Decomposition of Matrices into Three Involutions

Kang-Man Liu*
Department of Applied Mathematics
National Chiao Tung University
Hsinchu, Taiwan, Republic of China

Submitted by Richard A. Brualdi

ABSTRACT

In this paper, we generalize some results of C. S. Ballantine concerning products of three \( n \times n \) complex involutions. We prove that each \( n \times n \) complex matrix \( A \) with determinant \( \pm 1 \) and \( \dim \ker(A - \alpha) \leq \lfloor n/2 \rfloor \) for all \( \alpha \in \mathbb{C} \) is the product of three involutions. On the other hand, we show that if an \( n \times n \) complex matrix \( A \) is the product of three involutions, then \( m \leq (2n + r)/3 \) and \( m \leq \lfloor 3n/4 \rfloor \), where \( m = \dim \ker(A - \beta) \) and \( r = \dim \ker(A - \beta^{-3}) \) for any \( \beta, \beta \neq 0 \) and \( \beta^4 \neq 1 \). We also completely characterize products of three \( 5 \times 5 \) complex involutions.

0. INTRODUCTION

A square matrix \( A \) over some field is an \textit{involution} if \( A^2 \) is the identity matrix. Wonenburger [9] proved that an \( n \times n \) matrix \( A \) over a field with characteristic \( \neq 2 \) is the product of two involutions if and only if \( A \) is similar to \( A^{-1} \). Djokovic [5] proved it for arbitrary fields. Since then, it has also been proved by other people independently [1, 2, 7]. In [6], Gustafson, Halmos, and Radjavi showed that every \( n \times n \) matrix over a field \( F \) with determinant \( \pm 1 \) is the product of at most four involutions. Moreover, four is the smallest such number. In 1985, Sourour [8] gave a short proof for the special case when \( F \) has at least \( n + 2 \) elements.

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In [3], Ballantine proved that every matrix over an arbitrary field \( F \) with determinant \( \pm 1 \) having no more than two nontrivial invariant factors is the product of three involutions over \( F \). Moreover, he showed that if an \( n \times n \) matrix \( A \) over a field \( F \) is the product of three involutions, then \( m \leq 3n/4 \), where \( m = \dim \ker(A - \beta I) \) for any \( \beta \in F \). He also characterized products of three \( n \times n \) involutions for the special cases when \( n \leq 4 \) or \( F \) has prime order \( \leq 5 \).

In this paper, we generalize all these results for matrices over the complex field \( \mathbb{C} \). More precisely, we prove that each \( n \times n \) complex matrix \( A \) with determinant \( \pm 1 \) and \( \dim \ker(A - \alpha) \leq [n/2] \) for any \( \alpha \in \mathbb{C} \) is the product of three involutions (Theorem 2.5). Moreover, we show that if an \( n \times n \) complex matrix \( A \) is the product of three involutions, then \( m \leq (2n + r)/3 \) and \( m \leq [3n/4] \), where \( m = \dim \ker(A - \beta) \) and \( r = \dim \ker(A - \beta^{-3}) \) for any \( \beta \), \( \beta \neq 0 \) and \( \beta^4 \neq 1 \) (Theorem 3.1). We also completely characterize products of three \( 5 \times 5 \) complex involutions (Theorem 3.2).

1. NOTATION AND PRELIMINARY DEFINITIONS

A matrix is called cyclic if its characteristic and minimal polynomials coincide. By an elementary Jordan matrix \( J_k(\lambda) \) is meant a square matrix of size \( k \) of the form

\[
\begin{bmatrix}
\lambda & & \\
1 & \lambda & \\
& \ddots & \ddots \\
& & 1 & \lambda
\end{bmatrix}.
\]

Let \( \text{tr}(A) \) denote the trace of \( A \), and \( \sigma(A) \) denote the set of all eigenvalues of a matrix \( A \). Denote by \( I \) the identity matrix, by \( I_n \) the \( n \times n \) identity matrix, and by \( 0_n \) the \( n \times n \) zero matrix. Denote by \( T(n) \) the set of all \( n \times n \) complex involutions, and by \( T(n)^k \) the set of all matrices which are products of \( k \) matrices from \( T(n) \).

For complex matrices, Djokovic [5] proved the following theorem, which is also our main tool in proving results for products of three involutions.

**Theorem 1.1.** Let \( A \) be a complex invertible matrix. Assume that \( A \) is similar to \( \sum_i \oplus J_{k_i}(\lambda_i) \), where each \( J_{k_i}(\lambda_i) \) is an elementary Jordan matrix.
belonging to \( \lambda_i \) of size \( k_i \). Then the following are equivalent:

1. \( A \) is similar to \( A^{-1} \);
2.除那些\( f_k(\lambda_i) \)与\( \lambda_i = \pm 1 \), 其余的所有都是成对的\( f_k(\lambda_j) \)和\( f_k(\lambda_j) \) 使得\( k_j = k_i \) 和\( \lambda_j \lambda_l = 1 \);
3. \( A \) is the product of two involutions.

2. SUFFICIENT CONDITIONS

Our main result in this section is Theorem 2.5, which gives a sufficient condition for a complex matrix expressible as the product of three involutions and generalizes a sufficient condition in [3]. To prove this theorem, we need the following lemmas.

**Lemma 2.1.** Let \( T \) be an invertible cyclic matrix of order \( n \). If \( \alpha_1, \alpha_2, \ldots, \alpha_n \) are complex numbers satisfying \( \alpha_1 \alpha_2 \cdots \alpha_n = -\det T \), then there exist an involution \( P \) and a cyclic \( B \) with \( \sigma(B) = \{ \alpha_1, \alpha_2, \ldots, \alpha_n \} \) such that \( T = PB \).

**Proof.** Since \( T \) is cyclic, \( T = STS^{-1} \), where \( S \) is invertible and

\[
T_1 = \begin{bmatrix}
0 & a_0 \\
I_{n-1} & a_1 \\
& \ddots \\
& & a_{n-1}
\end{bmatrix}.
\]

Let \( \{ \alpha_1, \alpha_2, \ldots, \alpha_{n-1}, \alpha_n \} \) be the roots of the polynomial equation

\[
\lambda^n + d_{k-1} \lambda^{n-1} + \cdots + d_2 \lambda^2 + d_1 \lambda + a_0 = 0.
\]

If

\[
B_1 = \begin{bmatrix}
0 & -a_0 \\
I_{n-1} & -d_1 \\
& \ddots \\
& & -d_{n-1}
\end{bmatrix},
\]
then $B_1$ is cyclic, $\sigma(B_1) = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and $\det B_1 = \alpha_1 \alpha_2 \cdots \alpha_n = -\det T$.

Let

$$P_1 = \begin{bmatrix}
-1 & 0 \\
\begin{array}{c}
x_1 \\
x_2 \\
\vdots \\
x_{n-1}
\end{array} & I_{n-1}
\end{bmatrix},$$

where $x_i = -a_0^{-1}(d_i + a_i)$, $1 \leq i \leq n - 1$. Note that $P_1$ is an involution and

$$P_1 T_1 = \begin{bmatrix}
-1 & 0 \\
\begin{array}{c}
x_1 \\
x_2 \\
\vdots \\
x_{n-1}
\end{array} & I_{n-1}
\end{bmatrix} \begin{bmatrix}
0 & a_0 \\
\begin{array}{c}
0 \\
I_{n-1} \\
\vdots \\
I_{n-1}
\end{array} & \begin{array}{c}
a_1 \\
a_2 \\
\vdots \\
a_{n-1}
\end{array}
\end{bmatrix} = \begin{bmatrix}
0 & -a_0 \\
\begin{array}{c}
0 \\
I_{n-1} \\
\vdots \\
I_{n-1}
\end{array} & \begin{array}{c}
a_0 x_1 + a_1 \\
a_0 x_2 + a_2 \\
\vdots \\
a_0 x_{n-1} + a_{n-1}
\end{array}
\end{bmatrix} = \begin{bmatrix}
0 & -a_0 \\
\begin{array}{c}
0 \\
I_{n-1} \\
\vdots \\
I_{n-1}
\end{array} & \begin{array}{c}
d_1 \\
d_2 \\
\vdots \\
d_{n-1}
\end{array}
\end{bmatrix} = B_1.$$

Hence $T_1 = P_1 B_1$. Let $P = S P_1 S^{-1}$ and $B = S B_1 S^{-1}$. We have $T = S T_1 S^{-1} = S P_1 S^{-1} S B_1 S^{-1} = P B$ and $\sigma(B) = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$.

