NOTE

A NOTE ON THE ASCENDING SUBGRAPH DECOMPOSITION PROBLEM

Hung-Lin FU

Department of Applied Mathematics, National Chiao Tung University, Hsinchu, Taiwan, People's Rep. of China

Received 16 September 1987
Revised 14 October 1988

Let G be a graph with \((n+1)/2\) edges. We say G has an ascending subgraph decomposition (ASD) if the edge set of G can be partitioned into n sets generating graphs G_1, G_2, ..., G_n such that \(|E(G_i)| = i\) (for i = 1, 2, ..., n) and G_i is isomorphic to a subgraph of G_{i+1} for i = 1, 2, ..., n - 1.

In this note, we prove that if G is a graph of maximum degree \(d \leq \lfloor (n+1)/2 \rfloor\) on \((n+1)/2\) edges, then G has an ASD. Moreover, we show that if \(d \leq \lfloor (n-1)/2 \rfloor\), then G has an ASD with each member a matching. Subsequently, we also verify that every regular graph of degree a prime power has an ASD.

1. Introduction

In [1] the authors give the following decomposition conjecture.

**Conjecture.** Let G be a graph with \((n+1)/2\) edges. Then the edge set of G can be partitioned into n sets generating graphs G_1, G_2, ..., G_n such that \(|E(G_i)| = i\) (for i = 1, 2, ..., n) and G_i is isomorphic to a subgraph of G_{i+1} for i = 1, 2, ..., n - 1.

A graph G that can be decomposed as described in the conjecture will be said to have an ascending subgraph decomposition (abbreviated ASD). The graphs G_1, G_2, ..., G_n are said to be members of such a decomposition.

In [1, 2], the conjecture has been verified for star forests. Also, in [2] it is proved that if G is a graph of maximum degree \(d \leq \lfloor (n+1)/2 \rfloor\) on \((n+1)/2\) edges and \(n \geq 4d^2 + 6d + 3\), then G has an ASD with each member a matching.

In this note, we prove that if G is a graph of maximum degree \(d \leq \lfloor (n+1)/2 \rfloor\) on \((n+1)/2\) edges, then G has an ASD. Moreover, we show that if \(d \leq \lfloor (n-1)/2 \rfloor\), then G has an ASD with each member a matching. As a special case we also verify that every regular graph of degree a prime power has an ASD.

2. Main results

Let \(N\) be the set \(\{1, 2, \ldots, n\}\), and \(A_1, A_2, \ldots, A_k\) be mutually disjoint subsets of \(N\) such that \(\bigcup_{i=1}^k A_i = N\). Let \(s(A_i)\) be the sum of all elements in

\[0012-365X/90/$03.50 \copyright 1990 - Elsevier Science Publishers B.V. (North-Holland)\]
We will say that $N$ can be decomposed into subsets of type $(s_1, s_2, \ldots, s_k)$ if there exists a collection of mutually disjoint subsets of $N$, $A_1, A_2, \ldots, A_k$, such that their union is $N$ and $s(A_i) = s_i$, $i = 1, 2, \ldots, k$. Obviously, $\sum_{i=1}^{k} s_i = \binom{n+1}{2}$. For example $\{1, 2, \ldots, 6\}$ can be decomposed into subsets of type $(3, 5, 6, 7)$. ($A_1 = \{3\}$, $A_2 = \{1, 4\}$, $A_3 = \{6\}$, $A_4 = \{2, 5\}$.)

An edge-coloring of a graph is an assignment of colors to its edges so that no two incident edges have the same color. If a graph $G$ has an edge-coloring with $k$ colors, then $G$ is called $k$-colorable. (Let $\delta_i$ denote the number of edges in $G$ which are colored $c_i$, $i = 1, 2, \ldots, k$.)

After a bit of reflection, we have the following proposition. (Unless stated otherwise, we assume that $G$ has $\binom{n+1}{2}$ edges and that the number of edges that are colored $c_i$ is $\delta_i$.)

**Proposition 1.** Let $G$ be a $k$-colorable graph. If $N$ can be decomposed into subsets of type $(\delta_1, \delta_2, \ldots, \delta_k)$, then $G$ has an ASD with each member a matching.

**Proof.** Since $N$ can be decomposed into subsets of type $(\delta_1, \delta_2, \ldots, \delta_k)$, it follows that $s(A_i) = \delta_i$, $i = 1, 2, \ldots, k$. We can choose $G_i$ as the collection of $i$ edges that are colored $c_j$ if $i \in A_j$. \hfill $\square$

We call an edge-coloring equalized if $|\delta_i - \delta_j| \leq 1$ ($1 \leq i < j \leq k$). McDiarmid [3] and de Werra [5] independently proved that if a graph has an edge-coloring with $k$ colors then it has an equalized edge-coloring with $k$ colors. We can easily prove the following result by using the above fact.

**Proposition 2.** Let $G$ be a graph with maximum degree $d \leq \lfloor (n - 1)/2 \rfloor$, then $G$ has an ASD with each member a matching.

