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Published by: American Mathematical Society
Stable URL: http://www.jstor.org/stable/2000734

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ON THE ELLIPTIC EQUATIONS $\Delta u = K(x)u^\sigma$ AND $\Delta u = K(x)e^{2u}$

KUO-SHUNG CHENG AND JENN-TSANN LIN

ABSTRACT. We give some nonexistence results for the equations $\Delta u = K(x)u^\sigma$ and $\Delta u = K(x)e^{2u}$ for $K(x) > 0$.

1. Introduction. In this paper we study the elliptic equations

\begin{align}
\Delta u &= K(x)u^\sigma \quad \text{in } \mathbb{R}^n \\
\Delta u &= K(x)e^{2u} \quad \text{in } \mathbb{R}^n,
\end{align}

where $\sigma > 1$ is a constant, $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ and $K(\cdot)$ is a bounded Hölder continuous function in $\mathbb{R}^n$. We are concerned with the existence problems of locally bounded and positive solutions for (1.1) and locally bounded solutions for (1.2).

These problems come from geometry. We give a brief description and refer the details to Kazdan and Warner [5] and Ni [13, 14]. Let $(M, g)$ be a Riemannian manifold of dimension $n$, $n \geq 2$, and $K(\cdot)$ be a given function on $M$. We ask the following question: can one find a new metric $g_1$ on $M$ such that $K$ is the scalar curvature of $g_1$ and $g_1$ is conformal to $g$ (i.e., $g_1 = \psi g$ for some function $\psi > 0$ on $M$)? In the case $n \geq 3$, we write $\psi = u^{4/(n-2)}$. Then this problem is equivalent to the problem of finding positive solutions of the equation

$$
\frac{4(n-1)}{n-2} \Delta u - ku + K u^{(n+2)/(n-2)} = 0,
$$

where $\Delta$, $k$ are the Laplacian and scalar curvature in the $g$ metric, respectively. In the case $M = \mathbb{R}^n$ and $g = (\delta_{ij})$, then $k = 0$ and equation (1.3) reduces to (1.1) with $\sigma = (n+2)/(n-2)$, after an appropriate scaling and sign changing of $K(\cdot)$. In the case $n = 2$, we write $\psi = e^{2u}$. Then this problem is equivalent to the problem of finding locally bounded solutions of the equation

$$
\Delta u - k + Ke^{2u} = 0,
$$

where $\Delta$, $k$ are the Laplacian and Gaussian curvature on $M$ in the $g$ metric. In the case $M = \mathbb{R}^2$ and $g = (\delta_{ij})$, we have $k = 0$ and equation (1.4) reduces to (1.2), after a sign changing of $K$. 

Received by the editors July 23, 1986 and, in revised form, December 23, 1986.
1980 Mathematics Subject Classification (1985 Revision). Primary 35J60; Secondary 45G10.
Key words and phrases. Semilinear elliptic equations.

Work of the first author was supported by the National Science Council of the Republic of China under contract NSC75-0208-M009-05.

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0002-9947/87 $1.00 + $.25 per page
In [13 and 14], Ni makes major contributions to the existence of solutions for (1.1) and (1.2). After these two papers, there are many improved results published, such as McOwen [10, 11], Naito [12], Kawano, Kusano and Naito [3], Kawano and Kusano [4], Kusano and Oharu [7], Ding and Ni [1], Kusano, Swanson and Usami [8] and Lin [9].

In this paper, we consider the case \( K(x) > 0 \) in (1.1) and (1.2). We obtain some nonexistence results which make the understanding of the case \( K(x) > 0 \) almost complete. We divide this paper into two parts. In Part I, we consider (1.1). Thus we consider the case (1.1) with \( n \geq 3 \) in §2, (1.1) with \( n = 2 \) in §3 and (1.1) with \( n = 1 \) in §4. We consider (1.2) in Part II. Thus we consider the case (1.2) with \( n \geq 3 \) in §5, (1.2) with \( n = 2 \) in §6 and (1.2) with \( n = 1 \) in §7.

We remark that the technique of the proof of the main nonexistence theorem is essentially equivalent to the proof of Keller [6]. We thank the referee for bringing the reference [6] to our attention.

PART I. \( \Delta u = K(x)u^\sigma \)

2. The case \( n \geq 3 \). In this case, Ni [13] proves the main existence result: Let \( K \) be bounded. If \( |K(x)| \leq C/|x|^{2+\varepsilon} \) at \( \infty \) for some constants \( C > 0 \) and \( \varepsilon > 0 \), then equation (1.1) has infinitely many bounded solutions in \( \mathbb{R}^n \) with positive lower bounds. Later on, Naito [12] improves the result: If \( |K(x)| \leq \varphi(|x|) \) for all \( x \in \mathbb{R}^n \) and \( \int_0^\infty t\varphi(t) \, dt < \infty \), then equation (1.1) has infinite many bounded positive solutions which tend to a positive constant at \( \infty \). On the other hand, when \( K(x) \geq 0 \), Ni [13] proves a nonexistence result: If \( K(x) \geq C/|x|^{2-\varepsilon} \) at \( \infty \) for some constants \( C > 0 \) and \( \varepsilon > 0 \), then (1.1) does not possess any positive solution in \( \mathbb{R}^n \). Lin [9] proves that it is still true even \( \varepsilon = 0 \). In view of Naito’s existence result, we expect that the following conjecture be true.

**Conjecture.** Let \( K(x) \geq \overline{K}(|x|) \geq 0 \) for all \( x \in \mathbb{R}^n \) and \( \int_0^\infty s\overline{K}(s) \, ds = \infty \). Then (1.1) does not possess any positive solution in \( \mathbb{R}^n \).

We give three theorems which almost answer this conjecture completely. Following Ni [13], we define the averages of \( u(x) > 0 \) and \( K(x) \geq 0 \) by \( \overline{u}(r) \) and \( \overline{K}(r) \),

\[
\overline{u}(r) = \frac{1}{\omega_n r^{n-1}} \int_{|x|=r} u(x) \, dS,
\]

\[
\overline{K}(r) = \left( \frac{1}{\omega_n r^{n-1}} \int_{|x|=r} \frac{dS}{K(x)^{\mu/\sigma}} \right)^{-\sigma/\mu},
\]

where \( dS \) denotes the volume element in the surface integral, \( \omega_n \) denotes the surface area of the unit sphere in \( \mathbb{R}^n \) and \( 1/\mu + 1/\sigma = 1 \).

For the sake of completeness, we give another proof of Lin’s result of nonexistence [9] in the following.

**Theorem 2.1.** Let \( K(x) \) be a locally Hölder continuous function. If \( K(x) \geq 0 \) and \( \overline{K}(r) \geq C/r^2 \) for \( r \) large for some constant \( C > 0 \), then equation (1.1) does not possess any positive solution in \( \mathbb{R}^n \).
PROOF. Let $u$ be a positive solution of (1.1) in $\mathbb{R}^n$. Then from Ni [12, Lemma 3.21], we have

$$
\begin{cases}
\tilde{u}''(r) + \frac{n-1}{r} \tilde{u}'(r) \geq \bar{K}(r) \tilde{u}^\sigma(r) & \text{in } (0, \infty), \\
\tilde{u}(0) = \alpha > 0, \quad \tilde{u}'(0) = 0.
\end{cases}
$$

Hence we have

$$
\tilde{u}(r) \geq \alpha + \frac{1}{n-2} \int_0^r s \bar{K}(s) \left[ 1 - \left( \frac{s}{r} \right)^{n-2} \right] \tilde{u}^\sigma(s) \, ds.
$$

Now assume that $\bar{K}(r) \geq C/r^2$ for $r \geq R_0$. Let $r > R_0$. Then from (2.4), we have

$$
\begin{align*}
\tilde{u}(r) & \geq \alpha + \frac{1}{n-2} \int_0^{R_0} s \bar{K}(s) \left[ 1 - \left( \frac{s}{r} \right)^{n-2} \right] \tilde{u}^\sigma(s) \, ds \\
& \quad + \frac{1}{n-2} \int_{R_0}^r s \bar{K}(s) \left[ 1 - \left( \frac{s}{r} \right)^{n-2} \right] \tilde{u}^\sigma(s) \, ds \\
& \geq \alpha + \frac{1}{n-2} \int_{R_0}^r s \bar{K}(s) \left[ 1 - \left( \frac{s}{r} \right)^{n-2} \right] \tilde{u}^\sigma(s) \, ds \\
& \geq \alpha + \frac{\alpha^\sigma}{n-2} \cdot C \left[ 1 - \left( \frac{1}{2} \right)^{n-2} \right] \cdot \int_{R_0}^{r/2} \frac{1}{s} \, ds \\
& \geq C_1 \log r
\end{align*}
$$

for some $C_1 > 0$ and $r \geq R_1 > 2R_0$. For $R > R_1$ and $R \leq s \leq r \leq 2R$, we have

$$
\frac{1}{2} \leq \frac{s}{r} \leq 1.
$$

Hence

$$
\frac{1}{r} \left[ 1 - \left( \frac{s}{r} \right)^{n-2} \right] = \frac{1}{r} \left[ r^{n-2} - s^{n-2} \right] \geq \left( n - 2 \right) \left( \frac{1}{2} \right)^{n-2} (r-s).
$$

From (2.4), (2.5) and (2.7), we obtain

$$
\tilde{u}(r) \geq C_1 \log R + \frac{C_2}{R^2} \int_R^r (r-s) \tilde{u}^\sigma(s) \, ds
$$

for $R > R_1$ and $R \leq r \leq 2R$, where $C_2 > 0$ is a constant. Let

$$
g(r) = C_1 \log R + \frac{C_2}{R^2} \int_R^r (r-s) \tilde{u}^\sigma(s) \, ds.
$$

Then

$$
g(R) = C_1 \log R, \quad g'(R) = 0,
$$

and

$$
g''(r) = \frac{C_2}{R^2} \tilde{u}^\sigma(r) \geq \frac{C_2}{R^2} (g(r))^\sigma.
$$
From (2.10) and (2.11), we have

\[ 2g''(r)g'(r) \geq \frac{2C_2}{R^2} (g(r))^\sigma g'(r), \]

or

\[ \frac{d}{dr} \left( [g'(r)]^2 \right) \geq \frac{2C_2}{R^2} \frac{d}{dr} \left[ \frac{1}{\sigma + 1} g^{\sigma + 1}(r) \right]. \]

Hence

\[ (2.12) \quad [g'(r)]^2 \geq \left( \frac{2C_2}{(\sigma + 1)R^2} \right) [g^{\sigma + 1}(r) - g^{\sigma + 1}(R)]. \]

Let \( \beta = C_1 \log R = g(R) \) and \( \delta = C_2/R^2 \). Then we have

\[ [g'(r)]^2 \geq \frac{2\delta}{\sigma + 1} [g^{\sigma + 1}(r) - \beta^{\sigma + 1}]. \]

Thus

\[ (2.13) \quad \int_B^{g(r)} \frac{dg}{\sqrt{g^{\sigma + 1} - \beta^{\sigma + 1}}} \geq \left( \frac{2\delta}{\sigma + 1} \right)^{1/2} \int_R^r ds. \]

Let \( g(r) = \beta z \), we have

\[ (2.14) \quad \int_1^z \frac{dz'}{\sqrt{(z')^{\sigma + 1} - 1}} \geq \left( \frac{2\delta}{\sigma + 1} \right)^{1/2} \beta^{(\sigma - 1)/2} (r - R). \]

Now if we choose \( R \) so large that

\[ (2.15) \quad \left( \frac{2\delta}{\sigma + 1} \right)^{1/2} \beta^{(\sigma - 1)/2} \cdot R = \left( \frac{2C_2}{(\sigma + 1)R^2} \right)^{1/2} (C_1 \log R)^{(\sigma - 1)/2} \cdot R \]

\[ = \left( \frac{2C_2}{\sigma + 1} \right)^{1/2} (C_1 \log R)^{(\sigma - 1)/2} \]

\[ > \int_1^\infty \frac{dz}{\sqrt{z^{\sigma + 1} - 1}}. \]

Then there is a \( R_e \leq 2R \), such that

\[ (2.16) \quad \lim_{r \to R_e} g(r) = \infty. \]

But \( u(R_e) \geq g(R_e) = \infty \). This is a contradiction. This completes the proof of this theorem.

Now we can state our main nonexistence results.