**Lemma 2.2.** Let $A$ be an $n \times n$ complex matrix with determinant $\pm 1$. If $A = A_1 \oplus A_2 \oplus \cdots \oplus A_m$, where each $A_i$ is a square matrix of order $l_i$ at least 2 and each $A_i$ is cyclic, then $A$ is the product of three involutions.
Proof. We distinguish two cases. Let \( a_i = -\det A_i, 1 \leq i \leq m \), and \( \alpha \) be sufficiently large, say, \( \alpha > h^m \), where \( h = \max_{1 \leq i \leq m} |a_i| + 1 \).

Case 1: \( m \) is odd and \( \det A = -1 \), or \( m \) is even and \( \det A = 1 \).

Applying Lemma 2.1, we obtain \( m \) involutions \( P_i \) such that \( \sigma(P_iA_i) = \{ \alpha, \alpha^{-1}a_1, 1, \ldots, 1 \} \) and \( \sigma(P_iA_i) = \{ \alpha(a_1 \cdots a_{i-1})^{-1}, \alpha^{-1}(a_1a_2 \cdots a_i), 1, \ldots, 1 \} \), \( 2 \leq i \leq m \), where the 1's may be absent. Let \( P = P_1 \oplus P_2 \oplus \cdots \oplus P_m \). Then \( P \) is an involution and \( PA = P_1A_1 \oplus P_2A_2 \oplus \cdots \oplus P_mA_m \).

Note that by our choice of \( \alpha \) the eigenvalues of \( PA \) except the 1's are pairwise reciprocal and distinct. Hence \( PA \) is the product of two involutions by Theorem 1.1, and \( A \) is the product of three involutions.

Case 2: \( m \) is odd and \( \det A = 1 \), or \( m \) is even and \( \det A = -1 \).

(I) If there exists some \( A_i \), with \( l_i \geq 3 \), say \( l_1 \geq 3 \), by the same method as in Case 1, there exist \( m \) involutions \( P_i \) such that \( \sigma(P_iA_i) = \{ -\alpha, \alpha^{-1}a_1, -1, 1, \ldots, 1 \} \) and \( \sigma(P_iA_i) = \{ \alpha(a_1 \cdots a_{i-1})^{-1}, \alpha^{-1}(a_1a_2 \cdots a_i), 1, \ldots, 1 \} \), \( 2 \leq i \leq m \), where the 1's may be absent. As before, \( A \) is the product of three involutions.

(II) If \( A \) is not as in (I), then \( l_i = 2 \) for \( i = 1, 2, \ldots, m \). Again, we consider two cases:

1. If \( a_i \neq a_j \) for some \( i \neq j \), say \( a_1 \neq a_2 \), then \( A_1 \oplus A_2 \) is similar to either \( B_2 \) or \( B_1 \oplus \beta \), where \( B_2 \) is cyclic, \( B_1 \) is a \( 3 \times 3 \) cyclic matrix, and \( \beta \) is a scalar.

   If \( A_1 \oplus A_2 \) is similar to \( B_2 \), then \( A \) is similar to \( B_2 \oplus A_3 \oplus \cdots \oplus A_m \), which reduce to Case 1.

   If \( A_1 \oplus A_2 \) is similar to \( B_1 \oplus \beta \), then choose \( \delta \) such that \( \beta, \beta^{-1} \), \( -\delta, \delta^{-1}a_1a_2 \) are distinct and \( \delta \geq \alpha \). Applying Lemma 2.1, we obtain \( m \) involutions \( P_i \) such that \( \sigma(P_iB_i) = \{ \beta^{-1}, -\delta, \delta^{-1}a_1a_2 \} \), \( \sigma(P_2\beta) = \{ \beta \} \), and \( \sigma(P_3A_i) = \{ \sigma(a_1a_2 \cdots a_{i-1})^{-1}, \sigma^{-1}(a_1a_2 \cdots a_i), 1, \ldots, 1 \} \), \( 3 \leq i \leq m \), where the 1's may be absent. Using the same technique as in Case 1, we can prove this case.

2. If all \( a_i \)'s are equal, say \( a_i = a \) for all \( i \), then \( a^m = -1 \). By Lemma 2.1, there exist \( m \) involutions \( P_i \) such that

\[
\sigma(P_iA_i) = \{ a^{1-i}, a^i \}, \quad 1 \leq i \leq m \quad \text{and} \quad a^{2i-1} \neq 1.
\]

Let \( P = P_1 \oplus P_2 \oplus \cdots \oplus P_m \). By the same method as in Case 1, the proof is complete.

The main idea of constructing a new basis in the proofs of Lemmas 2.3 and 2.4 comes from [4].
Lemma 2.3. Let $A_1$ be a complex invertible matrix of order $2k$. If $A_1 = B_1 \oplus D_1$, where $B_1$ is a square matrix of order $k \geq 2$. 

\[ B_1 = \begin{bmatrix} \beta & & & \\ 1 & \beta & & \\ & 1 & \ddots & \\ & & \ddots & 1 & \beta \end{bmatrix}, \]

and $D_1 = \alpha I_k$ is a $k \times k$ scalar matrix, $\alpha \neq \beta$, then, for any $\alpha_1, \ldots, \alpha_k$, $\beta_1, \ldots, \beta_k$ satisfying $\alpha_i \beta_i = -\alpha \beta$, $1 \leq i \leq k$, there exist an involution $P$ and a $2k \times 2k$ matrix $C$ with $\sigma(C) = \{ \alpha_1, \alpha_2, \ldots, \alpha_k, \beta_1, \beta_2, \ldots, \beta_k \}$ and $PA_1 = C$.

Proof. Let $A = A_1 - \beta I_{2k}$ and $\gamma = \alpha - \beta$. Then

\[ A = \begin{bmatrix} 0 & & & 0 \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \\ 0 & & & \gamma I_k \end{bmatrix}. \]

Let $e_i = \langle \delta_{1i}, \delta_{2i}, \ldots, \delta_{ni} \rangle$, $1 \leq i \leq n$, where $n = 2k$ and

\[ \delta_{ik} = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}, \]

and let $f = e_1 + e_2 + \cdots + e_k$. Then the vectors $f, Af, \ldots, A^{k-1}f$ are linearly independent. Let $e'_i = A^{-1}f + e_{k+i}$ and $e'_{k+i} = Af + \gamma e_{k+i}$ for $1 \leq i \leq k$. Then

\[ e'_{i+1} - e'_{k+i} = A^2f + e_{k+i+1} - (Af + \gamma e_{k+i}) \]

\[ = e_{k+i+1} - \gamma e_{k+i} \quad (\ast) \]

for $i = 1, 2, \ldots, k - 1$, and $Ae'_i = e_{k+i}$, $1 \leq i \leq k$. 

