0 & 0 \\
1 & 0
\end{bmatrix}
\]

is not the product of any two nilpotent matrices.

Proof. Since $2 \times 2$ nilpotent matrices must be of the following forms

\[
a \begin{bmatrix}
1 & x \\
-1/x & -1
\end{bmatrix}, \quad b \begin{bmatrix}
0 & 0 \\
1 & 0
\end{bmatrix}, \quad \text{and} \quad c \begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix},
\]

it is easily seen that the product of any two of them cannot be equal to

\[
\begin{bmatrix}
0 & 0 \\
1 & 0
\end{bmatrix}.
\]

Lemma 2. For any $n \neq 2$, the $n \times n$ matrix

\[
J = \begin{bmatrix}
0 & 0 & \cdots & 0 & 0 & 0 \\
1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 1 & 0
\end{bmatrix}
\]

is the product of two nilpotent matrices with ranks equal to rank $J$.

Proof.

\[
J = \begin{bmatrix}
0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 1 \\
1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0
\end{bmatrix}
\]

for odd $n$ and

\[
J = \begin{bmatrix}
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 1 \ 1 \\
1 & -1 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 1 \\
0 & 0 & \cdots & 0 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 1 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0
\end{bmatrix}
\]

for even $n$. 

for even \( n \) are the required product. [To get some insight into the factorizations above, it is instructive to observe that for odd \( n \) the \( n \)-cycle \((1 2 3 \cdots n)\) is the product of the two \( n \)-cycles \((1 3 5 \cdots n 2 4 6 \cdots n-1)\) and \((1 n n-1 \cdots 3 2)\) and that the factorization of \( J \) in this case can be obtained by replacing appropriate ones by zeros in the corresponding permutation matrices.] The nilpotency of the factors can be verified by showing that their characteristic polynomials are all \( x^n \).

**Lemma 3.** Any \( n \times n \) \((n \neq 2)\) nilpotent matrix \( T \) is the product of two nilpotent matrices with ranks equal to rank \( T \).

**Proof.** Since nilpotency is preserved under the similarity of matrices, we need only consider a nilpotent Jordan matrix. In view of Lemma 2 we may further reduce the problem to the factorizations of \( J_k \oplus J_2 \) \((k \geq 2)\) and \( J_2 \oplus J_2 \oplus J_2 \), where \( J_i \) denotes the nilpotent Jordan block of size \( i \):

\[
J_i = \begin{bmatrix}
0 & & & \\
1 & & & \\
& \ddots & & \\
& & 1 & 0
\end{bmatrix}.
\]

For the former, we have

\[
\begin{bmatrix} J_k & 0 \\ 0 & J_2 \end{bmatrix} = \begin{cases}
\begin{bmatrix} 0 & J_k \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & J_2 \\ 0 & 0 \end{bmatrix} & \text{if } k \text{ is even}, \\
\begin{bmatrix} 0 & J_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & J_2 \\ 0 & 0 \end{bmatrix} & \text{if } k \text{ is odd},
\end{cases}
\]
and for the latter

\[
\begin{bmatrix}
J_2 & 0 & 0 \\
0 & J_2 & 0 \\
0 & 0 & J_2
\end{bmatrix}
= \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
J_2 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
J_2 & 0 & 0
\end{bmatrix}.
\]

Finally, we are ready for the proof of our main theorem.

**Proof of Theorem.** We need only consider Jordan matrices. Let \( J \) be an \( n \times n \) singular Jordan matrix. We will prove by induction on \( n \) that if \( J \) is not \( 2 \times 2 \) nilpotent, then \( j \) is the product of two nilpotent matrices \( J = AB \) such that

(a) the \( i \)th row of \( A \) (\( j \)th column of \( B \)) is zero if and only if the \( i \)th row of \( J \) (\( j \)th column of \( J \)) is zero, \( i, j = 1, 2, \ldots, n \), and

(b) the nonzero rows of \( A \) (nonzero columns of \( B \)) are independent.

In particular, \( \text{rank} \, A \) and \( \text{rank} \, B \) will both equal \( \text{rank} \, J \). Note that the factorizations for nilpotent \( J \) given in Lemmas 2 and 3 do satisfy these properties.