**Theorem 2.2.** Let \( K(x) \geq 0 \) be a locally Hölder continuous function. If \( \bar{K}(r) \) satisfies

1. there exist \( \alpha > 0, R_0 > 0 \) and \( C > 0 \), such that

\[ \bar{K}(r) \geq C/r^\alpha \quad \text{for } r \geq R_0, \]
(2) there exist \( \varepsilon > 0 \) and \( P > 2 \), such that

\[
\int_{R}^{(P-1)R} r\bar{K}(r) \, dr \geq \varepsilon \quad \text{for} \quad R \geq R_0,
\]

then equation (1.1) does not possess any positive solution in \( \mathbb{R}^n \).

**Proof.** Assume that (1.1) has a positive solution \( u(x) \) in \( \mathbb{R}^n \). Then as in the proof of Theorem 2.1, we have

\[
\bar{u}(r) \geq \alpha + \frac{1}{n-2} \int_{0}^{s\bar{K}(s)} \left[ 1 - \left( \frac{s}{r} \right)^{n-2} \right] \bar{\sigma}(s) \, ds.
\]

From assumption (2), we have

\[
\int_{0}^{\infty} s\bar{K}(s) \, ds = \infty.
\]

Hence

\[
\bar{u}(r) \geq \alpha + C \int_{0}^{r/2} \alpha s\bar{K}(s) \, ds
\]

and

\[
\lim_{r \to \infty} \bar{u}(r) = \infty.
\]

Thus we can choose \( R_0 \) so large that

\[
\bar{u}(R_0) > 1.
\]

Now let \( R > R_0 \). From assumption (2), we have

\[
\bar{u}(PR) \geq \bar{u}(R) + \frac{1}{n-2} \int_{R}^{PR} s\bar{K}(s) \left[ 1 - \left( \frac{s}{PR} \right)^{n-2} \right] \bar{\sigma}(R) \, ds
\]

\[
\geq \bar{u}(R) + \frac{1}{n-2} \cdot \bar{\sigma}(R) \cdot \left[ 1 - \left( \frac{P-1}{P} \right)^{n-2} \right] \cdot \int_{R}^{(P-1)R} s\bar{K}(s) \, ds
\]

\[
\geq \bar{u}(R) + C_1 u_0(R),
\]

where \( 1 > C_1 > 0 \) and \( C_1 \) is a constant.

From (2.20), (2.21) and the fact that \( \sigma > 1 \), we have

\[
\bar{u}(P^mR) \geq (1 + C_1)^m \quad \text{for all} \quad R \geq R_0 \quad \text{and} \quad m \geq 1.
\]

Choose \( \alpha_1 > 0 \) so small that

\[
\log(1 + C_1) \geq \alpha_1 \left[ \log P + \log(PR_0) \right].
\]

Then

\[
m \log(1 + C_1) \geq \alpha_1 \left[ m \log P + \log(PR_0) \right].
\]

Hence \((1 + C_1)^m \geq (P^mR)^{\alpha_1}\) for all \( m \geq 1 \) and \( PR_0 \geq R \geq R_0 \). This means that \( \bar{u}(P^mR) \geq (P^mR)^{\alpha_1}\) for all \( m \geq 1 \) and \( PR_0 \geq R \geq R_0 \). Hence

\[
\bar{u}(r) \geq r^{\alpha_1} \quad \text{for} \quad R \geq R_0.
\]

Now we return to (2.21). We have for \( R \geq R_0 \)

\[
\bar{u}(P^mR) \geq C_1 \bar{u}^{\sigma}(P^{m-1}R) \geq C_1^{(1+\sigma+\cdots+\sigma^{m-1})} \cdot \bar{u}^{\sigma}(R)
\]

\[
= C_1^{(\sigma^m-1)/(\sigma-1)} \cdot \bar{u}^{\sigma}(R), \quad m \geq 1.
\]
Hence
\begin{equation}
\log(u(P^mR)) > \sigma^m \left[ \log u(R) + \frac{1 - 1/\sigma^m}{\sigma - 1} \log C_1 \right] \geq \sigma^m \left[ a_1 \log R - \frac{1}{\sigma - 1} |\log C_1| \right].
\end{equation}

Choose \( C_2 > 0 \) and \( R_1 \) sufficiently large, such that
\begin{equation}
a_1 \log R_1 > \frac{1}{\sigma - 1} |\log C_1| + C_2.
\end{equation}

Then
\begin{equation}
\log(\bar{u}(P^mR)) > C_2 \sigma^m
\end{equation}
for \( R > R_1 \) and \( m > 1. \)

Now we can choose \( \alpha_2 \) sufficiently small, such that
\[
\log \sigma > \alpha_2 (\log P + \log PR_1).
\]

Then
\[
m \log \sigma > \alpha_2 (m \log P + \log PR_1), \quad m > 1.
\]

Hence \( \sigma^m \geq (P^mR)^{\alpha_2} \) for \( m > 1 \) and \( PR_1 > R > R_1. \) Hence from (2.29), we have
\[
\bar{u}(P^mR) > \exp[C_2(P^mR)^{\alpha_2}]
\]
for \( m > 1 \) and \( PR_1 > R > R_1. \) That is,
\begin{equation}
\bar{u}(r) > \exp[C_2 r^{\alpha_2}]
\end{equation}
for \( r > R_1. \) Hence from (2.17), for \( r > R_1, \) we have
\[
\bar{u}(r) > \bar{u}(R_1) + \frac{1}{n - 2} \int_{R_1}^r s \bar{K}(s) \left[ 1 - \left( \frac{s}{r} \right)^{n-2} \right] \bar{u}^\sigma(s) \, ds
\]
\[
= \bar{u}(R_1) + \frac{1}{n - 2} \int_{R_1}^r s \left[ 1 - \left( \frac{s}{r} \right)^{n-2} \right] \left[ \bar{K}(s) \cdot \bar{u}^{(\sigma-1)/2}(s) \right] \bar{u}^{(\sigma+1)/2}(s) \, ds.
\]

Now from (2.30) and the assumption (1), we can choose \( R_2 > R_1 \) so large that
\[
\bar{K}(s) \bar{u}^{(\sigma-1)/2}(s) > C_3 s^{-2}
\]
for \( s > R_2 \) for some constant \( C_3 > 0. \) Hence we have
\begin{equation}
\bar{u}(r) > \bar{u}(R_1) + \frac{1}{n - 2} \int_{R_1}^r s \left[ \bar{K}(s) \cdot \bar{u}^{(\sigma-1)/2}(s) \right] \left[ 1 - \left( \frac{s}{r} \right)^{n-2} \right] \bar{u}^{(\sigma+1)/2}(s) \, ds.
\end{equation}

But from the proof of Theorem 2.1, this is impossible. Hence we complete the proof of this theorem.

**Theorem 2.3.** Let \( K(x) \geq 0 \) be a locally Hölder continuous function. If \( \bar{K}(r) \) satisfies

1. \( \int_0^s s \bar{K}(s) \, ds \) is strictly increasing in \([0, \infty)\) and \( \int_0^\infty s \bar{K}(s) \, ds = \infty, \)
2. \((s/r)^m < \int_0^t \bar{K}(t) \, dt / \int_0^t t \bar{K}(t) \, dt \) for some finite \( m > 0 \) and for all \( r > s > R_0 > 0, \)

then equation (1.1) does not possess any positive solution in \( \mathbb{R}^n. \)
In particular, if $\overline{K}(r)$ satisfies (1) and $0 \leq \overline{K}(r) \leq C/r^2$ for $r \geq R_1$ for some constants $C > 0$ and $R_1 > 0$, then $\overline{K}(r)$ also satisfies (2) and hence (1.1) does not possess any positive solution in $\mathbb{R}^n$.

PROOF. Assume that (1.1) has a positive solution $u(x)$ in $\mathbb{R}^n$. Then as in the proof of Theorem 2.2, we have (2.17). Let

$$f(r) = \int_0^r s\overline{K}(s) \, ds = \eta.$$

Then $f: [0, \infty) \to [0, \infty)$ is one-one and onto. Hence $f^{-1}$ exists and let it be denoted by $g$. Let

$$t = f(s), \quad \eta = f(r), \quad \overline{u}(g(\eta)) = v(\eta).$$

Then from (2.17), we have

$$(2.32) \quad v(\eta) \geq \alpha + \frac{1}{n-2} \int_0^\eta \left[ 1 - \left( \frac{g(t)}{g(\eta)} \right)^{(n-2)/m} \right] v^\alpha(t) \, dt.$$

From the assumption (2), we have

$$(2.33) \quad \frac{g(t)}{g(\eta)} \leq \left( \frac{t}{\eta} \right)^{1/m} \text{ for all } \eta \geq t \geq f(R_0).$$

Hence from (2.32) and (2.33), we have

$$(2.34) \quad v(\eta) \geq \overline{u}(R_0) + \frac{1}{n-2} \int_{f(R_0)}^\eta \left[ 1 - \left( \frac{t}{\eta} \right)^{(n-2)/m} \right] v^\alpha(t) \, dt.$$

But from Theorem 2.1, this is impossible. Hence (1.1) does not possess any positive solution.

If in addition to condition (1), $\overline{K}(r)$ also satisfies $0 \leq \overline{K}(r) \leq C/r^2$ for $r \geq R_1$.

Then we have

$$\frac{d}{dr} \left( \int_0^r t\overline{K}(t) \, dt \right) = \frac{r^2\overline{K}(r) - \int_0^r t\overline{K}(t) \, dt}{r^2} \leq \frac{C - \int_0^r t\overline{K}(t) \, dt}{r^2}.$$

for $r \geq R_1$. Thus we can choose $R_2 \geq R_1$ so large that

$$C - \int_0^r t\overline{K}(t) \, dt < 0 \quad \text{for } r \geq R_2.$$

Hence $\int_0^r t\overline{K}(t) \, dt/r$ is monotonically decreasing for $r \geq R_2$. Thus $\overline{K}(r)$ satisfies condition (2) for $r \geq s \geq R_2$.

This completes the proof of this theorem.

**THEOREM 2.4.** Let $K(x) \geq 0$ be a locally H"{o}lder continuous function in $\mathbb{R}^n$ and $\tilde{K}(t)$ be a locally H"{o}lder continuous function in $[0, \infty)$.

Let the average $\overline{K}(r)$ of $K(x)$ in the sense of (2.2) satisfy:

- $\overline{K}(r) \geq \tilde{K}(r - \beta_i)$ if $\alpha_i + \beta_i \leq r < \alpha_{i+1} + \beta_i$,
- $\overline{K}(r) \geq 0$ if $\alpha_{i+1} + \beta_i < r < \alpha_{i+1} + \beta_{i+1}$.
for $i = 0, 1, 2, \ldots$, where $\{\alpha_i\}_{i=0}^\infty$ is a strictly increasing sequence satisfying $\alpha_0 = 0$ and $\lim_{n \to \infty} \alpha_n = \infty$ and $\{\beta_i\}_{i=0}^\infty$ is a nondecreasing sequence satisfying $\beta_0 = 0$ and $\beta_i / \alpha_i \leq M$ for some constant $M > 0$ and $i = 1, 2, \ldots$. If

$$
\begin{cases}
    u''(r) + \frac{n-1}{r}u'(r) = \bar{K}(r)u^\sigma(r) \quad \text{in } (0, \infty), \\
    u(0) = \alpha > 0, \quad u'(0) = 0
\end{cases}
$$

does not possess any solution in $[0, \infty)$ for all $\alpha > 0$, then (1.1) does not possess any positive solution in $\mathbb{R}^n$.