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Since the set \( \{ f, A f, \ldots, A^{k-1} f, e_{k+1}, e_{k+2}, \ldots, e_{2k} \} \) forms a basis, so does the set \( D = \{ e_1', e_2', \ldots, e_k', Ae_1', Ae_2', \ldots, Ae_k' \} \). Moreover,

\[
Ae_{k+i}' = A^{i+1} f + \gamma^2 e_{k+i} \\
= e_{k+i+1}' - \gamma e_{k+i+1} + \gamma^2 e_{k+i} \\
= e_{k+i+1}' - \gamma(e_{k+i+1} - \gamma e_{k+i}) \\
= e_{k+i+1}' - \gamma(e_i' - e_{k+i}) \quad \text{[by (**)]} \\
= -\gamma e_{i+1} + e_{k+i+1} + \gamma e_{k+i}, \quad 1 \leq i \leq k - 1,
\]

and

\[
Ae_{2k}' = A(\gamma e_{2k}) = \gamma^2 e_{2k} = \gamma e_{2k}'.
\]

Relative to the basis \( D \), the matrix representation of \( A \) is of the form

\[
A_2 = \begin{bmatrix} 0 & E_1 \\ I_k & E_2 \end{bmatrix},
\]

where

\[
E_1 = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ -\gamma & 0 & \cdots & 0 \\ & \ddots & \ddots & \ddots \\ & & -\gamma & 0 \end{bmatrix} \quad \text{and} \quad E_2 = \begin{bmatrix} \gamma & \gamma & \cdots & \gamma \\ 1 & \gamma & \cdots & \gamma \\ \vdots & \ddots & \ddots & \ddots \\ \gamma & \gamma & \cdots & 1 \end{bmatrix}
\]

are both of size \( k \).

We conclude that \( A \) is similar to \( A_2 \). Let

\[
S = \begin{bmatrix} I_k & -\beta I_k \\ 0 & I_k \end{bmatrix}.
\]
Then

$$SA_2S^{-1} = \begin{bmatrix} I_k & -\beta I_k \\ 0 & I_k \end{bmatrix} \begin{bmatrix} 0 & E_1 \\ I_k & E_2 \end{bmatrix} \begin{bmatrix} 0 & \beta I_k \\ I_k & I_k \end{bmatrix}$$

$$= \begin{bmatrix} -\beta I_k & -\beta^2 I_k + E_1 - \beta E_2 \\ I_k & \beta I_k + E_2 \end{bmatrix}.$$ 

Hence $A_1 = \Lambda + \beta I_{2k}$ is similar to the matrix

$$A_3 = \begin{bmatrix} 0 & -\beta^2 I_k + E_1 - \beta E_2 \\ I_k & 2\beta I_k + E_2 \end{bmatrix},$$

say, $A_1 = VA_3V^{-1}$. Let $E_3 = -\beta^2 I_k + E_1 - \beta E_2$ and $E_4 = 2\beta I_k + E_2$. Then

$$E_3 = \begin{bmatrix} -\alpha \beta \\ -\alpha & -\alpha \beta \\ & \ddots & \ddots \\ & & -\alpha & -\alpha \beta \end{bmatrix},$$

$$E_4 = \begin{bmatrix} \alpha + \beta \\ 1 & \alpha + \beta \\ & \ddots & \ddots \\ & & 1 & \alpha + \beta \end{bmatrix},$$

and

$$\Lambda_3 = \begin{bmatrix} 0 & E_3 \\ I_k & E_4 \end{bmatrix}.$$ 

If

$$P_1 = \begin{bmatrix} -I_k & 0 \\ (Q - E_4)E_3^{-1} & I_k \end{bmatrix},$$
where \( Q \) is an arbitrary \( k \times k \) matrix, then \( P_1 \) is an involution and

\[
P_1 A_3 = \begin{bmatrix} 0 & -E_3 \\ I_k & Q \end{bmatrix}.
\]

If \( \lambda \neq 0 \), we have

\[
P_1 A_3 - \lambda I_{2k} = \begin{bmatrix} -\lambda I_k & -E_3 \\ I_k & Q - \lambda I_k \end{bmatrix} = \begin{bmatrix} -\lambda I_k & 0 \\ I_k & \lambda^{-1} E_3 \end{bmatrix}.
\]

where \( E_5 = Q - \lambda I_k - \lambda^{-1} E_3 \). If we choose

\[
Q = \begin{bmatrix} c_1 \\ & c_2 \\ & & \ddots \\ & & & c_k \end{bmatrix},
\]

where \( c_i = \alpha_i + \beta_i, \ 1 \leq i \leq k \). then

\[
E_5 = \begin{bmatrix} d_1 \\ -\alpha_1 & d_2 \\ & -\alpha & \ddots \\ & & \ddots & \ddots \\ & & & -\alpha & d_k \end{bmatrix},
\]

where \( d_i = c_i - \lambda + \lambda^{-1}\alpha \beta \). Since

\[
\det \begin{bmatrix} I_k & \lambda^{-1} E_3 \\ 0 & E_5 \end{bmatrix} = \det E_5,
\]

we have \( \sigma(P_1 A_3) = \{ \alpha_1, \alpha_2, \ldots, \alpha_k, \beta_1, \beta_2, \ldots, \beta_k \} \).

Let \( P = VP_1 V^{-1} \) and \( PA_1 = C \). Then \( PA_1 = VP_1 V^{-1} V A_3 V = VP_1 A_3 V^{-1} \)
and \( \sigma(PA_1) = \sigma(P_1 A_3) \). The proof is complete.
LEMMA 2.4. Let $A_1$ be a $2n \times 2n$ complex invertible matrix. If $A_1 = B \oplus D$, where $D$ is a scalar matrix $\alpha I_m$ of order $m$, and

$$B = \begin{bmatrix}
\alpha & 0 & \cdots & 0 \\
0 & \alpha & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \alpha
\end{bmatrix}$$

is a square matrix of order $l \geq 3$, $l - m \geq 2$, then, for any $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ satisfying $\alpha_i \beta_i = -\alpha_i^2$, $1 \leq i \leq n$, there exist an involution $P$ and a $2n \times 2n$ matrix $C$ with $\sigma(C) = \{ \alpha_1, \alpha_2, \ldots, \alpha_n, \beta_1, \beta_2, \ldots, \beta_n \}$ and $PA_1 = C$. Moreover, we may choose some $\lambda_1 \in \sigma(C)$ such that $2 \leq \dim \ker(C - \lambda_1) \leq n$.

Proof. Let $A = A_1 - \alpha I_{2n}$. Then

$$A = \begin{bmatrix}
0 & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{bmatrix}.$$ 

Let $e_i = (\delta_{1,i}, \delta_{2,i}, \ldots, \delta_{2n,i})^t$, $1 \leq i \leq 2n$, and $f = e_1 + e_2 + \cdots + e_l$. Then the vectors $f, Af, \ldots, A^{l-1}f$ are linearly independent.

Let $k = n - m$, $A^0f = f$, $e'_i = A^{2i-1}f$ if $1 \leq i \leq k$, and $e''_{1+i} = A^{2k-2+i}f + e'_{1+i}$ if $1 \leq i \leq m$. Let $e''_{n+i} = Ae'_i$, $1 \leq i \leq n$. Then $e''_{n+i} = A^{2i-1}f$ if $1 \leq i \leq k$, and $e''_{n+k+i} = A^{2k-1+i}f$ if $1 \leq i \leq m$. Since the set of vectors \{ $f, Af, \ldots, A^{l-1}f, e_{l+1}, e_{l+2}, \ldots, e_{2n}$ \} forms a basis, so does the set

\{ $f, Af, \ldots, A^{l-1}f, A^{2k-1}f + e_{l+1}, A^{2k}f + e_{l+2}, \ldots, A^{l-2}f + e_{2n}$ \}.

