Wilson loop and contour shapes

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By computing the Wilson loop expectation value $W[C]$ in the two-dimensional Schwinger model on $R^1 \times S^1$, we show that nonleading terms depend on the shape of the contours both on $R^1 \times R^1$ and $R^1 \times S^1$. We also find that the rhombic contour and triangular contour lead to the same static potentials both on $R^1 \times R^1$ and $R^1 \times S^1$. The binding energy is also affected by the choice of contour shapes on the $R^1 \times S^1$ model. This indicates that the model on $R^1 \times S^1$ deserves more study.

I. INTRODUCTION

It is known that the gauge-invariant Wilson loop expectation value

$$W[C] = \text{Tr} P \exp \left[ \oint \mathcal{A}_\mu dx^\mu \right]$$

(1.1)

is related to the binding energy of a quark-antiquark pair. Therefore, $W[C]$ has been employed by a number of authors to test the idea of quark confinement. Moreover, the gauge-invariant dynamical variable $W[C]$ is useful as a toy model in discussing hadronic physics.

Furthermore, many articles have argued that by averaging over different contours $C$, the Green's function of the quark currents is derivable from the gauge-invariant Wilson loop expectation value $W[C]$ in the large-$N_c$ limit. Accordingly, a study of the equations for the loop averages was first proposed by Polyakov and Nambu in order to make transparent the relevant relation between QCD and the dual resonance model.

It was shown in Ref. 2 that $W[C]$ depends nontrivially on the shapes of the contour $C$. This property was also discussed by several authors to analyze the effect of the long-wavelength string fluctuations before we get into the details of computing $W[C]$. Moreover, by considering a smoothly shaped contour one can avoid singularities usually plagued with a rectangular contour. Indeed, it was shown in Ref. 2 that nonleading terms depend sensitively on the contour shapes $C$.

Also, the Wilson loop expectation value $W[C]$ is expected to be a linearly increasing quark-antiquark pair static potential of the form

$$V(d) = \frac{Q^2 d}{2} .$$

(1.2)

Here $d$ is the distance between the quark and antiquark. Here $\rho$ and $\theta$ denote, respectively, the radial and polar coordinates on $S^1$.

For simplicity, we will consider a two-dimensional Schwinger model on $R^1 \times S^1$ with massless fermions. Extending our result from two-dimensional QED (QED$_2$) to diagonal QCD$_2$ (DQCD$_2$) is straightforward and will only bring in an additional group-theoretical factor

$$\ln W_{\text{DQCD}_2}[C] = \frac{N-1}{2N} \ln W_{\text{QED}_2}[C] .$$

(1.3)

Here $N$ stands for the dimension of the fundamental representation of the symmetry group $SU(N)$.

There are several advantages for considering $R^1 \times S^1$ instead of $R^1 \times R^1$. First of all, it is much easier to consider the model on $R^1 \times S^1$ since most of the relevant physics remains unaffected. Secondly, the annoying infrared divergence usually plagued with most two-dimensional gauge theories can be shown to be absent on $R^1 \times S^1$. Moreover, the flat-space limit can be easily reproduced by letting $\rho$, the radial coordinate, go to infinity. Also the fermions are introduced (in contrast with a pure gauge theory) in order to study not only the gauge-invariant dynamical variable $W[C]$ in the loop space more directly but also the mechanism for quark trapping. We reported part of the results concerning the binding energy in QED$_2$ in a previous paper. In this paper, we will present complete and consistent details and make transparent some technical tricks in showing the decoupling theorem.

Therefore, we will study the shape dependence of $W[C]$ in QED$_2$. In Sec. II, we will briefly review the loop calculations of $W[C]$ by the path-integral method. In Sec. III, we will review the computation of the anomalous mass $\mu^2$ on the $R^1 \times S^1$ model using Fujikawa's method. In Sec. IV, we present the details of the calculations for four different contours $C$: namely, the rectangular, rhombic, triangular, and elliptical contours. We also demonstrate a special trick in taking limits. Finally, several comments and discussions are in order in Sec. V.

II. THE THEORY ON $R^1 \times S^1$

We will give a brief review of the derivation of the chiral anomaly using Fujikawa's path-integral method before we get into the details of computing $W[C]$. The Lagrangian for the two-dimensional (Euclidean) Schwinger model is given by

$$\mathcal{L} = -i \bar{\psi} \gamma^\mu D_\mu \psi + \frac{1}{2} F_{\mu\nu} F^{\mu\nu} + \mathcal{L}_{\text{gauge fixing}} .$$

(2.1)

Here $D_\mu = \partial_\mu + e A_\mu$. Also the $\gamma$ matrices satisfy...
In this paper we will work on the Euclidean (Riemannian) base manifold after a Wick rotation. Writing \( y^2 = i y^0 \), we find that \( y^p D_p \) becomes a Hermitian operator after a Wick rotation \( x^0 \rightarrow -ix^2 \) and \( A_0 \rightarrow i A_2 \). Note that \( y^p y^p = -2g_{\mu \nu} \). (2.2)