**Proof.** Assume that (1.1) has a positive solution $u(x)$ in $\mathbb{R}^n$. Then as in the proof of Theorem 2.2, we have

$$(2.36) \quad \bar{u}(r) \geq \alpha + \frac{1}{n-2} \int_0^r s \bar{K}(s) \left[1 - \left(\frac{s}{r}\right)^{n-2}\right] \bar{u}^\sigma(s) \, ds.$$ 

Now we define the function $v$ by

$$(2.37) \quad v(r) = \bar{u}(r + \beta_i) \quad \text{if } \alpha_i \leq r < \alpha_{i+1}$$

for $i = 0, 1, 2, \ldots$. We shall prove that

$$(2.38) \quad v(r) \geq \alpha + \frac{A}{n-2} \int_0^r s \bar{K}(s) \left[1 - \left(\frac{s}{r}\right)^{n-2}\right] \bar{u}^\sigma(s) \, ds,$$

where $A$ is a positive constant depending only on the constant $M$. To prove (2.38), let $\alpha_i \leq r \leq \alpha_{i+1}$. Then from (2.36), we have

$$\bar{u}(r + \beta_i) \geq \alpha + \frac{1}{n-2} \int_0^{r+\beta_i} s \bar{K}(s) \left[1 - \left(\frac{s}{r+\beta_i}\right)^{n-2}\right] \bar{u}^\sigma(s) \, ds$$

$$\geq \alpha + \frac{1}{n-2} \int_0^{\alpha_i+\beta_i} s \bar{K}(s) \left[1 - \left(\frac{s}{r+\beta_i}\right)^{n-2}\right] \bar{u}^\sigma(s) \, ds$$

$$+ \frac{1}{n-2} \int_{\alpha_i+\beta_i}^{r+\beta_i} s \bar{K}(s) \left[1 - \left(\frac{s}{r+\beta_i}\right)^{n-2}\right] \bar{u}^\sigma(s) \, ds$$

$$+ \cdots$$

$$+ \frac{1}{n-2} \int_{r+\beta_i}^{r+\beta_i} s \bar{K}(s) \left[1 - \left(\frac{s}{r+\beta_i}\right)^{n-2}\right] \bar{u}^\sigma(s) \, ds$$

$$= \alpha + \frac{1}{n-2} \int_0^{\alpha_i} s \bar{K}(s) \left[1 - \left(\frac{s}{r+\beta_i}\right)^{n-2}\right] \bar{u}^\sigma(s) \, ds$$

$$+ \frac{1}{n-2} \int_{\alpha_i}^{\alpha_i+\beta_i} (s + \beta_i) \bar{K}(s + \beta_i) \left[1 - \left(\frac{s+\beta_i}{r+\beta_i}\right)^{n-2}\right] \bar{u}^\sigma(s + \beta_i) \, ds$$

$$+ \cdots$$

$$+ \frac{1}{n-2} \int_{r+\beta_i}^{r+\beta_i} (s + \beta_i) \bar{K}(s + \beta_i) \left[1 - \left(\frac{s+\beta_i}{r+\beta_i}\right)^{n-2}\right] \bar{u}^\sigma(s + \beta_i) \, ds.$$
But for \(1 \leq j \leq i\),
\[
1 - \left(\frac{s + \beta_j}{r + \beta_i}\right)^{n-2} \geq 1 - \left(\frac{s + \beta_j}{r + \beta_i}\right)^{n-2} = \frac{(1 + \beta_j/r)^{n-2} - (s/r + \beta_j/r)^{n-2}}{(1 + \beta_j/r)^{n-2}}
\]
\[
\geq \frac{1 - (s/r)^{n-2}}{(1 + \beta_j/\alpha_i)^{n-2}} \geq A\left[1 - (s/r)^{n-2}\right].
\]
Hence we have
\[
\bar{u}(r + \beta_j) \geq \alpha + \frac{A}{n-2} \int_0^r s\tilde{K}(s) \left[1 - \left(\frac{s}{r}\right)^{n-2}\right] \bar{u}\sigma(s) \, ds
\]
\[
+ \frac{A}{n-2} \int_{\alpha_i}^r s\tilde{K}(s) \left[1 - \left(\frac{s}{r}\right)^{n-2}\right] \bar{u}\sigma(s + \beta_j) \, ds
\]
\[
+ \cdots + \frac{A}{n-2} \int_0^r s\tilde{K}(s) \left[1 - \left(\frac{s}{r}\right)^{n-2}\right] \bar{u}\sigma(s + \beta_i) \, ds.
\]
Hence (2.38) is true for all \(r \in [0, \infty)\). Let \(\bar{\alpha} = A^{1/(\alpha - 1)}v\) and \(\alpha = A^{1/(\alpha - 1)}\). Then (2.38) becomes
\[
\bar{\alpha}(r) \geq \alpha + \frac{1}{n-2} \int_0^r s\tilde{K}(s) \left[1 - \left(\frac{s}{r}\right)^{n-2}\right] \bar{\alpha}\sigma(s) \, ds.
\]
Now let \(X\) denote the locally convex space of all continuous functions on \([0, \infty)\) with the usual topology and consider the set
\[
Y = \{ y \in X : \bar{\alpha} \leq y(r) \leq \bar{\alpha}(r) \text{ for } r \geq 0 \},
\]
where \(\bar{\alpha}\) is defined above. Clearly, \(Y\) is a closed convex subset of \(X\). Define the mapping \(T\) by
\[
Ty(r) = \alpha + \frac{1}{n-2} \int_0^r s\tilde{K}(s) \left[1 - \left(\frac{s}{r}\right)^{n-2}\right] y\sigma(s) \, ds.
\]
If \(y \in Y\), then \(\bar{\alpha} \leq y(r) \leq \bar{\alpha}(r)\). Hence we have
\[
Ty(r) = \alpha + \frac{1}{n-2} \int_0^r s\tilde{K}(s) \left[1 - \left(\frac{s}{r}\right)^{n-2}\right] y\sigma(s) \, ds \geq \alpha
\]
and
\[
Ty(r) \leq \alpha + \frac{1}{n-2} \int_0^r s\tilde{K}(s) \left[1 - \left(\frac{s}{r}\right)^{n-2}\right] \bar{\alpha}\sigma(s) \, ds \leq \bar{\alpha}(r).
\]
Thus \(T\) maps \(Y\) into itself. Let \(\{ y_m \}_{m=1}^\infty \subset Y \) be a sequence which converges to \(y\) in \(X\). Then \(\{ y_m \}\) converges uniformly to \(y\) on any compact interval of \([0, \infty)\). Since
\[
|Ty_m(r) - Ty(r)| \leq \frac{1}{n-2} \int_0^r s\tilde{K}(s) \left[1 - \left(\frac{s}{r}\right)^{n-2}\right] |y_m\sigma(s) - y\sigma(s)| \, ds,
\]
we have \(\{Ty_m\}\) converges uniformly to \(Ty\) on any compact interval of \([0, \infty)\). Hence \(T\) is a continuous mapping from \(Y\) into \(Y\). On the other hand, we have
\[
(Ty)'(r) = \int_0^r \left(\frac{s}{r}\right)^{n-1} \tilde{K}(s) y\sigma(s) \, ds.
\]
Hence for any fixed $R > 0$, $TY$ is a uniformly bounded and equicontinuous family of functions defined on $[0, R]$. Hence $TY$ is relatively compact. Thus we can use the Schauder-Tychonoff fixed point theorem (see Edwards [2, p. 161]) to conclude that $T$ has a fixed point $y \in Y$. This fixed point $y$ satisfies the integral equation

$$y(r) = \tilde{a} + \frac{1}{n-2} \int_0^r s \tilde{K}(s) \left[ 1 - \left( \frac{s}{r} \right)^{n-2} \right] y^\sigma(s) \, ds.$$  

Hence (2.35) has a solution for this $\tilde{a}$. This is a contradiction. The theorem is proved. Q.E.D.

3. The case $n = 2$. In this case, we consider only the situation $K(x) \geq 0$ in (1.1). Kawano, Kusano and Naito [3] obtain the following existence result: Let $K(x) \geq 0$ be a locally Hölder continuous function which is positive in some neighborhood of the origin. If

$$K(x) \leq \tilde{K}(|x|) \quad \text{for all } x \in \mathbb{R}^2$$

and

$$\int_1^\infty s (\log s)^\sigma \tilde{K}(s) \, ds < \infty.$$

Then equation (1.1) has infinitely many positive solutions in $\mathbb{R}^2$ with logarithmic growth at infinity.

To our knowledge, there seems no known nonexistence result. Our nonexistence results are

**Theorem 3.1.** Let $K(x) \geq 0$ be a locally Hölder continuous function in $\mathbb{R}^2$. Let the average $\overline{K}(r)$ of $K(x)$ in the sense of (2.2) satisfy

$$\overline{K}(r) \geq C/r^2 (\log r)^{\sigma+1} \quad \text{for } r \geq R_0.$$  

Then equation (1.1) does not possess any positive solution in $\mathbb{R}^2$.

**Proof.** Assume that (1.1) has a positive solution $u(x)$ in $\mathbb{R}^2$. Then we have

$$\begin{aligned}
\tilde{u}''(r) + \frac{u'(r)}{r} \geq \overline{K}(r) \tilde{u}^\sigma(r), \\
u(0) = \alpha > 0, \quad u'(0) = 0,
\end{aligned}$$

where $\tilde{u}$ and $\overline{K}$ are defined in (2.1) and (2.2). From (3.2), $\overline{u}(r)$ satisfies the integral equation

$$\overline{u}(r) \geq \alpha + \int_0^r s \log \left( \frac{r}{s} \right) \overline{K}(s) \tilde{u}^\sigma(s) \, ds.$$  

Without loss of generality, we assume that $K(0) > 0$ and hence $\overline{K}(0) > 0$. Thus we have from (3.3)

$$\begin{aligned}
\overline{u}(r) &\geq \alpha + \int_0^1 s \log \left( \frac{r}{s} \right) \overline{K}(s) \tilde{u}^\sigma(s) \, ds + \int_1^r s \log \left( \frac{r}{s} \right) \overline{K}(s) \tilde{u}^\sigma(s) \, ds \\
&\geq \alpha + \int_0^1 s \log r \overline{K}(s) \tilde{u}^\sigma(s) \, ds \\
&\geq \alpha + \alpha^\sigma \cdot \log r \cdot \int_0^1 s \overline{K}(s) \, ds \\
&\geq \alpha + C_1 \log r
\end{aligned}$$

for $r \geq 1$ and a constant $C_1 > 0$. 


Now consider $r \geq e$. We have
\begin{equation}
\bar{u}(r) \geq \alpha + \int_{e}^{r} s \log \left( \frac{r}{s} \right) \tilde{K}(s) \bar{u}^\sigma(s) \, ds \\
+ \int_{e}^{r} s \log \left( \frac{r}{s} \right) \tilde{K}(s) \bar{u}^\sigma(s) \, ds \\
\geq C_1 \log r + \int_{e}^{r} s \log \left( \frac{r}{s} \right) \tilde{K}(s) \bar{u}^\sigma(s) \, ds.
\end{equation}

Let $v(r) = u(r)/\log r$ for $r \geq e$. Then from (3.5), we have
\begin{equation}
v(r) \geq C_1 + \int_{e}^{r} s \left( 1 - \frac{\log s}{\log r} \right) \tilde{K}(s)(\log s)^{\sigma} \bar{v}^\sigma(s) \, ds.
\end{equation}

Let $t = \log s$, $\eta = \log r$ and $v(e^n) = v(r) = \bar{v}(\eta)$. Then (3.6) becomes
\begin{equation}
\bar{v}(\eta) \geq C_1 + \int_{1}^{n} t \left( 1 - \frac{t}{\eta} \right) e^{2t \tilde{K}(e^t)} t^{(\sigma-1)\bar{v}^\sigma(t)} \, dt.
\end{equation}

Let $\tilde{K}(t) = e^{2t} \tilde{K}(e^t) t^{(\sigma-1)}$. Then from (3.1), we have $\tilde{K}(t) \geq C/t^2$ for $t \geq \exp(R_0)$ and
\begin{equation}
\bar{v}(\eta) \geq C_1 + \int_{1}^{n} t \left( 1 - \frac{t}{\eta} \right) \tilde{K}(t) \bar{v}^\sigma(t) \, dt.
\end{equation}

Using a similar argument as in the proof of Theorem 2.1, we obtain a contradiction. This completes the proof of this theorem. Q.E.D.

**Theorem 3.2.** Let $K(x) \geq 0$ be a locally Hölder continuous function in $\mathbb{R}^2$. Let the average $\tilde{K}(r)$ of $K(x)$ in the sense of (2.2) satisfy
\begin{equation}
\int_{e^R}^{e^{(P-1)R}} s \tilde{K}(s)(\log s)^{\sigma} \, ds \geq \varepsilon \quad \text{for all} \quad R \geq R_0.
\end{equation}

Then equation (1.1) does not possess any positive solution in $\mathbb{R}^2$.