That is, the set $D = \{ e'_1, e'_2, \ldots, e'_{2n} \}$ forms a basis. Under this change of basis, $A$ is similar to

$$A_2 = \begin{bmatrix}
0_n & E_1 \\
I_n & E_2
\end{bmatrix}.$$
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where

\[
E_1 = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
1 & 0 & & & \\
\vdots & & \ddots & & \\
0 & & & 0 & 1 \\
0 & & & & 0
\end{bmatrix}
\text{ and }
\]

\[
E_2 = \begin{bmatrix}
0_{k-1} & 0 \\
0 & 0 & 1 & 0 & \cdots & \cdots & 0 \\
0 & & & 1 & 0
\end{bmatrix}
\]

and thus \( A_1 = A + \alpha I_{2n} \) is similar to the matrix

\[
A_3 = \begin{bmatrix}
0 & -\alpha^2 I_n + E_1 - \alpha E_2 \\
I_n & -2\alpha I_n + E_2
\end{bmatrix}
\]

Say, \( A_1 = VA_3V^{-1} \). By the same method as in proving Lemma 2.3, the proof is complete.

**Theorem 2.5.** Let \( A \) be an \( n \times n \) complete matrix with determinant \( \pm 1 \). If \( \dim \ker (A - \alpha I) \leq [n/2] \) for all \( \alpha \in \mathbb{C} \), then \( A \) is the product of three involutions.

**Proof.** \( A \) is similar to the Jordan canonical form \( A_1 = V_1 \oplus V_2 \oplus \cdots \oplus V_k \oplus C_1 \oplus C_2 \oplus \cdots \oplus C_m \), where each

\[
V_i = \begin{bmatrix}
\lambda_i \\
1 & \lambda_i \\
\cdots & \cdots \\
1 & \lambda_i
\end{bmatrix}
\]

is a square matrix of order \( \omega_i \geq 2 \), and each \( C_i \) is a scalar matrix \( \alpha_i I_{l_i} \) of order \( l_i \) with \( l_i \geq l_{i+1} \), \( 1 \leq i \leq m - 1 \). It is understood that \( \alpha_i \neq \alpha_j \) if \( i \neq j \) and that either the \( V_i \)'s or the \( C_i \)'s may be absent.
Case I: If the $C_i$'s are absent or $l_1 \leq \sum_{i=2}^{m} l_i$, then $A_1$ is similar to $A_2 = D_1 \oplus D_2 \oplus \cdots \oplus D_m$, where each $D_i$ is cyclic and of size $\geq 2$. The conclusion follows from Lemma 2.2.

Case II: If $A_1$ is not as in Case I, then $l_1 > \sum_{i=2}^{m} l_i$. Let $h_2 = \sum_{i=2}^{m} l_i$ and $h_1 = l_1 - h_2$. Then $C_1 \oplus C_2 \oplus \cdots \oplus C_m$ is similar to $C_1 \oplus E_1 \oplus E_2 \oplus \cdots \oplus E_{h_2}$, where $C_1'$ is a scalar matrix $\alpha_1 I_{h_1}$ of order $h_1$ and

$$E_i = \begin{bmatrix} \alpha_i & 0 \\ 0 & e_i \end{bmatrix},$$

where $e_i$ is one of $\alpha_2, \alpha_3, \ldots, \alpha_m$ for each $i = 1, 2, \ldots, h_2$. Hence $A_1$ is similar to $A_2 = V_1 \oplus V_2 \oplus \cdots \oplus V_k \oplus C_1 \oplus E_1 \oplus E_2 \oplus \cdots \oplus E_{h_2}$.

Let $k_2$ be the number of $V_i$'s with eigenvalue $\alpha_1$, $k_1 = k - k_2$, and $s$ be the number of $V_i$'s with order $\omega_i \geq 3$. We may assume that $\alpha_1$ is not the eigenvalue of $V_1, V_2, \ldots, V_k$. Then $A_2$ is similar to either

$$A_3 = R_1 \oplus R_2 \oplus \cdots \oplus R_t \oplus V_{t+1} \oplus \cdots \oplus V_k \oplus B_1 \oplus B_2 \oplus \cdots \oplus V_{k+s+1} \oplus \cdots \oplus V_k \oplus E_1 \oplus E_2 \oplus \cdots \oplus E_{h_2}$$

or

$$A_4 = B_1 \oplus B_2 \oplus \cdots \oplus B_s \oplus V_{k+s+1} \oplus \cdots \oplus V_k \oplus E_1 \oplus E_2 \oplus \cdots \oplus E_{h_2} \oplus \alpha_1,$$

where

$$R_i = V_i \oplus \alpha_i I_{d_i}, \quad 0 < d_i \leq \omega_i,$$

$$R_i = V_i \oplus \alpha_i I_{\omega_i}, \quad 2 \leq i \leq t,$$

$$B_j = V_{k_1+j} \oplus \alpha_1 I_{r(j)},$$

$$0 < r(j) \leq \omega_{k_1+j} - 2, \quad r(j) + \omega_{k_1+j} = 2q_j$$

for some integer $q_j \geq 2$. $j = 1, 2, \ldots, s$. It is understood that if $k_1 = 0$, then

$$A_3 = B_1 \oplus B_2 \oplus \cdots \oplus B_s \oplus V_{k+s+1} \oplus \cdots \oplus V_k \oplus E_1 \oplus E_2 \oplus \cdots \oplus E_{h_2}.$$

Let $a_i = (-1)^{\omega_i} \det R_i$ and $u_i = -\alpha_i \lambda_{i}, 1 \leq i \leq t$, $v_j = \det V_j, t + 1 \leq j \leq k_1 + s + 1 \leq j \leq k$. Let $u = -\alpha_1^0, b_j = (-1)^{q_j} \det B_j, 1 \leq j \leq s$, and $\varphi_i = -\det E_i, 1 \leq i \leq h_2$. 
For simplicity, we may assume that $u^n \neq 1$ if $0 < \eta < q_i$, $1 \leq i \leq s$, $u^n \neq 1$ if $0 < \eta_1 < d$, and $u^n \neq 1$ if $0 < \eta_2 < \omega_i$, $2 \leq i \leq t$. Let $\epsilon = 1$ and $l = d + \sum_{i=2}^{s} \omega_i + \sum_{j=1}^{s} q_j + h_2 + k - t - s$. To prove that there exists an involution $P$ such that $PA_3$ is similar to its inverse, we choose $\alpha$ sufficiently large, say $\alpha \geq \xi^n$, where $\xi = \max_{\lambda \in \sigma(A)} |\lambda| + 1$. We now distinguish five cases.