In the Lorentz gauge (i.e., \( \partial_\mu A^\mu = 0 \)), on the spatial circle \( S^1 \), we need to specify appropriate boundary conditions in order to define the theory properly. Therefore, by requiring that \( \mathcal{L} \) and \( F_{\mu \nu} \) (hence \( J^\mu = \psi_{\mu \nu} \psi \)) be single valued, one obtains the following constraints on the photon and fermion fields:

\[
A_\mu (t, x + 2\pi p) = A_\mu (t, x), \quad \psi (t, x + 2\pi p) = e^{i\epsilon (t, x)} \psi (t, x) .
\] (2.3)

Note that it was argued that one should take \( e^{i\epsilon} = -1 \) due to the stability of the vacuum. Hence (2.4) and (2.5) are sufficient to restrict ourselves to \( R^1 \times S^1 \). Note also that all the algebraic and differential operations are not affected by this restriction since the connection on \( R^1 \times S^1 \) is trivial, namely, \( F^{\mu \nu} = 0 \). In fact, \( g_{\mu \nu} = \text{diag}(1, \rho^2) \) on \( R^1 \times S^1 \). Hence the loop average is

\[
\Delta (x) = \lim_{M \to \infty} \text{Tr} M \int \frac{dk_2}{(2\pi)^2} \sum_{n = -\infty}^{\infty} \gamma \gamma \exp \left[ -k_2 k_2 - \frac{1}{4M^2} [\gamma prime, \gamma^\prime] F_{\mu \nu} \right] \exp \left[ -\frac{1}{4M^2} [\gamma \gamma^\prime] F_{\mu \nu} \right] \sum_{n = -\infty}^{\infty} e^{-\pi n / M} .
\] (3.5)

Note that we have scaled the momentum \( k_2 \) according to \( k_2 \rightarrow M k_2 \). Moreover, we have suppressed the trivial radial factor \( \rho \) since \( \rho \) will not appear in the final expression (3.12).

In order to carry on our computations, we need to know the asymptotic properties of the Jacobi \( \theta \) function of the form

\[
\theta (y) = \sum_{n = -\infty}^{\infty} e^{-\pi n^2 y} .
\] (3.6)

In fact, we want to know the large-\( M \) behavior of

\[
\theta (1/\pi M^2) = \sum_{n = -\infty}^{\infty} e^{-\pi^2 n^2 / M^2} .
\] (3.7)

If \( \theta (y) \) is in the Schwartz space (the set of \( C^\infty \) functions on \( R^1 \) which, along with their partial derivatives of all orders, tend to vanish rapidly at infinity), one can obtain the following functional equation of the Jacobi \( \theta \) function: namely,

\[
\theta (y^{-1}) = \sqrt{\pi} \theta (y) .
\] (3.8)

Note that Gaussian functions (3.6) are certainly in the Schwartz space. Hence, using the functional equation of the Jacobi \( \theta \) function (3.8), one obtains

\[
\theta (1/\pi M^2) = M \sqrt{\pi} \sum_{n = 1}^{\infty} e^{-\pi n^2 / M^2} .
\] (3.9)
Therefore, we have

\[ \lim_{M \to \infty} \theta(1/\pi M^2) = M \sqrt{\pi}. \] (3.10)

Hence the Jacobian factor can be shown to be

\[ \Delta(x) = \lim_{M \to \infty} \frac{\text{Tr} M^2 \sqrt{\pi} \int \frac{dk}{(2\pi)^2} \gamma_5}{\sqrt{\pi} \text{Tr} \gamma_5 \frac{1}{4} [\gamma^\mu, \gamma^\nu] F_{\mu\nu}} \times \exp \left( -k^2 \frac{1}{4M^2} [\gamma^\mu, \gamma^\nu] F_{\mu\nu} \right) = \sqrt{\pi} \text{Tr} \gamma_5 \frac{1}{4} [\gamma^\mu, \gamma^\nu] F_{\mu\nu} \int \frac{dk}{(2\pi)^2} e^{-k^2} \right) = \frac{1}{4\pi} F_{\mu\nu} \epsilon_{\mu\nu}. \] (3.11)

Note that (3.11) is exactly the same as the anomalous contribution in the flat-space Schwinger model. Therefore we have the same anomalous contribution: namely,

\[ \mathcal{D}\psi \mathcal{D}\bar{\psi} = \mathcal{D} \chi \mathcal{D} \bar{\chi} \exp \left( -\frac{\mu^2}{2} \int A^T A d^2 x \right), \] (3.12)

with \( \mu^2 \) given by

\[ \mu^2 = \frac{e^2}{\pi}. \] (3.13)