**Proof.** Assume that (1.1) has a positive solution $u(x)$ in $\mathbb{R}^2$. As in the proof of Theorem 3.1, we have (3.3)–(3.7). Hence
\begin{equation}
\bar{v}(\eta) \geq C_1 + \int_{1}^{n} t \left( 1 - \frac{t}{\eta} \right) \tilde{K}(t) \bar{v}^\sigma(t) \, dt.
\end{equation}

But from (3.9) and (3.10), $\tilde{K}(t)$ satisfies
\begin{equation}
\int_{R}^{(P-1)R} t \tilde{K}(t) \, dt \geq \varepsilon \quad \text{for all} \quad R \geq R_0,
\end{equation}
\begin{equation}
\tilde{K}(s) \geq C/t^{(1+\alpha)} \quad \text{for all} \quad t \geq \log R_1.
\end{equation}

Using a similar argument as in the proof of Theorem 2.2, we obtain a contradiction. This completes the proof. Q.E.D.
THEOREM 3.3. Let $K(x) \geq 0$ be a locally Hölder continuous function in $\mathbb{R}^2$. Let the average $\overline{K}(r)$ of $K(x)$ in the sense of (2.2) satisfy

\begin{align}
\int_0^r s\overline{K}(s)(\log s)^\alpha \, ds \text{ is strictly increasing on } [0, \infty) \text{ and} \\
\int_0^\infty s\overline{K}(s)(\log s)^\alpha \, ds = \infty ,
\end{align}

\begin{align}
\left( \frac{\log s}{\log r} \right)^m \leq \int_0^s t\overline{K}(t)(\log t)^\alpha \, dt \int_0^r t\overline{K}(t)(\log t)^\alpha \, dt
\end{align}

for some $m > 0$ and for all $r \geq s \geq R_0 > 0$. Then equation (1.1) does not possess any positive solution in $\mathbb{R}^2$. In particular, if $\overline{K}(r)$ satisfies (3.14) and $0 \leq \overline{K}(r) \leq C/r^2(\log r)^{\alpha+1}$ for $r \geq R_1$ for some constants $C > 0$ and $R_1 > 0$, then $\overline{K}(r)$ also satisfies (3.15) and hence (1.1) does not possess any positive solution in $\mathbb{R}^2$.

PROOF. Assume that (1.1) has a positive solution $u(x)$ in $\mathbb{R}^2$. As in the proof of Theorem 3.1, we have (3.3)–(3.7). Hence we obtain (3.8) or (3.11). But now $\overline{K}(t)$ satisfies

\begin{align}
\int_1^\infty t\overline{K}(t) \, dt \text{ is strictly increasing in } [1, \infty) \text{ and} \\
\int_1^\infty t\overline{K}(t) \, dt = \infty ,
\end{align}

\begin{align}
\left( \frac{s}{\eta} \right)^m \leq \int_1^s t\overline{K}(t) \, dt \int_1^\eta t\overline{K}(t) \, dt
\end{align}

for some $m > 0$ and for all $\eta \geq s \geq \log R_0$.

Using a similar argument as in the proof of Theorem 2.3, we obtain a contradiction. This completes the proof. Q.E.D.

THEOREM 3.4. Let $K(x) \geq 0$ be a locally Hölder continuous function in $\mathbb{R}^2$ and $\overline{K}(t)$ be a locally Hölder continuous function in $[0, \infty)$. Let the average $\overline{K}(r)$ of $K(x)$ in the sense of (2.2) satisfy

\begin{align}
\overline{K}(r) &\geq 0 \text{ if } \alpha_{i+1} + \beta_i < r < \alpha_{i+1} + \beta_{i+1} , \\
\overline{K}(r) &\geq \overline{K}(r-\beta_i) \text{ if } \alpha_i + \beta_i \leq r \leq \alpha_{i+1} + \beta_i
\end{align}

for $i = 0, 1, 2, \ldots$, where $\{ \alpha_i \}_{i=0}^\infty$ is a strictly increasing sequence satisfying $\alpha_0 = 0$ and $\lim_{n \to \infty} \alpha_n = \infty$ and $\{ \beta_i \}_{i=0}^\infty$ is a nondecreasing sequence satisfying $\beta_0 = 0$ and $\beta_i/\alpha_i \leq M$ for some $M > 0$ for all $i \geq 1$. If

\begin{align}
\begin{cases}
u''(r) + u'(r)/r = \overline{K}(r)u^\alpha(r) \text{ in } (0, \infty) , \\ u(0) = \alpha > 0 , \quad u'(0) = 0
\end{cases}
\end{align}

does not possess any solution in $[0, \infty)$ for all $\alpha > 0$, then (1.1) does not possess any positive solution in $\mathbb{R}^2$.

PROOF. The proof is very similar to that of Theorem 2.4. Hence we only sketch the proof. Assume that (1.1) has a positive solution in $\mathbb{R}^2$. Then we have

\begin{align}
\overline{u}(r) \geq \alpha + \int_0^r s \log \left( \frac{r}{s} \right) \overline{K}(s) \overline{u}^\alpha(s) \, ds.
\end{align}
Let
\[ v(r) = \bar{u}(r + \beta_i) \quad \text{if} \quad \alpha_i \leq r < \alpha_{i+1} \]
for \( i = 0, 1, 2, \ldots \). Then
\[ v(r) \geq \alpha + A \cdot \int_0^r s \log \left( \frac{r}{s} \right) \tilde{K}(s) v^o(s) \, ds. \]  

(3.19)

Let \( X \) denote the locally convex space of all continuous function on \([0, \infty)\) with the usual topology and consider the set
\[ Y = \{ y \in X: \tilde{\alpha} \leq y(r) \leq \bar{\beta}(r) \text{ for } r \geq 0 \}. \]

Define the mapping \( T \) by
\[ (Ty)(r) = \tilde{\alpha} + \int_0^r s \log \left( \frac{r}{s} \right) \tilde{K}(s) y^o(s) \, ds. \]  

(3.20)

We can prove that \( TY \subset Y \) and \( T \) is continuous. Furthermore \( TY \) is relatively compact. Hence \( T \) has a fixed point in \( Y \). Thus (3.17) has a solution for this given \( \tilde{\alpha} > 0 \). This is a contradiction. The proof is complete. Q.E.D.

4. The case \( n = 1 \). In this case, we also consider only the situation \( K(x) > 0 \) in (1.1). We give a main existence result which have an extension to the higher-dimensional case. We also give some nonexistence results which may have applications.

**Theorem 4.1.** Let \( K(x) > 0 \) be a Hölder continuous (actually only continuous is sufficient) function in \( \mathbb{R} \). If \( K(0) > 0 \)
\[ \int_{-\infty}^{\infty} |x|^\sigma K(x) \, dx < \infty, \]
then (1.1) has infinitely many positive solutions in \( \mathbb{R} \) with linear growth at \( |x| = \infty \).

**Proof.** We shall seek solutions \( u \) such that \( u(0) = \alpha > 0 \) and \( u'(0) = 0 \). Consider now \( x \geq 0 \). Then equation (1.1) with \( u(0) = \alpha > 0 \) and \( u'(0) = 0 \) is equivalent to the integral equation
\[ u(x) = \alpha + \int_0^x (x - t) K(t) u^o(t) \, dt, \quad x \geq 0. \]  

(4.2)

Now choose \( \alpha \) so small that
\[ 2^\sigma \alpha^{(\sigma - 1)} \int_0^1 K(t) \, dt \leq \frac{1}{2}, \]
\[ 2^\sigma \alpha^{(\sigma - 1)} \int_1^{\infty} K(t) t^\sigma \, dt \leq \frac{1}{2}. \]  

(4.3)

(4.4)

Let
\[ A(x) = \begin{cases} 2\alpha, & \text{if } 0 \leq x \leq 1, \\ 2\alpha x, & \text{if } 1 \leq x. \end{cases} \]

As in the proofs of Theorems 2.4 and 3.4, we let \( X \) denote the locally convex space of all continuous functions on \([0, \infty)\) with the usual topology and consider the set
\[ Y = \{ y \in X: \alpha \leq y(x) \leq A(x) \text{ for } x \geq 0 \}. \]
Clearly, $Y$ is a closed convex subset of $X$. Let the mapping $T$ be defined by

$$ (Ty)(x) = \alpha + \int_0^x (x-t)K(t)y^\sigma(t) \, dt, \quad x \geq 0. $$

If $y \in Y$, then $\alpha \leq y(x) \leq A(x)$. Hence we have

$$ (Ty)(x) = \alpha + \int_0^x (x-t)K(t)y^\sigma(t) \, dt $$
$$ \geq \alpha + \int_0^x (x-t)K(t)\alpha^\sigma dt \geq \alpha. $$

On the other hand, for $0 < x < 1$, we have

$$ (Ty)(x) = \alpha + \int_0^x (x-t)K(t)y^\sigma(t) \, dt $$
$$ \leq \alpha + \int_0^1 K(t)(2\alpha)^\sigma dt $$
$$ = \alpha \left[ 1 + 2\alpha(\sigma-1) \int_0^1 K(t) \, dt \right] $$
$$ \leq \alpha \left[ 1 + \frac{1}{2} + \frac{1}{2} \right] \leq 2\alpha = A(x). $$

For $1 \leq x$, we have

$$ (Ty)(x) = \alpha + \int_0^1 (x-t)K(t)y^\sigma(t) \, dt + \int_1^x (x-t)K(t)y^\sigma(t) \, dt $$
$$ \leq \alpha + x \int_0^1 K(t)(2\alpha)^\sigma dt + x \int_1^\infty K(t)(2\alpha)^\sigma dt $$
$$ \leq \alpha x + \alpha x \left[ 2\alpha(\sigma-1) \int_0^1 K(t) \, dt \right] + \alpha x \left[ 2\alpha(\sigma-1) \int_1^\infty K(t)t^\sigma dt \right] $$
$$ \leq \alpha x \left[ 1 + \frac{1}{2} + \frac{1}{2} \right] \leq 2\alpha x = A(x). $$

Thus $T$ maps $Y$ into itself. Now let $\{y_m\}_{m=1}^\infty \subset Y$ be a sequence which converges to $y$ in $X$. Then $\{y_m\}$ converges uniformly to $y$ on any compact interval of $[0, \infty)$. But

$$ |Ty_m(x) - Ty(x)| \leq \int_0^x (x-t)K(t) |y_m^\sigma(t) - y^\sigma(t)| \, dt, $$

we conclude that $\{Ty_m\}$ converges uniformly to $Ty$ on any compact interval of $[0, \infty)$. Hence $T$ is a continuous mapping from $Y$ into $Y$. As in the proof of Theorem 2.4, the precompactness of $T$ can be verified by

$$ |(Ty)'(x)| \leq \int_0^x K(t)y^\sigma(t) \, dt $$
$$ \leq \int_0^\infty K(t)(2\alpha)^\sigma t^\sigma dt < \infty. $$

Thus $T$ has a fixed point $y \in Y$. This fixed point $y$ is a solution of equation (1.1) for $x \geq 0$ with $y(0) = \alpha$ and $y'(0) = 0$.

Similarly, we can find a solution of equation (1.1) for $x \leq 0$ with $y(0) = \alpha$ and $y'(0) = 0$ if $\alpha$ is sufficiently small. Now let $y(x)$ be the solution of (1.1) in $R$ with
y(0) = α, \ y'(0) = 0. Then
\begin{equation}
2ax \geq y(x) = α + \int_0^x (x - t) K(y) y'^{\sigma}(t) \, dt
\end{equation}
\begin{align*}
\geq α + \int_0^1 (x - 1) K(t) α^{\sigma} \, dt \\
\geq α + k_1(x - 1) \geq k_2x
\end{align*}
for x large. Hence y grows linearly at |x| = ∞. Now we can choose a smaller y(0), such as y(0) = α/2 to obtain another solution. This completes the proof of this theorem. Q.E.D.

We can apply this theorem to the higher-dimensional case as used in Ni [13, 14] and Kawano, Kusano and Naito [3].

**THEOREM 4.2.** Let \( K(x) \geq 0 \) be a locally Hölder continuous function in \( \mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1} \). Let \( \phi_*(x_1) \) and \( \phi^*(x_1) \) be two locally Hölder continuous function in \( \mathbb{R} \). If
\begin{equation}
0 \leq \phi_*(x_1) \leq K(x) \leq \phi^*(x_1)
\end{equation}
for all \( x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1} \),
\begin{equation}
\phi_*(0) > 0 \text{ and } \int_{-\infty}^{\infty} |x_1|^{\sigma} \phi^*(x_1) \, dx_1 < \infty,
\end{equation}
then equation (1.1) has infinitely many positive solutions in \( \mathbb{R}^n \) which are unbounded.