(1) Assume that $d = \omega_1$ and $(-1)^d \det A_3 = 1$. Applying Lemmas 2.1, 2.3, and 2.4, we obtain $k + h_2$ involutions $P_i$ such that

$$\sigma(P_i R_i) = \{ \rho_i(1, u_{i-1}^{-1}, u_{i-2}^{-1}, \ldots, u_{i}^{-\omega_i + 1}), \rho_i^{-1}(u_i, u_{i+1}^{2}, \ldots, a_i) \},$$

where $\rho_i = \rho(a_0 a_1 \cdots a_{i-1})^{-1}$, $a_0 = 1$, $1 \leq i \leq t$;

$$\sigma(P_j V_j) = \{ \xi_j, \xi_j^{-1} v_j, \epsilon, \ldots, \epsilon \},$$

where $\xi_j = \rho_i(a_i v_i v_{i+1} \cdots v_{j-1})^{-1}$, $v_i = 1$, $t + 1 \leq j \leq k_1$, and $\epsilon$ may be absent;

$$\sigma(P_{k_i} B_i) = \{ \xi_i(1, u_{i-1}^{-1}, u_{i-2}^{-1}, \ldots, u_{i}^{-\omega_i + 1}), \xi_i^{-1}(u, u_{i}^{2}, \ldots, b_i) \},$$

where $\xi_i = \xi_{k_i}(b_0 b_1 \cdots b_{i-1})^{-1}$, $b_0 = 1$, $1 \leq i \leq s$;

$$\sigma(P_j V_j) = \{ \theta_j, \theta_j^{-1} v_j, \epsilon, \ldots, \epsilon \},$$

where $\theta_j = \xi_s(b_s v_{k_1 + s + 1} \cdots v_{j-1})^{-1}$, $v_{k_1 + s + 1} = 1$, $k_1 + s + 1 \leq j \leq k$, and $\epsilon$ may be absent; and

$$\sigma(P_{k+s} E_i) = \{ \beta_i, \beta_i^{-1} \varphi_i \},$$

where $\beta_i = \theta_s(v_i \varphi_i \varphi_1 \cdots \varphi_{i-1})^{-1}$, $\varphi_0 = 1$, $1 \leq i \leq h_2$. Let $P = P_1 \oplus P_2 \oplus \cdots \oplus P_{k+h_2}$. Then $P$ is an involution and $PA$ is similar to its inverse by Theorem 1.1. Hence $\Lambda$ is the product of three involutions.

(2) Assume that $d = \omega_1$ and $(-1)^d \det A_3 = -1$. We want to show that there exists an involution $P_1$ with $\det P_1 = (-1)^{d+1}$ such that $\sigma(P_1 R_i) = \{ \alpha_1, \alpha_1^{-1}, \rho, \rho^{-1} u_2^1, -\rho(u_1^2, u_1^3, \ldots, u_{1}^{4-d}), -\rho^{-1}(u_3^1, \ldots, u_{4}^d) \}$. Let

$$P_1 = \begin{pmatrix} P_1' & 0 \\ 0 & P_2' \end{pmatrix},$$
where $P'_1$ is a $4 \times 4$ matrix and $P'_2$ is a $(2d - 4) \times (2d - 4)$ matrix. Since $R_1$ is similar to $W$.

$$W = \begin{bmatrix}
W_1 & 0 \\
W_3 & W_2
\end{bmatrix},$$

where

$$W_3 = \begin{bmatrix}
0 & 1 \\
0_{3 \times (d-2)} & 0
\end{bmatrix},$$

$$W_1 = \begin{bmatrix}
\alpha_1 & 0 & 0 & 0 \\
0 & \alpha_1 & 0 & 0 \\
0 & 0 & \lambda_1 & 0 \\
0 & 0 & 1 & \lambda_1
\end{bmatrix},$$

$$W_2 = \alpha_1 I_{d-2} \otimes \begin{bmatrix}
\lambda_1 & 1 & \lambda_1 & 1 \\
1 & \lambda_1 & 1 & \lambda_1 \\
\vdots & \vdots & \vdots & \vdots \\
1 & \lambda_1 & \cdots & \lambda_1
\end{bmatrix}_{(d-2) \times (d-2)},$$

Then

$$P'_1 W = \begin{bmatrix}
P'_1 & 0 \\
0 & P'_2
\end{bmatrix} \begin{bmatrix}
W_1 & 0 \\
W_3 & W_2
\end{bmatrix} = \begin{bmatrix}
P'_1 W_1 & 0 \\
P'_2 W_3 & P'_2 W_2
\end{bmatrix},$$

and there exist an involution $P'_1$ such that

$$\sigma(P'_1 W_1) = \{ \alpha_1, \alpha_1^{-1}, \rho, -\rho^{-1}u_1^2 \}$$

and an involution $P'_2$ such that

$$\sigma(P'_2 W_2) = \{-\rho(u_1^{-2}, u_1^{-3}, \ldots, u_1^{1-d}), -\rho^{-1}(u_1^3, u_1^4, \ldots, u_1^d)\}.$$

Hence 

$$\sigma(P_1 R_1) = \sigma(P'_1 W) = \{ \alpha_1, \alpha_1^{-1}, -\rho(-1, u_1^{-2}, u_1^{-3}, \ldots, u_1^{1-d}) \}.$$
By the same method as in Case (1), the proof is complete.

(3) Assume that \( d < \omega_1 \) and \( \eta = (-1)^l \det A_3 \), where \( \eta = \pm 1 \). We want to prove that there exists an involution \( P_1 \) with \( \det P_1 = (-1)^d \) such that 
\[
\sigma(P_1 R_1) = \{ \eta, \rho, \eta \rho^{-1} \tau, \eta \rho^{-1} \tau^{-1}(1, u_1^{d-1}, \ldots, u_1^2, \ldots, u_1), \eta \rho^{-1} \tau(u_1, u_1^{d-1}, \ldots, u_1^2, \ldots, u_1), \epsilon, \ldots, \epsilon \}.
\]

(i) If \( d = 1 \), then \( R_1 \) is cyclic and there exists an involution \( P_1 \) such that 
\[
\sigma(P_1 R_1) = \{ \eta, \rho, \eta \rho^{-1} b \}, \quad \text{where} \quad b = - \det R_1.
\]

(ii) \( d \geq 2 \). Let
\[
P_1 = \begin{bmatrix} P'_1 & 0 \\ 0 & P''_1 \end{bmatrix},
\]
where \( P'_1 \) is a \((\omega_1 - d + 2) \times (\omega_1 - d + 2)\) matrix and \( P''_1 \) is a \((2d - 2) \times (2d - 2)\) matrix. Since \( R_1 \) is similar to \( T \),
\[
T = \begin{bmatrix} T_1 & 0 \\ T_3 & T_2 \end{bmatrix},
\]
where
\[
T_3 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad r = (2d - 3) \times (\omega_1 - d + 1),
\]
\[
T_2 = \alpha I_{d-1} \oplus \begin{bmatrix} \lambda_1 & & \\ & 1 & \lambda_1 \\ & & \ddots & \ddots & \ddots \\ & & & 1 & \lambda_1 \end{bmatrix}_{(d-1) \times (d-1)},
\]
and
\[
T_1 = \begin{bmatrix} \alpha_1 & \lambda_1 & & \\ 0 & 1 & \lambda_1 & & \\ & 1 & \ddots & \ddots & \ddots \\ & & \ddots & \ddots & \ddots \\ & & & 1 & \lambda_1 \end{bmatrix}.
\]
is cyclic. Then
\[ P_1T = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \begin{bmatrix} T_1 & 0 \\ T_3 & T_2 \end{bmatrix} = \begin{bmatrix} P_1T_1 & 0 \\ P_2T_3 & P_2T_2 \end{bmatrix}, \]
and there exist an involution \( P_1' \) such that \( \sigma(P_1'T_1) = \{ \eta, \rho, \eta \rho^{-1} \alpha, \epsilon, \ldots, \epsilon \} \), where \( \tau = -\det T_1 \) and \( \epsilon = 1 \) may be absent, and an involution \( P_2' \) from Lemma 2.3 such that
\[ \sigma(P_2'T_2) = \{ \eta \rho^{-1} \tau^{-1}(1, u_1^{-1}, u_1^{-2}, \ldots, u_1^{-d}), \eta \rho^{-1} \tau(u_1, u_1^2, \ldots, u_1^{d-1}) \}. \]
So \( \sigma(P_1R_1) = \sigma(P_1T) = \{ \eta, \rho, \eta \rho^{-1} \tau, \eta \rho \tau^{-1}(1, u_1^{-1}, u_1^{-2}, \ldots, u_1^{-d}), \epsilon, \ldots, \epsilon \} \), \( \eta \rho^{-1} \tau(u_1, u_1^2, \ldots, u_1^{d-1}) \) and \( \det P_1 = (-1)^d \). As in the proof for case (1), we have \( A' \) as the product of three involutions.