**Proof.** By Vizing's Theorem [4] $G$ has edge chromatic number $\chi'(G)$ at most $\lfloor (n - 1)/2 \rfloor + 1$. Hence we can color $G$ with $n/2$ or $(n + 1)/2$ colors depending on whether $n$ is even or odd. By the theorem of McDiarmid and de Werra, we obtain an equalized edge-coloring with $n/2$ or $(n + 1)/2$ colors as the case may be. If $n$ is even, then each color occurs $n + 1$ times. Since, $N$ can be decomposed into subsets of type $(n + 1, n + 1, \ldots, n + 1)$ ($n/2$-tuple), we conclude that $G$ has an ASD with each member a matching by Proposition 1. Similarly, if $n$ is odd, then each color occurs $n$ times. Since $N$ can be decomposed into subsets of type $(n, n, \ldots, n)$ ($(n + 1)/2$-tuple), we have the proof. \hfill $\square$

As a matter of fact, if $G$ is of class one, i.e. $\chi'(G) = d$, then we can let $d \leq \lfloor (n + 1)/2 \rfloor$ in Proposition 2. Actually, if we simply want to prove that $G$ has an ASD, we can improve the upper bound of $d$ a bit.

**Proposition 3.** Let $G$ be a graph with maximum degree $d \leq \lfloor (n + 1)/2 \rfloor$, then $G$ has an ASD.
Proof. From Proposition 2, the only cases left are \( d = n/2 \) \((n \text{ is even})\) and \( d = (n + 1)/2 \) \((n \text{ is odd})\). If \( n \) is even, then \( G \) is \((n/2 + 1)\)-colorable. Since we have an equalized edge-coloring, hence we can color the edges by the way: \( n/2 \) colors occur \( n - 1 \) times and one color occurs \( n \) times. Since \( N \) can be decomposed into subsets of type \( \langle n - 1, n - 1, \ldots, n - 1, n \rangle \) \((n/2 + 1)\)-tuple), we are done. For the case when \( n \) is odd, \( G \) is \(((n + 1)/2 + 1)\)-colorable. Similarly, we can color the edges in the following way: \((n - 3)/2\) colors occur \((n - 2)\) times and 3 colors occur \((n - 1)\) times. Without loss of generality, we let those three colors which occur \((n - 1)\) times be \( c_1, c_2, \) and \( c_3 \). It is not difficult to see \( \{1, 2, \ldots, n - 3\} \) can be decomposed into subsets of type \( \langle n - 2, n - 2, \ldots, n - 2 \rangle \) \((n - 3)/2\)-tuple), therefore we can choose \( G_1, G_2, \ldots, G_{n-3} \) subsequently. We conclude the proof by letting \( G_{n-2} \) be the collection of edges colored \( c_1 \) except for one edge \( e \), \( G_{n-1} \) be the collection of edges colored \( c_2 \), and \( G_n \) be the collection of those edges colored \( c_2 \) and \( e \). □

From Proposition 3, it is easy to see every regular graph of degree a prime power has an ASD.

Proposition 4. Every regular graph of degree a prime power has an ASD.

Proof. Let the degree and order of \( G \) be \( d \) and \( v \) respectively. Then \( d \cdot v = n \cdot (n + 1) \). Hence we have \( d \mid n(n + 1) \). Since \( d \) is a prime power and the common divisor of \( n \) and \( n + 1 \) is 1, \( d \mid n \) or \( d \mid n + 1 \). If \( d < n \), then \( d = (n + 1)/2 \). By Proposition 3, \( G \) has an ASD. If \( d = n \), then \( G = K_{n+1} \). The theorem follows from the fact that \( K_{n+1} \) has an ASD. □

As we have seen above, if the maximum degree of the graph is not too large, it has an ASD. In what follow we suggest a slightly different approach to the problem.

A vertex covering in a graph is any set of vertices such that each edge of the graph has at least one of its end vertices in the set. We will say \( \langle \beta_1, \beta_2, \ldots, \beta_k \rangle \) is a covering pattern for a graph \( G \), if we can find a vertex covering \( \{v_1, v_2, \ldots, v_k\} \) such that there are \( \beta_i \) edges incident with the vertex \( v_i \), \( i = 1, 2, \ldots, k \) and each edge can be counted only once. For example, Fig. 1 has a covering pattern \( \langle 5, 4, 3, 3 \rangle \).

Since the following proposition is easy to see, it will be stated without proof.

Proposition 5. Let \( G \) be a graph with a covering pattern \( \langle \beta_1, \beta_2, \ldots, \beta_k \rangle \). If \( N \) can be decomposed into subsets of type \( \langle \beta_1, \beta_2, \ldots, \beta_k \rangle \), then \( G \) has an ASD with each member a star.
The following proposition is also easy to prove, we simply state it.

**Proposition 6.** If a graph can be partitioned into edge disjoint paths of length \( r_1, r_2, \ldots, r_k \) respectively, and the set \( N \) can be decomposed into subsets of type \( \langle r_1, r_2, \ldots, r_k \rangle \), then \( G \) has an ASD with each member a path.

3. Acknowledgement

The author would like to express his appreciation to the referee for his helpful comments and his patience in correcting errors.

References