**PROOF.** Consider the equations
\begin{equation}
d^2 v/dx_1^2 = \phi^*(x_1) v^\sigma,
\end{equation}
\begin{equation}
d^2 w/dx_1^2 = \phi_*(x_1) w^\sigma.
\end{equation}
From the proof of Theorem 4.1 we see that (4.14) and (4.15) have unbounded solutions (linear growth at \( \infty \)) \( \bar{v} \) and \( \tilde{w} \). We can choose \( \bar{v} \) and \( \tilde{w} \) such that \( \bar{v}(x_1) \leq \tilde{w}(x_1) \) for all \( x_1 \in \mathbb{R} \). Now let
\begin{equation}
v(x_1, x') = \bar{v}(x_1) \text{ and } w(x_1, x') = \tilde{w}(x_1).
\end{equation}
Then from (4.12), we have
\begin{align*}
\Delta v - K(x) v^\sigma &= \frac{d^2 \bar{v}(x_1)}{dx_1^2} - K(x) \bar{v}^\sigma(x_1) \\
&= [\phi^*(x_1) - K(x)] \bar{v}^\sigma(x_1) \geq 0,
\end{align*}
\begin{align*}
\Delta w - K(x) w^\sigma &= \frac{d^2 \tilde{w}(x_1)}{dx_1^2} - K(x) \tilde{w}^\sigma(x_1) \\
&= [\phi_*(x_1) - K(x)] \tilde{w}^\sigma(x_1) \leq 0
\end{align*}
in \( \mathbb{R}^n \). Hence \( v(x_1, x') \) and \( w(x_1, x') \) are, respectively, a subsolution and a supersolution of (1.1) in \( \mathbb{R}^n \). Since \( v(x_1, x') \leq w(x_1, x') \) in \( \mathbb{R}^n \), from Theorem 2.10 of Ni [13], it follows that (1.1) has a positive solution \( u(x) \) in \( \mathbb{R}^n \) such that \( \bar{v}(x_1) \leq u(x_1, x') \leq \tilde{w}(x_1) \). It is easy to see that \( k_1|x| \leq u(x_1, x') \leq k_2|x| \) for \( |x| \) large for some positive constants \( k_1 \) and \( k_2 \). This completes the proof of the theorem. Q.E.D.
Now let \( u \) be a positive function in \( \mathbb{R} \) and \( K(x) \geq 0 \) in \( \mathbb{R} \). Define for \( r \geq 0 \)
\[
(4.17) \quad \bar{u}(r) = (u(r) + u(-r))/2,
\]
\[
(4.18) \quad \bar{K}(r) = \left[ \frac{1}{2} \left( K(r)^{-\sigma/\alpha} + K(-r)^{-\sigma/\alpha} \right) \right]^{-\sigma/\alpha'},
\]
where \( 1/\alpha + 1/\alpha' = 1 \). It is easy to see that
\[
(4.19) \quad \bar{u}(0) = u(0) \quad \text{and} \quad \bar{u}'(0) = 0
\]
if \( u \) is also continuously differentiable.

**Theorem 4.3.** Let \( K(x) \geq 0 \) be a continuous function in \( \mathbb{R} \). If the average \( \bar{K}(r) \) of \( K(x) \) in the sense (4.18) satisfies
\[
(4.20) \quad \bar{K}(r) \geq C/r^{(\sigma+1)}
\]
for \( r \geq R_0 \) for some constant \( C > 0 \), then equation (1.1) does not possess any positive solution in \( \mathbb{R} \).

**Proof.** Assume that \( u(x) \) is a positive solution of (1.1) in \( \mathbb{R} \). Then we have
\[
(4.21) \quad \bar{u}''(r) = \frac{u''(r) + u''(-r)}{2} = \frac{1}{2} \left[ K(r)u^\sigma(r) + K(-r)u^\sigma(-r) \right].
\]
But
\[
(4.22) \quad \bar{u}(r) = \frac{1}{2} [u(r) + u(-r)]
\]
\[
\leq \left[ \frac{1}{2} \left( K(r)u^\sigma(r) + K(-r)u^\sigma(-r) \right) \right]^{1/\alpha}
\]
\[
\cdot \left[ \frac{1}{2} \left( K^{-\sigma/\alpha}(r) + K^{-\sigma/\alpha}(-r) \right) \right]^{1/\alpha'}.
\]
Hence
\[
(4.23) \quad \frac{1}{2} \left( K(r)u^\sigma(r) + K(-r)u^\sigma(-r) \right) \geq \bar{K}(r)\bar{u}^\sigma(r).
\]
Thus we have
\[
(4.24) \quad \begin{cases}
\bar{u}''(r) \geq \bar{K}(r)\bar{u}^\sigma(r) & \text{for } r > 0, \\
\bar{u}(0) = \alpha > 0, & \bar{u}'(0) = 0.
\end{cases}
\]
Hence \( \bar{u} \) satisfies
\[
(4.25) \quad \bar{u}(r) \geq \alpha + \int_0^r (r-t)\bar{K}(t)\bar{u}^\sigma(t) \, dt.
\]
Without loss of generality, we may assume that \( K(0) > 0 \) and hence \( \bar{K}(0) > 0 \). Thus for \( r \geq 2 \), we have
\[
(4.26) \quad \bar{u}(r) \geq \alpha + \int_0^1 (r-t)\bar{K}(t)\bar{u}^\sigma(t) \, dt + \int_1^r (r-t)\bar{K}(t)\bar{u}^\sigma(t) \, dt
\]
\[
\geq \alpha + \left( \alpha^\sigma \cdot \int_0^1 \left( 1 - \frac{t}{r} \right)\bar{K}(t) \, dt \right) \cdot r + \int_1^r (r-t)\bar{K}(t)\bar{u}^\sigma(t) \, dt
\]
\[
\geq C_1 \cdot r + \int_1^r (r-t)\bar{K}(t)\bar{u}^\sigma(t) \, dt,
\]
where
\[ C_1 = \alpha^a \cdot \int_{0}^{1} \left( 1 - \frac{1}{2} \right) \bar{K}(t) \, dt = \alpha^a \cdot \frac{1}{2} \cdot \int_{0}^{1} \bar{K}(t) \, dt > 0. \]

Now let \( \bar{u}(r) = v(r) \cdot r \) for \( r \geq 2 \). We obtain
\[ (4.27) \quad v(r) \geq C_1 + \int_{1}^{r} \frac{1}{t} \left( 1 - \frac{t}{r} \right) \bar{K}(t) t^{(a-1)} v^a(t) \, dt. \]
Letting \( \tilde{K}(t) = \bar{K}(t) t^{(\sigma-1)} \). Then from (4.20), we have
\[ (4.28) \quad \tilde{K}(t) \geq C/t^2 \quad \text{for } t \geq R_0 \]
and
\[ (4.29) \quad v(r) \geq C_1 + \int_{1}^{r} t \tilde{K}(t) \left( 1 - \frac{t}{r} \right) v^a(t) \, dt. \]
From the proof of Theorem 2.1, we see that it is impossible to have a function \( v \) defined in \([2, \infty)\) satisfying (4.29). This completes the proof. Q.E.D.

**Theorem 4.4.** Let \( K(x) \geq 0 \) be a continuous function in \( \mathbb{R} \). If the average \( \bar{K}(r) \) of \( K(r) \) in the sense (4.18) satisfies
\[ (4.30) \quad \text{there exist } \alpha > 0, R_0 > 0 \text{ and } C > 0 \text{ such that} \]
\[ \bar{K}(r) \geq C/r^{(\sigma+a)} \quad \text{for } r \geq R_0, \]
\[ (4.31) \quad \text{there exist } \varepsilon > 0 \text{ and } P > 2 \text{ such that} \]
\[ \int_{R}^{(P-1)R} r^{p} \bar{K}(r) \, dr \geq \varepsilon \quad \text{for } \mathbb{R} \geq R_0. \]
Then equation (1.1) does not possess any positive solution in \( \mathbb{R} \).

**Proof.** Assume on the contrary that (1.1) has a positive solution \( u(x) \) in \( \mathbb{R} \). Then as in the proof of Theorem 4.3, we have (4.24)-(4.27). But now \( \tilde{K}(r) = r^{(\sigma-1)} \bar{K}(r) \) satisfies
\[ (4.32) \quad \tilde{K}(r) \geq C/r^{(1+a)} \quad \text{for } r \geq R_0, \]
\[ (4.33) \quad \int_{R}^{(P-1)R} r \tilde{K}(r) \, dr \geq \varepsilon \quad \text{for } R \geq R_0. \]
But from the proof of Theorem 2.2, there is no positive function \( v \) satisfying (4.27). This contradiction proves the theorem. Q.E.D.

**Theorem 4.5.** Let \( K(x) \geq 0 \) be a continuous function in \( \mathbb{R} \). Let the average \( \bar{K}(r) \) of \( K(r) \) in the sense (4.18) satisfy
\[ (4.34) \quad \int_{0}^{r} s^{a} \bar{K}(s) \, ds \text{ is strictly increasing in } [0, \infty) \text{ and} \]
\[ \int_{0}^{\infty} s^{a} \bar{K}(s) \, ds = \infty, \]
\[ (4.35) \quad \left( \frac{s}{r} \right)^{m} \leq \int_{0}^{s} t^{a} \bar{K}(t) \, dt / \int_{0}^{r} t^{a} \bar{K}(t) \, dt \text{ for some } m > 0 \text{ and} \]
for all \( r \geq s \geq R_0 > 0 \).
Then equation (1.1) does not possess any positive solution in $\mathbb{R}$. In particular, if $\bar{K}(r)$ satisfies (4.34) and $0 \leq \bar{K}(r) \leq C/r^{(\sigma+1)}$ for $r \geq R_1$ for some constants $C > 0$ and $R_1 > 0$, then $\bar{K}(r)$ also satisfies (4.35) and hence (1.1) does not possess any positive solution in $\mathbb{R}$.

PROOF. Assume on the contrary that (1.1) has a positive solution $u(x)$ in $\mathbb{R}$. Then as in the proof of Theorem 4.3, we have (4.24)–(4.27). Now the function $\bar{K}(r) = r^{(\sigma-1)}\bar{K}(r)$ satisfies the assumptions of Theorem 2.3. Hence there is no positive function $v$ satisfying (4.27). This contradiction proves the theorem. Q.E.D.

THEOREM 4.6. Let $K(x) \geq 0$ be a continuous function in $\mathbb{R}$ and $\bar{K}(r)$ be a continuous function in $[0, \infty)$. Let the average $\bar{K}(r)$ of $K(x)$ in the sense (4.18) satisfy

$$\bar{K}(r) \geq 0 \quad \text{if} \quad \alpha_i + \beta_i < r < \alpha_{i+1} + \beta_{i+1},$$

$$\bar{K}(r) \geq \bar{K}(r - \beta_i) \quad \text{if} \quad \alpha_i + \beta_i \leq r \leq \alpha_{i+1} + \beta_{i+1}$$

for $i = 0, 1, 2, \ldots,$ where $\{\alpha_i\}_{i=0}^\infty$ is a strictly increasing sequence satisfying $\alpha_0 = 0$ and $\lim_{n \to \infty} \alpha_n = \infty$, and $\{\beta_i\}_{i=0}^\infty$ is a nondecreasing sequence satisfying $\beta_0 = 0$ and $\beta_i/\alpha_i \leq M$ for some $M > 0$ and for $i \geq 1$. If

\begin{align*}
\left\{ \begin{array}{l}
\varepsilon''(r) = \bar{K}(r)\varepsilon'(r) \quad \text{in} \quad (0, \infty), \\
\varepsilon(0) = \alpha > 0, \quad \varepsilon'(0) = 0
\end{array} \right.
\end{align*}

does not possess any positive solution in $[0, \infty)$ for all $\alpha > 0$, then (1.1) does not possess any positive solution in $\mathbb{R}$.