(4) Assume that \( k_1 = 0 \) and \( (-1)^d \det A_3 = -1 \). If \( \omega_{s+1} > 3 \), by the same method as in case (1), we take
\[ \sigma(P_{s+1}V_{s+1}) = \{ \theta_{s+1}, -\theta_{s+1}^{-1}v_{s+1}, -1, \epsilon, \ldots, \epsilon \}, \]
\[ \sigma(P_jV_j) = \{ -\theta_j, -\theta_j^{-1}v_j, \epsilon, \ldots, \epsilon \}, \quad s + 2 \leq j \leq k, \]
\[ \sigma(P_{k+i}E_i) = \{ -\beta_i, -\beta_i^{-1}v_{k+i} \}, \quad 1 \leq i \leq h_2, \]
and complete the proof as before. Hence we may assume that \( \omega_i = 2 \) for \( s + 1 \leq i \leq k \). Again, we consider two cases:

(4-1) If \( \varphi_i \neq \varphi_j \) for some \( i \neq j \), say \( \varphi_1 \neq \varphi_2 \), then \( E_1 \oplus E_2 \) is similar to \( E_1' \oplus \alpha_1 \), where \( E_1' \) is cyclic. By Lemma 2.1, there exists an involution \( P_{k+1} \) such that
\[ \sigma(P_{k+1}E_1') = \{ \alpha_1^{-1}, \beta_1, -\beta_1^{-1}\varphi_1 \varphi_2 \}. \]
So \( \sigma((P_{k+1} \oplus 1)(E_1 \oplus E_2)) = \{ \alpha_1, \alpha_1^{-1}, \beta_1, -\beta_1^{-1}\varphi_1 \varphi_2 \}. \) As in case (1), if we take \( P_{k+2} = 1 \) and \( \sigma(P_{k+1}E_i) = \{ -\beta_i, -\beta_i^{-1}v_i \}, 3 \leq i \leq h_2 \), then the proof is complete.

(4-2) Assume that all the \( E_i \)'s are absent or \( \varphi_1 = \varphi_i \), for \( i = 1, 2, \ldots, h_2 \). Since \( (-1)^d \det A_3 = -1 \), we have \( u_{l-h_2}^{\eta_2}q_2^{h_2} = -1 \). For simplicity, we may assume that \( u_{l-h_2}^{\eta_1}q_2^{h_2} \neq 1 \) for all positive integers \( \eta_1 \leq q - h_2 \) and \( \eta_2 \leq h_2 \). Let \( \varphi = \varphi_1 \). To choose in pairs \( c_i \) and \( d_i \), such that
c_i d_i = u$ and $c_i \neq d_i$, $1 \leq i \leq l - h_2$, or in pairs $c_i'$ and $d_j'$ such that $c_i' d_j' = \varphi$ and $c_j' \neq d_j'$, $1 \leq j \leq h_2$. We now distinguish seven sub-cases.

\begin{enumerate}
\item If $h_2 = 0$, we take $G_1 = \{-1, - u\}$ and $G_2 = \{-u^{-1}, - u^2, \ldots, - u^{1-h_2}, - u^i\}$.
\item If $h_2 = 1$ and $u^y \neq 1$ for $y = 1, 3, 5, \ldots, 2l - 3$, we take $G_1 = \{1, \varphi\}$ and $G_2 = \{-1, - u, - u^{-1}, - u^2, \ldots, - u^{2-l}, - u^{l-1}\}$.
\item If $h_2 = 1$ and there exists an integer $y$ with $y = 2y_1 + 1$ and $l - h_2 < y < 2l - 2h_2 - 1$ such that $u^y = 1$, we take $G_1 = \{\alpha_i, - \alpha_i^{-1} \varphi\}$ and $G_2 = \{\alpha_i^{-1}, \alpha_i u, \alpha_i u^{-1}, \alpha_i u^2, \ldots, \alpha_i u^{2-l}, \alpha_i u^{l-1}\}$.
\item If $h_2 \geq 2$ and $\varphi^x \neq 1$ for $x = 1, 3, 5, \ldots, 2h_2 - 1$ and $u^y \neq 1$ for $y = 1, 3, 5, \ldots, 2l - 2h_2 - 1$, we take
\begin{equation*}
G_1 = \{1, \varphi, \varphi^{-1}, \varphi^2, \ldots, \varphi^{1-h_2}, \varphi^{h_2}\}
\end{equation*}
and
\begin{equation*}
G_2 = \{-1, - u, - u^{-1}, - u^2, \ldots, - u^{1-h_2}, - u^{l-h_2}\}.
\end{equation*}
\item If $h_2 \geq 2$ and $\varphi^x \neq 1$ for $x = 1, 3, 5, \ldots, 2h_2 - 1$ and there exists an integer $y$ with $y = 2y_1 + 1$ and $l - h_2 < y < 2l - 2h_2 - 1$ such that $u^y = 1$, we take $G_1 = \{1, \varphi, \varphi^{-1}, \varphi^2, \ldots, \varphi^{2-h_2}, \varphi^{h_2-1}\}$ and $G_2 = \{-1, - u, - u^{-1}, - u^2, \ldots, - u^{1-y_1}, - u^{y_1}, - u^{-y_1}, - u^{y_1} \varphi, - u^{-y_1} \varphi^{-1}, - u^{y_1+1} \varphi, \ldots, - u^{1+h_2} \varphi^{-1}, - u^{l-h_2} \varphi\}$.
\item If $h_2 > 2$ and there exists an integer $x$ with $x = 2s_1 + 1$ and $h_2 < x < 2h_2 - 1$ such that $\varphi^x = 1$, and $u^y \neq 1$ for $y = 1, 3, 5, \ldots, 2l - 2h_2 - 1$, we take
\begin{equation*}
G_1 = \{1, \varphi, \varphi^{-1}, \varphi^2, \ldots, \varphi^{-s_1}, \varphi^{s_1} u, \varphi^{-s_1} u^{-1}, \varphi^{s_1+1} u, \varphi^{-s_1-1} u^{-1}, \varphi^{s_1+2} u, \ldots, \varphi^{1-h_2} u^{-1}, \varphi^{h_2} u\}
\end{equation*}
and

\[ G_2 = \{-1, -u, -u^{-1}, -u^2, \ldots, -u^{h_2-l+2}, -u^{l-h_2-1}\}. \]

\(7\) If \(h_2 \geq 2\) and there exists an integer \(x\) with \(x = 2s_1 + 1\) and 
\(h_2 < x \leq 2h_2 - 1\) such that \(g_i^x = 1\), and there exists an integer \(y\) with \(y = 2y_1 + 1\) and \(l - h_2 < y \leq 2l - 2h_2 - 1\) such that \(u^y = 1\), we take

\[ G_1 = \{1, q, q^{-1}, q^2, \ldots, q^{-s_1}, q^{s_1}u, q^{-s_1}u^{-1}, q^{s_1}u^1u, \]
\[-q^{-s_1}u^{-1}, q^{s_1+2}u, \ldots, q^{2-h_2}u^{-1}, q^{h_2-l}u\} \]

and

\[ G_2 = \{-1, -u, -u^{-1}, -u^2, \ldots, -u^{l-y_1}, -u^{y_1}, -u^{-y_1}, \]
\[-u^{y_1}q, -u^{-y_1}q^{-1}, -u^{-y_1-1}q, \ldots, -u^{2+h_2-l}q^{-1}, -u^{l-h_2-1}q\} \]

By Lemmas 2.1 and 2.4, there exist \(k + h_2\) involutions \(P_i\) such that

\[ \left( \bigcup_{i=1}^{s} \sigma(P_iB_i) \right) \cup \left( \bigcup_{u=-s+1}^{k} \sigma(P_iV_i) \right) \cup \left( \bigcup_{i=1}^{h_2} \sigma(P_iE_i) \right) = G_1 \cup G_2. \]

Let \(P = P_1 \oplus P_2 \oplus \cdots \oplus P_{k+h_2}\). Then \(P\) is an involution and \(PA\) is similar to its inverse by Theorem 1.1 Hence \(A\) is the product of three involutions.