Accordingly, the Wilson loop expectation value becomes

\[ W[C] = N \int \mathcal{D} A^T \mathcal{D} \chi \exp(-S_{\text{eff}}) \exp \left( \int F_{\mu\nu} \right) \exp \left[ \frac{1}{2} \int A^T A d^2 x \right]. \] (3.14)

Here

\[ S_{\text{eff}} = \int d^2 x \left[ -i \bar{\chi} \gamma^\mu \partial_\mu \chi + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right] + \frac{\mu^2}{2} A^T A + \mathcal{L}'(A^T, \chi). \] (3.15)

Note that the conserved current \( J^\mu \) (i.e., \( \partial \cdot J = 0 \)) will not couple to \( A^T_{\mu} \). Therefore, we can collect all irrelevant \( A^L_{\mu} \) and \( \chi \) terms in \( \mathcal{L}' \). Thus, by integrating out the trivial \( A^L_{\mu} \) and \( \chi \) degrees of freedom and introducing the current

\[ J^\mu(x) = i \mathcal{Q} \bar{\psi} \gamma^\mu(x) \frac{dx^\mu(x)}{dt}, \] (4.1)

we have

\[ W[C] = N \int \mathcal{D} A^T \exp \left[ -S_{\text{eff}}(A^T) + \int d^2 x \mathcal{J} \cdot A \right] = N \exp \left[ \frac{1}{2} \int d^2 x d^2 y J^\mu(x) \Delta_{\mu\nu}(x-y) J^\nu(y) \right]. \] (3.17)

Note that the propagator for \( A^T_{\mu} \) is

\[ \Delta_{\mu\nu}(x-y) = \int d^2 k e^{-ik \cdot (x-y)} \frac{\delta_{\mu\nu} - k \mu k \nu}{(2\pi)^2 k^2 + m^2}. \] (3.18)

Transforming into momentum space by

\[ J^\mu(k) = \int d^2 x J^\mu(x) e^{-ik \cdot x}, \] (3.19)

we have, from (3.16),

\[ J^\mu(k) = i \mathcal{Q} \bar{\psi} \gamma^\mu e^{-ik \cdot x(t)} dx^\mu(t). \] (3.20)

Therefore,

\[ \ln W[C] = \frac{1}{8\pi^2} \int d^2 k J^\mu(-k) \frac{1}{k^2 + m^2} J^\mu(k). \] (3.21)

Note that the spatial momentum \( k_1 \) becomes discretized, namely,

\[ k_1 = \frac{n}{\rho}, \quad n \in \mathbb{Z}, \] (3.22)

due to the periodic boundary condition (2.4) and (2.5) in the spatial direction. Accordingly, the integration on \( k_1 \) becomes a discrete sum,

\[ \int_{-\infty}^\infty dk_1 \to \frac{1}{\rho} \sum_{n = -\infty}^\infty. \] (3.23)

IV. CONTOUR SHAPE DEPENDENCE

Let us first consider a rectangular contour given by the oriented boundary of the rectangular disk

\[ \{(x_1, x_2) \mid |x_1| \leq d/2, \quad |x_2| \leq T/2 \}. \]

The current can be shown to be

\[ J^\mu(k) = i \mathcal{Q} \bar{\psi} \gamma^\mu e^{-ik \cdot x(t)} dx^\mu(t). \] (4.1)

\[ = \epsilon_{\mu\nu} k \mathcal{Q} \rho \sin \frac{nd}{2\rho} \frac{k_2 T}{2}. \] (4.2)

Here \( k_\mu = (n/\rho, k_2) \). Note that the above current is indeed conserved. Therefore,

\[ \ln W[C] = \frac{1}{8\pi^2} \sum_{n = -\infty}^\infty \int dk_2 \frac{1}{k_2^2 + (n/\rho)^2 + \mu^2} \left[ -\frac{1}{k_2^2} \sin^2 \frac{k_2 T}{2} \sin^2 \frac{nd}{2\rho} - \left( \frac{\rho}{n} \right)^2 \sin^2 \frac{nd}{2\rho} \sin^2 \frac{k_2 T}{2} \right]^2 \] (4.3)

\[ = -\frac{Q^2}{\pi^2} \sum_{n = -\infty}^\infty \sin^2 \left( nd/2\rho \right) \left[ T + \mu^2 \frac{1 - \exp(\frac{T(\rho/\rho^2 + \mu^2)^{1/2})}{(\rho/\rho^2 + \mu^2)^{1/2}} \right]. \] (4.4)
The effective static potential of charge $\pm Q$ separated by a mean distance $d$ can be defined as

$$V(d) = -\lim_{T \to \infty} \left[ \frac{1}{T} \ln W[C] \right].$$

Hence this effective static potential becomes

$$V_\circ(d) = \frac{2Q^2}{\pi \rho^2} \sum_{n=-\infty}^{\infty} \sin^2\left(\frac{\pi d}{2\rho}\right) \left(\frac{n}{\rho}\right)^2 + \frac{\mu^2}{4\mu}$$

$$= \frac{Q^2}{2\mu} \left[ \coth \mu \pi - \coth \mu \pi \cos \mu d + \sinh \mu d \right],$$

(4.7)
in the case of the rectangular contour.