PROOF. Assume that (1.1) has a positive solution $u(x)$ in $\mathbb{R}$. Then we have as in the proof of Theorem 4.3,

$$\bar{u}(r) \geq \alpha + \int_0^r (r - t)\bar{K}(t)\bar{u}^\sigma(t) \, dt.$$

Let

$$v(r) = \bar{u}(r + \beta_i) \quad \text{if} \quad \alpha_i \leq r < \alpha_{i+1}$$

for $i = 0, 1, 2, \ldots$. As in the proof of Theorem 2.4, we have

$$v(r) \geq \alpha + \int_0^r (r - t)\bar{K}(t)v^\sigma(t) \, dt.$$

Now we can let $X$ denote the locally convex space of all continuous functions on $[0, \infty)$ with the usual topology and consider the set

$$Y = \{ y \in X : \alpha \leq y(r) \leq v(r) \text{ for } r \geq 0 \},$$

where $v$ is defined in (4.38). Clearly, $Y$ is a closed convex subset of $X$. We define the mapping $T$ by

$$T(y)(r) = \alpha + \int_0^r (r - t)\bar{K}(t)y^\sigma(t) \, dt.$$

Then it is easy to verify that (i) $TY \subset Y$, (ii) $T$ is continuous and (iii) $TY$ is precompact. Hence $T$ has a fixed point in $Y$. Thus (4.36) has a solution for this $\alpha$. This contradiction completes the proof. Q.E.D.
5. The case \( n \geq 3 \). In this case, the existence results are very similar to that of §2. Ni [14] proves that, if \( |K(x)| \leq C/|x_1|^l \) for \( |x_1| \) large and uniformly in \( x_2 \) for some \( l > 2 \), then equation (1.2) possesses infinitely many bounded solutions in \( \mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^{n-m} \), where \( x = (x_1, x_2) \) and \( m \geq 3 \). Later on, Kusano and Oharu [7] extend the result to the case where \( |K(x)| \leq K(|x_1|) \) for all \( x = (x_1, x_2) \in \mathbb{R}^m \times \mathbb{R}^{n-m} \) and \( \int_0^\infty tK(t) \, dt < \infty \). On the other hand, when \( K(x) \geq 0 \) in (1.2), Oleinik [15] shows that if \( K(x) \geq C/|x|^P \) at infinity for some \( P < 2 \), then (1.2) has no solution in \( \mathbb{R}^n \). The case when \( K(x) \) behaves like \( C/|x|^2 \) at infinity is left unsettled for \( n \geq 3 \). In this section, we give several theorems to settle the nonexistence question of (1.2), in particular we settle the case when \( K(x) \) behaves like \( C/|x|^2 \) at infinity.

We need some notations first. Let \( u \) be a smooth function in \( \mathbb{R}^n \) and \( K(x) \geq 0 \) be a continuous function in \( \mathbb{R}^n \). Following Ni [13] and Sattinger [16], we define the averages of \( u \) and \( K \) by \( \bar{u}(r) \) and \( \bar{K}(r) \),

\[
\bar{u}(r) = \frac{1}{\omega_n r^{n-1}} \int_{|x|=r} u(x) \, dS,
\]

\[
\bar{K}(r) = \left( \frac{1}{\omega_n r^{n-1}} \int_{|x|=r} \frac{dS}{K(x)} \right)^{-1}.
\]

We have

**Lemma 5.1.** Let \( u(x) \) be a solution of (1.2) in \( \mathbb{R}^n \) and \( K(x) \geq 0 \). Then \( \bar{u}(r) \) satisfies

\[
\begin{aligned}
&\bar{u}''(r) + \frac{n-1}{r} \bar{u}'(r) \geq \bar{K}(r) e^{2\bar{u}(r)}, \quad r \in (0, \infty), \\
&\bar{u}(0) = u(0), \quad \bar{u}'(0) = 0.
\end{aligned}
\]

**Proof.** From the definition of \( \bar{u} \), we have

\[
\bar{u}'(r) = \frac{1}{\omega_n} \int_{|\xi|=1} \nabla u(r\xi) \cdot \xi \, dS = \frac{1}{\omega_n r^{n-1}} \int_{|x|=r} \sum_i u_{x_i} \xi_i \, dS.
\]

Thus,

\[
\omega_n \left( r^{n-1} \bar{u}'(r) - R^{n-1} \bar{u}'(R) \right) = \int_D \Delta u \, dx = \int_R \left( \int_{|x|=r} \Delta u \, dS \right) \, dt
\]

where \( D = \{ x \in \mathbb{R}^n: R < |x| < r \} \). Hence we have

\[
\omega_n \left( r^{n-1} \bar{u}'(r) \right)' = \int_{|x|=r} \Delta u \, dS = \int_{|x|=r} K(x) \, e^{2u(x)} \, dS.
\]

Now Jensen’s and Cauchy-Schwarz’s inequalities give

\[
e^{2\bar{u}(r)} = \left( e^{\bar{u}(r)} \right)^2 \leq \left( \frac{1}{\omega_n r^{n-1}} \int_{|x|=r} e^{u(x)} \, dS \right)^2 \leq \left( \frac{1}{\omega_n r^{n-1}} \int_{|x|=r} K(x) \, e^{2u(x)} \, dS \right) \left( \frac{1}{\omega_n r^{n-1}} \int_{|x|=r} \frac{dS}{K(x)} \right).
\]
Hence
\[ \frac{1}{\omega_n r^{n-1}} \int_{|x|=r} K(x) e^{2u(x)} dS \geq \overline{K}(r) e^{2\overline{u}(r)}. \]
Combining (5.5) and (5.7), we obtain the first equation of (5.3). \( \overline{u}(0) = u(0) \) and \( \overline{u}'(0) = 0 \) can also be easily obtained. This completes the proof. Q.E.D.

Now we can state our main nonexistence theorems.

**THEOREM 5.1.** Let \( K(x) \geq 0 \) be a locally Hölder continuous function in \( \mathbb{R}^n \). If \( \overline{K}(r) \), as defined in (5.2), satisfies
\[ K(r) \geq C/r^2 \]
for \( r \geq R_0 \) for some constant \( C > 0 \), then equation (1.2) does not possess any locally bounded solution in \( \mathbb{R}^n \).

**PROOF.** Assume that \( u \) is a locally bounded solution of (1.2) in \( \mathbb{R}^n \). Then the average \( \overline{u} \) satisfies (5.3) from Lemma 5.1. Let \( \overline{u}(0) = u(0) = \alpha \). Then \( \overline{u} \) also satisfies
\[ \overline{u}'(r) \geq \int_0^r \left( \frac{s}{r} \right)^{n-1} \overline{K}(s) e^{2\overline{u}(s)} ds, \]
\[ \overline{u}(r) \geq \alpha + \int_0^r s\overline{K}(s) \left[ 1 - \left( \frac{s}{r} \right)^{n-2} \right] e^{2\overline{u}(s)} ds. \]
Hence
\[ \overline{u}(r) \geq \alpha + \frac{1}{n-2} \int_0^{r/2} \overline{K}(s) s \left[ 1 - \left( \frac{1}{2} \right)^{n-2} \right] e^{2\overline{u}(s)} ds, \]
\[ = \alpha + \frac{1}{n-2} \cdot e^{2\alpha} \cdot \left[ 1 - \left( \frac{1}{2} \right)^{n-2} \right] \int_0^{\sqrt{r}/2} \overline{K}(s) ds. \]
Thus there exists a constant \( R_0 \), such that \( \overline{u}(R_0) \geq 1 \). For \( r \geq R_0 \), we have
\[ \overline{u}(r) \geq 1 + \frac{1}{n-2} \int_{R_0}^{r} \overline{K}(s) \left[ 1 - \left( \frac{s}{r} \right)^{n-2} \right] e^{2\overline{u}(s)} ds \]
\[ \geq 1 + \frac{1}{n-2} \int_{R_0}^{r} \overline{K}(s) \left[ 1 - \left( \frac{s}{r} \right)^{n-2} \right] \overline{u}^2(s) ds. \]
In view of (5.8) and the proof of Theorem 2.1, we conclude that no function \( \overline{u} \) can satisfy (5.12) in \([R_0, \infty)\). This completes the proof. Q.E.D.

**THEOREM 5.2.** Let \( K(x) \geq 0 \) be a locally Hölder continuous function in \( \mathbb{R}^n \). If \( \overline{K}(r) \), as defined in (5.2), satisfies
\[ \text{there exist } \alpha > 0, R_0 > 0 \text{ and } C > 0, \text{ such that } \]
\[ \overline{K}(r) \geq C/r^\alpha \text{ for } r \geq R_0, \]
(5.14) 
\[ \text{there exist } \epsilon > 0 \text{ and } P > 2, \text{ such that } \]
\[ \int_{R}^{(P-1)R} r\overline{K}(r) dr \geq \epsilon \text{ for } R \geq R_0, \]
then equation (1.2) does not possess any locally bounded solution in \( \mathbb{R}^n \).
PROOF. Assume that $u$ is a locally bounded solution of (1.2) in $\mathbb{R}^n$. Then as in the proof of Theorem 5.1, we have (5.9)–(5.12). But from (5.13), (5.14) and Theorem 2.2, there is no function $\bar{u}(r)$ defined on $[R_0, \infty)$ satisfying (5.12). This contradiction proves the theorem. Q.E.D.

**THEOREM 5.3.** Let $K(x) \geq 0$ be a locally Hölder continuous function. If $\bar{K}(r)$, as defined in (5.2), satisfies

$$\int_0^r s \bar{K}(s) \, ds \text{ is strictly increasing in } [0, \infty) \text{ and}$$

$$\int_0^\infty s \bar{K}(s) \, ds = \infty,$$

(5.15)

$$\left(\frac{s}{r}\right)^m \leq \int_0^s t \bar{K}(t) \, dt / \int_0^r t \bar{K}(t) \, dt \text{ for some } m > 0 \text{ and}$$

$$\text{for all } r \geq s \geq R_0 > 0.$$  

Then equation (1.2) does not possess any locally bounded solution in $\mathbb{R}^n$. In particular, if $\bar{K}(r)$ satisfies (5.15) and $0 \leq \bar{K}(r) \leq C/r^2$ for $r \geq R_1$ for some constants $C > 0$ and $R_1 > 0$, then $\bar{K}(r)$ also satisfies (5.16) and hence (1.2) does not possess any locally bounded solution in $\mathbb{R}^n$.

**PROOF.** Using the proofs of Theorems 5.1 and 2.3, we can easily obtain a proof. We omit the details. Q.E.D.

**THEOREM 5.4.** Let $K(x) \geq 0$ be a locally Hölder continuous function in $\mathbb{R}^n$ and $\bar{K}(t)$ be a locally Hölder continuous function on $[0, \infty)$. Let the average $\bar{K}(r)$ of $K(x)$ in the sense of (5.2) satisfy

$$\bar{K}(r) \geq 0 \text{ if } \alpha_i+1 + \beta_i < r < \alpha_i+1 + \beta_i+1,$$

$$\bar{K}(r) \geq \bar{K}(r-\beta_i) \text{ if } \alpha_i + \beta_i \leq r \leq \alpha_i+1 + \beta_i,$$

for $i = 0, 1, 2, \ldots$, where $\{\alpha_i\}_{i=0}^\infty$ and $\{\beta_i\}_{i=0}^\infty$ are two sequences satisfying the same conditions as in Theorem 2.4. If

$$u''(r) + \frac{n-1}{r} u'(r) = \bar{K}(r)e^{2u(r)} \text{ in } (0, \infty),$$

$$u(0) = \alpha, \quad u'(0) = 0$$

(5.17)

does not possess any locally bounded solution in $[0, \infty)$ for any real number $\alpha$, then (1.2) does not possess any locally bounded solution in $\mathbb{R}^n$.

**PROOF.** The proof is similar to that of Theorem 2.4. Hence we omit the details.

Q.E.D.

6. The case $n = 2$. In the case $n = 2$ and $K(x) \geq 0$, Ni [14] shows that: If $K(x) \not\equiv 0$ and $K(x) \leq C/|x|^l$ at infinity for some $l > 2$, then for every $\alpha \in (0, \beta)$ where $\beta = \min\{8, (l-2)/3\}$, there exists a solution $u$ of (1.2) such that

$$\log|x|^a - C' \leq u(x) \leq \log|x|^a + C''$$

for $|x|$ large, where $C'$ and $C''$ are two constants.
Later, McOwen [10, 11] improves this result by giving a sharp bound on $\beta$ and sharp behavior of $u$ at infinity. For the nonexistence results, Sattinger [16] proves

Let $K$ be a smooth function on $\mathbb{R}^2$. If $K \geq 0$ on $\mathbb{R}^2$ and $K(x) \geq C/|x|^2$ at infinity, then (1.2) has no solution on $\mathbb{R}^2$. Ni [14] improves Sattinger’s result to include the $K$ such as $K = (1 + \sin r)/r^2$.