If \( n = 2 \), then

\[
J = \begin{bmatrix}
0 & 0 \\
0 & a
\end{bmatrix} = \begin{bmatrix}
0 & 0 \\
a & 0
\end{bmatrix} \begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix} \quad (a \neq 0),
\]

as asserted. For \( n = 3 \), a nonnilpotent \( J \) can be factored as one of the following:

\[
\begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & a
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 1/a \\
1 & -a & -1
\end{bmatrix} \begin{bmatrix}
a & 0 & a \\
0 & 2 & 0 \\
-a & 0 & -a
\end{bmatrix} \quad (a \neq 0),
\]

\[
\begin{bmatrix}
0 & a & 0 \\
0 & c & b \\
0 & 0 & a
\end{bmatrix} = \begin{bmatrix}
a & 0 & 0 \\
c & b & 0 \\
0 & 0 & a
\end{bmatrix} \begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix} \quad (a, b \neq 0),
\]

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & a
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
a & 0 & 0
\end{bmatrix} \begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} \quad (a \neq 0).
\]
Now assume that our assertion holds for \( n - 1 \) \((n \geq 4)\). Let \( J = J_1 \oplus \cdots \oplus J_m \), where, for \( k = 1,2,\ldots, m \),

\[
J_k = \begin{bmatrix}
  c_k & 0 \\
  1 & 0 \\
  \vdots & \vdots \\
  0 & 1
\end{bmatrix}
\]

is a Jordan block associated with the eigenvalue \( c_k \). We may assume that \( c_m \neq 0 \). Let \( J' \) be the singular \((n - 1) \times (n - 1)\) matrix obtained from \( J \) by deleting its \( n \)th row and \( n \)th column. By the induction hypothesis, \( J' = A'B' \), where \( A' \) and \( B' \) are nilpotent matrices with properties (a) and (b). Let \( a_i^T \) \((b_j)\) denote the \( i \)th row of \( A' \) \((j \)th column of \( B' \)), \( i, j = 1,2,\ldots, n - 1 \). To complete the proof, we need only determine \( a_n^T \) and \( b_n \) such that they are independent of \( a_1^T,\ldots,a_{n-1}^T \) and \( b_1,\ldots,b_{n-1} \), respectively, and satisfy

\[
J = \begin{bmatrix}
J' & 0 \\
0 & d
\end{bmatrix} = \begin{bmatrix}
A' & 0 \\
B' & b_n
\end{bmatrix} = \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix} \equiv AB,
\]

where \( d \) is either 0 or 1. In other words, \( a_n^T \) and \( b_n \) are to satisfy

\[
a_n^T b_j = 0 \quad (1 \leq j \leq n - 2), \quad a_n^T b_{n-1} = d, \quad a_n^T b_n = c_m, \quad (1)
\]

and

\[
a_i^T b_n = 0 \quad (1 \leq i \leq n - 1) \quad (2)
\]

besides the independence property. Note that the \( A \) and \( B \) thus defined are indeed nilpotent with properties (a) and (b).

We first prove the existence of \( b_n \) which satisfies (2) and is independent of \( b_1,\ldots,b_{n-1} \). Indeed, by the induction hypothesis we have \( \text{rank } A' = \text{rank } J' \leq n - 2 \). Hence there exists a nonzero \( b_n \) satisfying (2). If \( b_n \) is dependent on \( b_1,\ldots,b_{n-1} \), say,

\[
b_n = \sum_{j=1}^{n-1} x_j b_j
\]
—where we may assume that \( x_j = 0 \) whenever \( b_j = 0 \), or, equivalently,

\[
b_n = B'x,
\]

where

\[
x = \begin{bmatrix} x_1 & \cdots & x_{n-1} \end{bmatrix}^T
\]

—then

\[
J'x = A'B'x = A'b_n = 0.
\]

From the structure of the Jordan matrix \( J' \), it is easily seen that \( x_j = 0 \) for all \( j \) and therefore \( b_n = 0 \), contradicting our choice of \( b_n \).

On the other hand, since \( \text{rank } B' = \text{rank } J' \leq n - 2 \), the number of nonzero independent \( b_j \)'s is at most \( n - 1 \). Note that \( c_m \neq 0 \) implies that if \( d = 1 \), then the size of \( J_m \) is at least \( 2 \), whence \( b_{n-1} \neq 0 \) by property (a). Hence there exists an \( a_n^T \) satisfying (1). If \( a_n^T \) is dependent on \( a_1^T, \ldots, a_{n-1}^T \), say

\[
a_n^T = \sum_{i=1}^{n-1} y_i a_i^T,
\]

then

\[
c_m = a_n^T b_n = \sum_{i=1}^{n-1} y_i a_i^T b_n = 0,
\]

a contradiction. This completes the proof. \( \blacksquare \)

REFERENCES


Received 14 October 1986; revised 19 January 1987