(5) \text{Assume that } A \text{ is similar to } A_4. \text{ As in the proof for case (4-1), we may assume that } \omega_i = 2 \text{ for } s + 1 \leq i \leq k. \text{ Again, we consider two cases:}

(5-1) Assume that each \(E_i\) is absent or \(q_i = q_j\) for \(i = 1, 2, \ldots, h_2\). As in the proof for case (4-2), we have \(A\) is the product of three involutions.

(5-2) If \(h_2 \geq 2\) and \(q_i \neq q_j\) for some \(i \neq j\), say, \(q_1 \neq q_2\), then

\[ E_1 \oplus E_2 = \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix} \oplus \begin{bmatrix} q_2 & 0 \\ 0 & q_3 \end{bmatrix} \]
is similar to $E_2' \oplus \alpha_1$, where

$$E_2' = \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{bmatrix}.$$ 

Hence $A_4$ is similar to

$$A_4' = B_1 \oplus B_2 \oplus \cdots \oplus B_s \oplus V_{s+1} \oplus \cdots \oplus V_k \oplus E_2' \oplus E_3 \oplus E_4 \oplus \cdots \oplus E_{h_2} \oplus \alpha_1 I_2,$$

and $A_5$ is similar to

$$A_5' = B_1' \oplus B_2' \oplus \cdots \oplus B_{t_1}' \oplus V_{t_1+1} \oplus \cdots \oplus V_{t} \oplus E_2' \oplus E_3 \cdots \oplus E_{h_2},$$

where each $B_i, B_i' = V_i \oplus \alpha_1 I_{\delta(i)}$, $0 < \delta(i) \leq \omega_i - 2$, and $\delta(i) + \omega_i = 2d_i$ for some integer $d_i \geq 2$, $1 \leq i \leq t$. By the same method as in case (1), the proof is thus complete.

3. NECESSARY CONDITIONS

**Theorem 3.1.** Let $A$ be an $n \times n$ complex matrix, $\beta^4 \neq 1$, $\beta \neq 0$, $m = \dim \ker(A - \beta I)$, and $r = \dim \ker(A - \beta^{-3} I)$. If $A$ is the product of three involutions, then $m \leq (2n + r)/3$ and $m \leq [3n/4]$.

**Proof.** Let $P_1$ be an involution and $l = \dim \ker(P_1 A - \beta I) + \dim \ker(P_1 A + \beta I)$. Then $2m - n \leq l$. If $A = P_1 P_2 P_3$, where $P_i$'s are involutions, then $P_1 A = P_2 P_3$. Since $P_1 A$ is similar to its own inverse, we have

$$P_1 A = SDS^{-1}, \quad (1)$$

where $D$ is in Jordan canonical form and $D$ is similar to $D^{-1}$.

By (1), we obtain

$$P_1 A - \beta^{-2} A^{-1} P_1 = SD^{-1} S^{-1}. \quad (2)$$

Multiplying (2) by $\beta^{-2}$ and subtracting it from (1), we get

$$A^{-1} P_1 = S(D - \beta^{-2} D^{-1}) S^{-1}.$$
For simplicity, we will assume that $D$ is of the form

$$
\begin{bmatrix}
\alpha & 0 & 0 & 0 \\
0 & 1/\alpha & 0 & 0 \\
0 & 0 & V & 0 \\
0 & 0 & 0 & V_1
\end{bmatrix}
$$

(3)

where

$$V = \begin{bmatrix} t & 0 & 0 \\ 1 & t & 0 \\ 0 & 1 & t \end{bmatrix} \quad \text{and} \quad V_1 = \begin{bmatrix} t^{-1} & 0 & 0 \\ 1 & t^{-1} & 0 \\ 0 & 1 & t^{-1} \end{bmatrix}.$$

Then

$$D^{-1} = \begin{bmatrix}
\alpha^{-1} & 0 & 0 & 0 \\
0 & \alpha & 0 & 0 \\
0 & 0 & V^{-1} & 0 \\
0 & 0 & 0 & V_1^{-1}
\end{bmatrix}
$$

(4)

where

$$V^{-1} = \begin{bmatrix} t^{-1} & 0 & 0 \\ -t^{-2} & t^{-1} & 0 \\ t^{-3} & -t^{-2} & t^{-1} \end{bmatrix} \quad \text{and} \quad V_1^{-1} = \begin{bmatrix} t & 0 & 0 \\ -t^2 & t & 0 \\ t^3 & -t^2 & t \end{bmatrix}.$$

It follows from (3) and (4) that

$$D - \beta^{-2}D^{-1} = \begin{bmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & 0 \\
0 & E & 0 \\
0 & 0 & F
\end{bmatrix}
$$

where

$$E = \begin{bmatrix} \eta_1 & 0 & 0 \\ \epsilon & \eta_1 & 0 \\ -\beta^{-2}t^{-3} & \epsilon & \eta_1 \end{bmatrix}, \quad F = \begin{bmatrix} \eta_2 & 0 & 0 \\ \zeta & \eta_2 & 0 \\ -\beta^{-2}t^3 & \zeta & \eta_2 \end{bmatrix}.$$
\[\lambda_1 = \alpha - \beta^{-2}\alpha^{-1}, \quad \lambda_2 = \alpha^{-1} - \beta^{-2}\alpha, \quad \eta_1 = t - \beta^{-2}t^{-1}, \quad \eta_2 = t^{-1} - \beta^{-2}t, \quad \epsilon = \beta^{-2}t^2 + 1, \quad \text{and} \quad \zeta = \beta^{-2}t^2 + 1.\]

Since \(\lambda_1 = 0\) if and only if \(\alpha^2 = \beta^{-2}\), \(\lambda_2 = 0\) if and only if \(\alpha^2 = \beta^{-2}\), \(\eta_1 = 0\) if and only if \(t^2 = \beta^{-2}\), and \(\eta_2 = 0\) if and only if \(t^2 = \beta^{-2}\), we have

\[
\text{rank}(D - \beta^{-2}D^{-1}) = \text{rank}(P_1A - \beta^{-2}A^{-1}P_1) = n - l.
\]

Since \(P_1A - \beta^{-2}A^{-1}P_1 = P_1(A - \beta^{-3}I) + \beta^{-3}(A - \beta I)A^{-1}P_1\), we have \(\text{rank}(P_1(A - \beta^{-3}I)) \leq \text{rank}(P_1A - \beta^{-2}A^{-1}P_1) + \text{rank}(\beta^{-3}(A - \beta I)A^{-1}P_1)\). By \(n - l \leq 2n - 2m\), \(\text{rank}(\beta^{-3}(A - \beta I)A^{-1}P_1) = n - m\), and \(\text{rank}(P_1(A - \beta^{-3}I)) = n - r\), we obtain

\[n - r \leq n - m + 2n - 2m,\]

i.e., \(m \leq (2n + r)/3\) and \(m \leq [3n/4]\).