In this section, we give an existence result which overlaps parts of the results of Ni [14] and McOwen [10, 11] but with different method. We also give some nonexistence results improving Ni’s result.

**Theorem 6.1.** Let $K(x) \geq 0$ be a locally Hölder continuous function on $\mathbb{R}^2$. Let $K_1(r)$ and $K_2(r)$ be two locally Hölder continuous functions on $[0, \infty)$. If

\begin{align*}
(6.1) & 
K_1(0) > 0, \\
(6.2) & 
0 \leq K_1(|x|) \leq K(x) \leq K_2(|x|) \quad \text{for all } x \in \mathbb{R}^2, \\
(6.3) & 
\text{there exists } \alpha > 0 \text{ such that } \int_0^\infty s^{(1+2\alpha)}K_2(s) \, ds < \infty,
\end{align*}

then (1.2) has infinitely many solutions on $\mathbb{R}^2$ with logarithmic growth at infinity.

**Proof.** Consider the equations

\begin{align*}
(6.4) & 
\Delta v = K_1(|x|) e^{2v}, \quad x \in \mathbb{R}^2, \\
(6.5) & 
\Delta w = K_2(|x|) e^{2w}, \quad x \in \mathbb{R}^2.
\end{align*}

From (6.2), it is easy to see that a solution $v$ of (6.4) is a supersolution of (1.2) and a solution $w$ of (6.5) is a subsolution of (1.2) in $\mathbb{R}^2$. It is natural to seek solutions of $v$ and $w$ depending only on $|x|$. Considser now (6.5). We try to find a solution $w(|x|)$ of (6.5) with $w(0) = \beta$ and $w'(0) = 0$. Then (6.5) is equivalent to the following integral equation

\begin{align*}
(6.6) & 
w(r) = \beta + \int_0^r s \log\left(\frac{r}{s}\right) K_2(s) e^{2w(s)} \, ds.
\end{align*}

Now we choose $0 < \alpha' < \alpha$ and $\beta$ such that

\begin{align*}
(6.7) & 
\int_0^e s \log\left(\frac{e}{s}\right) K_2(s) e^{2(\beta+1)} \, ds < \frac{1}{2}, \\
(6.8) & 
\int_0^e s K_2(s) e^{2(\beta+1)} \, ds < \frac{\alpha'}{2}, \\
(6.9) & 
\int_e^\infty s^{(1+2\alpha')} K_2(s) e^{2(\beta+1)} \, ds < \frac{\alpha'}{2}, \\
(6.10) & 
\int_e^\infty s^{(1+2\alpha')} \log\left(\frac{e}{s}\right) K_2(s) e^{2(\beta+1)} \, ds < \frac{1}{2}.
\end{align*}

Define the function $A_\beta(r)$ by

\begin{align*}
(6.11) & 
A_\beta(r) = (\beta + 1) \quad \text{if } 0 \leq r \leq e, \\
A_\beta(r) = (\beta + 1) + \alpha' \log(r/e) \quad \text{if } e \leq r.
\end{align*}
Now let $X$ denote the locally convex space of all continuous functions on $[0, \infty)$ with the usual topology and consider the set

$$Y = \{ w \in X : \beta \leq w(r) \leq A_\beta(r), \ r \in [0, \infty) \}.$$

It is easy to see that $Y$ is a closed convex subset of $X$. Let $T$ be the mapping

$$T(w)(r) = \beta + \int_0^r s \log \left( \frac{r}{s} \right) K_2(s) e^{2w(s)} ds.$$

We shall prove that $T$ is a continuous mapping from $Y$ into itself such that $TY$ is relatively compact.

First, we verify that $TY \subseteq Y$. Assume $w \in Y$. Hence we have

$$\beta \leq w(r) \leq A_\beta(r) \quad \text{for} \ r \in [0, \infty).$$

It is easy to see that $T(w)$ is also continuous and $\beta \leq T(w)(r)$ for $r \in [0, \infty)$. Now for $0 \leq r \leq e$, we have

$$T(w)(r) = \beta + \int_0^r s \log \left( \frac{r}{s} \right) K_2(s) e^{2w(s)} ds \leq \beta + \int_0^e s \log \left( \frac{e}{s} \right) K_2(s) e^{2(\beta + 1)} ds < (\beta + 1) = A_\beta(r).$$

For $e \leq r$, we have

$$T(w)(r) = \beta + \int_e^r s \log \left( \frac{r}{s} \right) K_2(s) e^{2w(s)} ds + \int_0^e s \log \left( \frac{r}{s} \right) K_2(s) e^{2w(s)} ds \leq \beta + \int_0^e s \log \left( \frac{e}{s} \right) K_2(s) e^{2(\beta + 1)} ds + \int_0^e s \log \left( \frac{e}{s} \right) K_2(s) e^{2(\beta + 1)} ds + \log \left( \frac{r}{e} \right) \int_0^e s K_2(s) e^{2(\beta + 1)} ds + \int_0^e s \log \left( \frac{e}{s} \right) K_2(s) e^{2(\beta + 1)} ds + \log \left( \frac{r}{e} \right) \int_e^\infty s^{1 + 2\alpha'} K_2(s) e^{2(\beta + 1)} ds + \int_e^\infty s^{1 + 2\alpha'} \log \left( \frac{e}{s} \right) K_2(s) e^{2(\beta + 1)} ds < \beta + \frac{\alpha'}{2} \log \left( \frac{r}{e} \right) + \frac{1}{2} + \frac{\alpha'}{2} \log \left( \frac{r}{e} \right) + \frac{1}{2} = (\beta + 1) + \alpha' \log \left( \frac{r}{e} \right) = A_\beta(r).$$

This verifies that $TY \subseteq Y$. 
Now let \( \{ w_m \}_{m=1}^{\infty} \subset Y \) be a sequence converges to \( w \in Y \) in the space \( X \). Then \( \{ w_m \} \) converges to \( w \) uniformly on any compact interval on \([0, \infty)\). Now
\[
|T w_m(r) - T w(r)| \leq \int_0^r s \log \left( \frac{r}{s} \right) K_2(s) |e^{2 w_m(s)} - e^{2 w(s)}| \, ds
\]
But
\[
s \log \left( \frac{r}{s} \right) K_2(s) |e^{2 w_m(s)} - e^{2 w(s)}| \leq s \log \left( \frac{r}{s} \right) K_2(s) \left( e^{2 A_B(s)} - e^{2 \beta} \right)
\]
and \( s \log(r/s) K_2(s) e^{2 A_B(s)} \) is integrable. Hence from (6.17) and the uniform convergence of \( w_m \) to \( w \) on any compact interval, we conclude that \( T w_m \) converges to \( T w \) in \( X \). This verifies that \( T \) is continuous in \( Y \). We can easily compute that
\[
(T w)'(r) = \int_0^r \left( \frac{s}{r} \right) K_2(s) e^{2 w(s)} \, ds
\]
Hence, on any compact interval of \([0, \infty)\), \( T Y \) is uniformly bounded and equicontinuous. This proves that \( T Y \) is relatively compact in \( Y \). We can easily apply the Schauder-Tychonoff fixed point theorem to conclude that \( T \) has a fixed point \( w \) in \( Y \). This fixed point \( w \) is a solution of (6.6) and hence a solution of (6.5). Note that, when we have a solution \( w \) of (6.6) with a given \( \beta \), then we also have a solution \( w \) of (6.6) with \( \beta \) replaced by smaller \( \beta \)'s.

Similarly, we can construct solution \( v(|x|) \) of (6.4) such that \( v(0) = \beta' \) and \( v'(0) = 0 \). For a given \( \beta' \), since \( K_1(0) > 0 \), we can choose \( \beta < \beta' \), such that (6.6) has a solution \( w \) and \( w(r) < v(r) \) for all \( r \in [0, \infty) \). Using Theorem 2.10 of Ni [13], we conclude that (1.2) has a solution \( u(x) \) between \( w(|x|) \) and \( v(|x|) \). Now we can choose another \( \beta' \) smaller than this \( \beta \) to repeat the arguments. This completes the proof of this theorem. Q.E.D.

**Theorem 6.2.** Let \( K(x) \geq 0 \) be a locally Hölder continuous function in \( \mathbb{R}^2 \). If \( \bar{K}(r) \), as defined in (5.2), satisfies
\[
\bar{K}(r) \geq C/r^2 (\log r)^a
\]
for \( r \geq R_0 \), for some constants \( C > 0 \) and \( a > 0 \), then equation (1.2) does not possess any locally bounded solution in \( \mathbb{R}^2 \).

**Proof.** Assume that \( u \) is a locally bounded solution of (1.2) in \( \mathbb{R}^2 \). Then the average \( \bar{u} \) satisfies (5.3) for \( n = 2 \). Letting \( \bar{u}(0) = \beta = u(0) \), we have
\[
\bar{u}'(r) \geq \int_0^r \left( \frac{s}{r} \right) \bar{K}(s) e^{2 u(s)} \, ds,
\]
\[
\bar{u}(r) \geq \beta + \int_0^r s \log \left( \frac{r}{s} \right) \bar{K}(s) e^{2 u(s)} \, ds.
\]
Without loss of generality, we may assume that \( K(0) > 0 \) and hence \( \bar{K}(0) > 0 \). For \( r \geq e \), we have

\[
\bar{u}(r) \geq \beta + \int_0^1 s \log\left( \frac{r}{s} \right) \bar{K}(s) e^{2u(s)} ds + \int_r^1 s \log\left( \frac{r}{s} \right) \bar{K}(s) e^{2u(s)} ds \\
\geq \beta + \int_0^1 s \log r \bar{K}(s) e^{2\beta} ds + \int_1^r s \log\left( \frac{r}{s} \right) \bar{K}(s) e^{2u(s)} ds \\
\geq \beta + C_1 \log r + \int_r^1 s \log\left( \frac{r}{s} \right) \bar{K}(s) e^{2\bar{u}(s)} ds.
\]

Thus there exists a constant \( R_0 \) such that, for \( r \geq R_0 \),

\[
\bar{u}(r) \geq C_2 \log r + \int_r^1 s \log\left( \frac{r}{s} \right) \bar{K}(s) e^{2u(s)} ds \\
\geq C_2 \log r + \int_{R_0}^r s \log\left( \frac{r}{s} \right) \bar{K}(s) e^{2\bar{u}(s)} ds
\]

for some \( C_2 > 0 \). Let

\[
\bar{u}(r) = \frac{1}{2} C_2 \log r + v(r) \quad \text{for} \quad r \geq R_0.
\]

From (6.24), we have

\[
v(r) \geq \frac{1}{2} C_2 \log r + \int_{R_0}^r s \log\left( \frac{r}{s} \right) \bar{K}(s) s^{C_2} e^{2v(s)} ds \\
\geq \frac{1}{2} C_2 \log r + \int_{R_0}^r s \log\left( \frac{r}{s} \right) \bar{K}(s) s^{C_2} v^2(s) ds.
\]

But from assumption (6.20), we have

\[
\bar{K}(s) s^{C_2} \geq C/s^{2-C_2} (\log s)^a \geq C/s^2
\]

for \( s \geq R_1 > R_0 \). Hence from Theorem 3.1, there is no \( v \) in \([R_0, \infty)\) satisfying (6.26). This completes the proof of this theorem.

**Theorem 6.3.** Let \( K(x) \geq 0 \) be a locally Hölder continuous function in \( \mathbb{R}^2 \). If \( \bar{K}(r) \), as defined in (5.2), satisfies

\[
\int_0^r s^{1+a} \bar{K}(s) ds \text{ is monotonically strictly increasing in } [0, \infty) \text{ for all } a > 0.
\]

(6.24)

For given any \( a > 0 \), there exists an \( R_a > 0 \) such that

\[
\left( \frac{\log s}{\log r} \right)^m \leq \int_0^r t^{1+a} \bar{K}(t) dt / \int_0^r t^{1+a} \bar{K}(t) dt
\]

for some \( m > 0 \) and for all \( r \geq s \geq R_a \), then (1.2) does not possess any locally bounded solution in \( \mathbb{R}^2 \).
PROOF. Assume that \( u \) is a locally bounded solution of (1.2) in \( \mathbb{R}^2 \). Then as in the proof of Theorem 6.2, we have (6.21)–(6.26). Now we can let \( w(r) \log r = v(r) \) for \( r \geq R_0 \). Then from (6.26), we have

\[
(6.27) \quad w(r) \geq \frac{1}{2} C_2 + \int_{R_0}^{r} s \left( 1 - \frac{\log s}{\log r} \right) K(s) s^{C_1 v^2(s)} ds.
\]

Now using a similar argument as in the proof of Theorem 3.3, we conclude that there is no function \( w \) satisfying (6.27). This contradiction proves the theorem. Q.E.D.

**Theorem 6.4.** Let \( K(x) \geq 0 \) be a locally Hölder continuous function in \( \mathbb{R}^2 \) and \( \tilde{K}(t) \) be a locally Hölder continuous function on \([0, \infty)\). Let the average \( \overline{K}(r) \) of \( K(x) \) in the sense of (5.2) satisfy the same assumptions as in Theorem 5.4. If

\[
(6.28) \quad \begin{cases} 
  u''(r) + \frac{u'(r)}{r} = \tilde{K}(r)e^{2u(r)} \quad \text{in } (0, \infty), \\
  u(0) = \alpha, \quad u'(0) = 0
\end{cases}
\]

does not possess any locally bounded solution in \([0, \infty)\) for any real number \( \alpha \), then (1.2) does not possess any locally bounded solution in \( \mathbb{R}^2 \).

PROOF. The proof is similar to that of Theorem 2.4. Hence we omit the details. Q.E.D.

7. The case \( n = 1 \). In this case, we consider only the situation \( K(x) \geq 0 \) in (1.2). We give a main existence result which has an extension to the higher-dimensional case. We also give some nonexistence results.

**Theorem 7.1.** Let \( K(x) \geq 0 \) be a Hölder continuous function in \( \mathbb{R} \). If \( K(0) > 0 \) and there exists an \( \alpha > 0 \), such that

\[
(7.1) \quad \int_{-\infty}^{\infty} e^{2\alpha x} K(x) \, dx < \infty,
\]

then (1.2) has infinitely many locally bounded solutions in \( \mathbb{R} \) with linear growth at \( x = 0 \).

**Proof.** We shall seek solution \( u \) such that \( u(0) = \beta \) and \( u'(0) = 0 \). Consider now \( x \geq 0 \). In this situation, (1.2) is equivalent to the integral equation

\[
(7.2) \quad u(x) = \beta + \int_{0}^{x} (x - t) K(t) e^{2u(t)} \, dt, \quad x \geq 0.
\]

Now choose \( \beta \in \mathbb{R} \) so that

\[
(7.3) \quad \int_{0}^{1} K(t) e^{2(\beta + 1)} \, dt \leq \min \left\{ \frac{-1}{2}, 1 \right\},
\]

\[
(7.4) \quad \int_{1}^{\infty} K(t) e^{2(\beta + 1)} \, dt \leq \frac{e^{2\alpha \beta}}{2}.
\]

Let

\[
A(x) = \begin{cases} 
  (\beta + 1) & \text{if } 0 \leq x \leq 1, \\
  (\beta + 1) + \alpha x & \text{if } 1 < x.
\end{cases}
\]
As in the proofs of Theorems 2.4 and 3.4, we let \( X \) denote the locally convex space of all continuous functions on \([0, \infty)\) with the usual topology and consider the set
\[
Y = \{ y \in X : \beta \leq y(x) \leq A(x) \text{ for } x \geq 0 \}.
\]
Clearly, \( Y \) is a closed convex subset of \( X \). Now define the mapping \( T \) by
\[
(Ty)(x) = \beta + \int_0^x (x-t)K(t)e^{2y(t)}dt.
\]
If \( y \in Y \), then \( \beta \leq y(x) \leq A(x) \). Hence we have
\[
(Ty)(x) = \beta + \int_0^x (x-t)K(t)e^{2y(t)}dt \geq \beta.
\]
On the other hand, for \( 0 \leq x \leq 1 \), we have
\[
(Ty)(x) = \beta + \int_0^x (x-t)K(t)e^{2y(t)}dt \leq \beta + 1 = A(x).
\]
For \( 1 < x \), we have
\[
(Ty)(x) = \beta + \int_0^1 (x-t)K(t)e^{2y(t)}dt + \int_1^x (x-t)K(t)e^{2y(t)}dt \\
\leq \beta + x \cdot \int_0^1 K(t)e^{2(\beta+1)}dt + x \cdot \int_1^\infty K(t)e^{2(\beta+1)}dt \\
\leq \beta + \frac{\alpha}{2} \cdot x + \frac{\alpha}{2} x \leq (\beta + 1) + \alpha x = A(x).
\]
Hence \( T \) maps \( Y \) into itself. As in the proofs of Theorems 2.4, 3.4 and 4.1, we can easily verify that \( T \) is continuous and \( TY \) is precompact. Hence \( T \) has a fixed point \( y \in Y \). This fixed point \( y \) is a solution of (1.2) for \( x \geq 0 \) with \( y(0) = \beta \) and \( y'(0) = 0 \).

Similarly, we can find a solution of (1.2) for \( x \leq 0 \) with \( y(0) = \beta \) and \( y'(0) = 0 \) provided that \( \beta \in \mathbb{R} \) is properly selected. It is also easy to see that if \( y \) is a solution of (1.2) with \( y(0) = \beta \) and \( y'(0) = 0 \), then there is also solution \( y \) with \( y(0) = \beta' \) and \( y'(0) = 0 \) provided that \( \beta' < \beta \). The linear growth of solutions at \( |x| = \infty \) can be easily established as in the proof of Theorem 4.1. This completes the proof of this theorem. Q.E.D.

We can apply this theorem to the higher-dimensional case as used in Ni [13, 14] and Kawano, Kusano and Naito [3].

**Theorem 7.2.** Let \( K(x) \geq 0 \) be a locally Hölder continuous function in \( \mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1} \). Let \( \phi_\ast(x_1) \) and \( \phi^\ast(x_1) \) be two locally Hölder continuous function in \( \mathbb{R} \). If
\[
0 \leq \phi_\ast(x_1) \leq K(x) \leq \phi^\ast(x_1) \text{ for all } x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1},
\]
\[
\phi_\ast(0) > 0 \text{ and } \int_{-\infty}^{\infty} e^{2\alpha|x_1|} \phi^\ast(x_1)dx_1 < \infty \text{ for some } \alpha > 0,
\]
then equation (1.2) has infinitely many locally bounded solutions in \( \mathbb{R}^n \).
PROOF. The proof is actually similar to that of Theorem 4.2. We omit the details. Q.E.D.

Now let $u$ be smooth function on $\mathbb{R}$ and $K(x) \geq 0$ be a continuous function on $\mathbb{R}$. We define the averages $\bar{u}$ and $\bar{K}$ by

\begin{align}
\bar{u}(r) &= \frac{1}{2}[u(r) + u(-r)], \quad r \geq 0, \\
\bar{K}(r) &= \left[\frac{1}{2}(K(r)^{-1} + K(-r)^{-1})\right]^{-1}, \quad r \geq 0.
\end{align}

Our nonexistence results are

**Theorem 7.3.** Let $K(x) \geq 0$ be a locally Hölder continuous function on $\mathbb{R}$. If the average $\bar{K}(r)$ of $K(x)$ in the sense of (7.12) satisfies

\begin{equation}
\bar{K}(r) \geq \frac{C}{r^a}
\end{equation}

for $r \geq R_0$ and for some constants $C > 0$, $a > 0$, then equation (1.2) does not possess any locally bounded solution on $\mathbb{R}$.

**Proof.** Assume that $u(x)$ be a solution of (1.2) in $\mathbb{R}$. Then we have

\begin{equation}
\bar{u}''(r) = \frac{1}{2}[u''(r) + u''(-r)]
\end{equation}

But we have

\begin{equation}
e^{2\bar{u}(r)} = (e^{\bar{u}(r)})^2 \leq \left[\frac{1}{2}(e^{u(r)} + e^{u(-r)})\right]^2
\leq \left[\frac{1}{2}(K(r)e^{2u(r)} + K(-r)e^{2u(-r)})\right]
\cdot \left[\frac{1}{2}(K(r)^{-1} + K(-r)^{-1})\right].
\end{equation}

Hence we have

\begin{equation}
\bar{u}''(r) \geq \bar{K}(r)e^{2\bar{u}(r)}, \quad r \geq 0.
\end{equation}

It is also easy to see that $\bar{u}(0) = u(0)$ and $\bar{u}'(0) = 0$. From (7.16), we have

\begin{align}
\bar{u}'(r) &\geq \int_0^r \bar{K}(t)e^{2\bar{u}(t)} dt, \\
\bar{u}(r) &\geq \beta + \int_0^r (r-t)\bar{K}(t)e^{2\bar{u}(t)} dt.
\end{align}

Without loss of generality, we may assume that $K(0) > 0$ and hence $\bar{K}(0) > 0$. For $r \geq 1$, we have

\begin{align}
\bar{u}(r) &\geq \beta + \int_0^1 (r-t)\bar{K}(t)e^{2\bar{u}(t)} dt + \int_1^r (r-t)\bar{K}(t)e^{2\bar{u}(t)} dt \\
&\geq \beta + r\int_0^1 (1-t)\bar{K}(t)e^{2\beta} dt + \int_1^r (r-t)\bar{K}(t)e^{2\bar{u}(t)} dt \\
&\geq 2C_1 \cdot r + \int_{R_1}^r (r-t)\bar{K}(t)e^{2\bar{u}(t)} dt.
\end{align}
for $r \geq R_1 > 1$ and for some $C_1 > 0$. Now let $v(r) = \tilde{u}(r) + C_1 \cdot r$. We have from (7.19)

$$v(r) \geq C_1 \cdot r + \int_{R_1}^{r} (r - t) \tilde{K}(t) e^{2C_1 t} \cdot e^{2u(t)} dt.$$  

Let $v(r) = w(r) \cdot r$, we have

$$w(r) \geq C_1 + \int_{R_1}^{r} \left(1 - \frac{t}{r}\right) \tilde{K}(t) e^{2C_1 t} \cdot e^{2w(t)} dt.$$  

Now let $\tilde{K}(t) = t^{-1} \tilde{K}(t) e^{2C_1 t}$. We have from (7.13)

$$\tilde{K}(t) \geq C/t^2$$  

for $t \geq R_2 > R_1$ for some $C > 0$. But (7.21) becomes

$$w(r) \geq C_1 + \int_{R_1}^{r} t \left(1 - \frac{t}{r}\right) \tilde{K}(t) w(t)^2 dt.$$  

From Theorem 2.1, there is no function $w$ satisfying (7.23). This contradiction proves the theorem. Q.E.D.

**Theorem 7.4.** Let $K(x) \geq 0$ be a locally Hölder continuous function on $\mathbb{R}$. If the average $\overline{K}(r)$ of $K(x)$ in the sense of (7.12) satisfies

$$\int_0^r e^{\alpha \overline{K}(s)} ds \text{ is strictly increasing and } \int_0^\infty e^{\alpha \overline{K}(s)} ds = \infty$$  

for all $\alpha > 0$. For any given $\alpha > 0$, there exists $R_\alpha > 0$, such that

$$\left(\frac{s}{r}\right)^m \leq \int_0^s e^{\alpha \overline{K}(t)} dt / \int_0^r e^{\alpha \overline{K}(t)} dt$$  

for some $m > 0$ and for $r \geq s \geq R_\alpha$, then equation (1.2) does not possess any locally bounded solution in $\mathbb{R}$.

**Proof.** Using the proofs of Theorems 7.3 and 2.3, we can easily prove this theorem. We omit the details. Q.E.D.

**Theorem 7.5.** Let $K(x) \geq 0$ be a locally Hölder continuous function in $\mathbb{R}$ and $\tilde{K}(t)$ be a locally Hölder continuous function in $[0, \infty)$. Let the average $\overline{K}(r)$ of $K(x)$ in the sense of (7.12) satisfy the same assumptions as in Theorem 5.4. If

$$\begin{cases} u''(r) = \tilde{K}(r) e^{2u(r)} & \text{in } (0, \infty), \\ u(0) = \beta, \quad u'(0) = 0 \end{cases}$$  

does not possess any locally bounded solution in $[0, \infty)$ for any real number $\beta$, then equation (1.2) does not possess any locally bounded solution in $\mathbb{R}$.

**Proof.** The proof is quite similar to that of Theorem 2.4. Hence we omit it. Q.E.D.

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