For \(5 \times 5\) matrices, we have the following characterization:

**Theorem 3.2.** Let \(A\) be a \(5 \times 5\) complex matrix. Then \(A \in T(5)^3\) if and only if one of the following holds:

1. \(\det A = -1\) and, for any \(\beta^4 \neq 1\), \(\dim \ker (A - \beta I_5) \leq 3\) and \(A\) is not similar to \(B = \beta I_3 \oplus I_2\);
2. \(\det A = 1\) and, for any \(\beta^4 \neq 1\), \(\dim \ker (A - \beta I_5) \leq 3\) and \(A\) is not similar to \(B = \beta I_3 \oplus (-I_2)\).

**Proof.** Since \(A \in T(5)^3\) if and only if \(-A \in T(5)^3\), we need only prove (1).

\(\Leftarrow\): In view of Theorem 2.5, we only need to show this for the case \(m = 3\). Here \(A\) is either similar to \(A_1 = D_1 \oplus \beta \oplus \beta\) or \(A_2 = C_1 \oplus C_2 \oplus \beta\), where \(D_1\) is cyclic, \(C_1 = C_2\), and \(C_1\) is either

\[
\begin{bmatrix}
\alpha & 0 \\
0 & \beta
\end{bmatrix}
\]

or

\[
\begin{bmatrix}
\beta & 0 \\
0 & \beta
\end{bmatrix}.
\]

Let \(a = \det C_1\). If \(\beta^2a = -1\), then \(\beta = 1\) or \(\alpha = 1\) from \(\det A = -1\), which contradicts that \(\beta^4 \neq 1\) and \(A\) is similar to \(B\). So \(\beta^2a \neq -1\). By Lemma 2.1, there exist three involutions \(P_i\) such that \(\sigma(P_1D_1) = \{ \beta^{-1}, -\beta^{-1}, -1 \}\), \(\sigma(P_2C_1) = \{ \beta^{-1}, -a\beta \}\), and \(\sigma(P_2C_2) = \{ -1, a \}\). Let \(V = P_1 \oplus 1 \oplus -1\) and \(V_2 = P_2 \oplus P_3 \oplus 1\). Then \(V_1A_1 = P_1C_1 \oplus \beta \oplus -\beta\), \(\sigma(V_1A_1) = \{ \beta, \beta^{-1}, -\beta, -\beta^{-1}, -1 \}\), \(V_2A_2 = P_2C_1 \oplus P_3C_2 \oplus \beta\), and \(\sigma(V_2A_2) = \{ -1, a, -a\beta, \beta, \beta^{-1} \}\).
Therefore $V_1A_1$ is similar to its inverse, and so is $V_2A_2$. That is, $A_1 \in T(5)^3$ and $A_2 \in T(5)^3$. Hence $A \in T(5)^3$.

$\Rightarrow$: By Theorem 3.1, we obtain $m \leq 3$. It remains to prove that $B \notin T(5)^3$.
If $B = \beta I_3 \oplus I_2 \in T(5)^3$, say $B = P_1P_2P_3$, where the $P_i$'s are involutions, then, assuming

$$P_1 = \begin{bmatrix} C & D \\ E & F \end{bmatrix}$$

with $C$ and $F$ of sizes 2 and 3, respectively,

$$P_1B = \begin{bmatrix} C & D \\ E & F \end{bmatrix} \begin{bmatrix} \beta I_3 & 0 \\ 0 & I_2 \end{bmatrix} = \begin{bmatrix} \beta C & D \\ \beta E & F \end{bmatrix}$$

is similar to

$$B^{-1}P_1 = \begin{bmatrix} \beta^{-1}I_3 & 0 \\ 0 & I_2 \end{bmatrix} \begin{bmatrix} C & D \\ E & F \end{bmatrix} = \begin{bmatrix} \beta^{-1}C & \beta^{-1}D \\ E & F \end{bmatrix}.$$ 

Since

$$(P_1B)^2 = \begin{bmatrix} \beta^2C^2 + \beta DE & \beta CD + DF \\ \beta^2EC + \beta FE & \beta ED + F^2 \end{bmatrix},$$

$$(B^{-1}P_1)^2 = \begin{bmatrix} \beta^{-2}C^2 + \beta^{-1}DE & \beta^{-2}CD + \beta^{-1}DF \\ \beta^{-1}C + FE & \beta^{-1}ED + F^2 \end{bmatrix},$$

$\text{tr}(P_1B) = \text{tr}(B^{-1}P_1)$, $\beta^3 = -1$, and $\text{tr}(P_1B)^2 = \text{tr}(B^{-1}P_1)^2$, we have

$$\text{tr} C = 0 \quad (1)$$

and

$$\text{tr} C^2 + 2\text{tr} DE = 0. \quad (2)$$

Since

$$P_1^2 = \begin{bmatrix} C^2 + DE & CD + DF \\ ED + FE & ED + F^2 \end{bmatrix} = \begin{bmatrix} I_3 & 0 \\ 0 & I_2 \end{bmatrix},$$
we have
\[ C^2 + DE = I_3, \quad ED + F^2 = I_2, \quad CD + DF = 0, \quad EC + FE = 0. \] (3)

From (3), we obtain
\[ \text{tr} C^2 + \text{tr} DE = 3 \quad \text{and} \quad \text{tr} DE + \text{tr} F^2 = 2. \] (4)

By (2) and (4), we get
\[ \text{tr} DE = -3, \quad \text{tr} F^2 = 5, \quad \text{and} \quad \text{tr} C^2 = 6. \] (5)

Since \( \text{tr} C = 0, \text{tr} C^2 = 6, \text{DE} = I_3 - C^2, \) and rank \( \text{DE} \leq 2, \) we have either \( \sigma(C) = \{1, 1, -2\} \) or \( \sigma(C) = \{-1, -1, 2\}. \) So rank \( \text{DE} = 1 \) which implies that
\[ \text{rank} D = 1 \quad \text{or} \quad \text{rank} E = 1. \] (\*)

As in the proof of Theorem 3.1, we have
\[ \text{rank} \left( P_1B - \beta^{-2}B^{-1}P_1 \right) = \text{rank} \left( P_1B - \beta^2B^{-1}P_1 \right). \] (6)

From (\*) and
\[ P_1B - \beta^2B^{-1}P_1 = \begin{bmatrix} 0 & (1 - \beta)D \\ (\beta - \beta^2)E & (1 - \beta^2)F \end{bmatrix}, \]
we obtain
\[ \text{rank} \left( P_1B - \beta^2B^{-1}P_1 \right) \leq 3. \] (7)

\[ P_1B - \beta^{-2}B^{-1}P_1 = \begin{bmatrix} (1 + \beta)C & 2D \\ 2\beta E & (1 + \beta)F \end{bmatrix} \]
\[ = \begin{bmatrix} (1 + \beta)C & 0 \\ 2\beta E & I_2 \end{bmatrix} \]
\[ \times \begin{bmatrix} I_3 & 2(1 + \beta)^{-1}C^{-1}D \\ 0 & (1 + \beta)F - 4\beta(1 + \beta)^{-1}EC^{-1}D \end{bmatrix}, \] (8)
so

\[
\text{rank}\left( P_1 B - \beta^{-2} B^{-1} P_1 \right) \geq 3. \tag{9}
\]

From (6), (7), (8), and (9), we get

\[
(1 + \beta) F - 4\beta (1 + \beta)^{-1} EC^{-1} D = 0_2. \tag{10}
\]

Multiplying (10) by \((1 + \beta) F\) and comparing with (3), we obtain

\[
(1 + \beta)^2 F^2 + 4\beta ED = 0_2,
\]

which contradicts (5). Hence \( R \not\cong T(5)^3 \). 

REFERENCES


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