Next, we will consider a rhombic contour defined by the equation $|x_1|/a + |x_2|/b = 1$. The current can be shown to be

$$J_\mu(k) = 4abQe^{\mu k} \frac{\cos k_+ b - \cos (na/\rho)}{(k_+ b)^2 - (na/\rho)^2}. \quad (4.8)$$

Also note that the current conserved is apparent. Therefore,

$$\ln W_\circ(a,b) = -\frac{2a^2bQ^2}{2\pi \rho} \int \sum_{n=Z} \frac{k^2 + (n^2/\rho^2)b^2}{k^2 + (n^2/\rho^2 + \mu^2)b^2}$$

$$= \frac{2a^2bQ^2}{2\pi \rho} \int \sum_{n=Z} \frac{\cos k - \cos (na/\rho)}{k^2 - (n^2/\rho^2)^2}.$$  

(4.9)

Note that in (38) we performed a change of integration variable $k_2 \to k/b$, in order to extract the $b$ dependence in the integrand. After doing this, it is straightforward to compute the effective static potential

$$\ln W_\circ(d,T) = -\frac{T^2Q^2}{2\pi \rho^2} \int d k_2 \sum_{n=Z} \frac{1}{k^2 + (n^2/\rho^2 + \mu^2)}$$

$$\left[ \frac{\sin k_+^1 k_+ - \sin k_-^1 k_-}{k_+^1 + k_-^1} - 2 \frac{\sin k_+^1 k_+ \sin k_-^1 k_- \cos (nd/\rho)}{k_+^1 k_-^1} \right].$$

(4.15)

Replacing $k_2$ by $k/T$, we have

$$V_\circ(d,T) = \frac{Q^2}{2\pi \rho} \sum_{n=Z} \frac{1}{n^2/\rho^2 + \mu^2} \int \frac{k^2 + k_2^2}{n^2} \frac{\sin k_+^1 k_+ + \sin k_-^1 k_-}{k_+^1 + k_-^1}$$

$$= \frac{2a^2bQ^2}{2\pi \rho} \int \sum_{n=Z} \frac{n^2/\rho^2 + \mu^2}{n^2} \frac{\sin k_+^1 k_+ - \sin k_-^1 k_- \cos (nd/\rho)}{k_+^1 k_-^1}.$$  

(4.16)

Here $k_+$ and $k_-$ become $k + nd/\rho$ and $k - nd/\rho$, respectively. After some algebra, one has

$$V_\circ(d) = \frac{Q^2}{2\mu} \left[ \coth \mu \pi + \frac{\sinh \mu (\rho \pi - 2d) - \sinh \mu \rho \pi}{2\mu d \sinh \mu \rho \pi} \right].$$  

(4.17)

Note that $V_\circ = V_\circ$ although their contour shapes are not the same. Note also that $V_\Delta$ (from a reversed triangular contour) equals to $V_\circ$ due to time-reversal invariance.

Finally, let us consider an elliptical contour sketched by the equation $x_1^2/a^2 + x_2^2/b^2 = 1$. The current can be shown to be

$$J_\mu(k) = 2\pi ab \epsilon_{\mu k} \frac{J_1(\sqrt{k_+^2 a^2 + k_-^2 b^2})}{\sqrt{k_+^2 a^2 + k_-^2 b^2}}.$$  

(4.18)
Here $J_i(y)$ is the Bessel's function. Again, the current is apparently conserved. Now, by inserting (4.18) into (3.21), one obtains

$$\ln W_{\text{ellip}}(a,b) = -\frac{a^2b^2Q^2}{2p} \sum_{n \in \mathbb{Z}} \int dk \frac{k_1^2 + k_2^2}{k_1^2 + k_2^2 + \mu^2} \times \frac{J_i^2(\sqrt{k_1^2 + k_2^2 + a^2})}{k_1^2 + k_2^2 + a^2} .$$

Here $k_1 = n/p$. By scaling $k_2 \rightarrow k/b$, one obtains

$$\ln W_{\text{ellip}}(a,b) = -\frac{a^2b^2Q^2}{2p} \sum_{n \in \mathbb{Z}} \int dk \frac{k_1^2 + k_2^2}{k_1^2 + (k_1^2 + \mu^2)b^2} \times \frac{J_i^2(\sqrt{k_1^2 + k_2^2 + a^2})}{k_1^2 + k_2^2 + a^2} .$$

(4.19)

The effective potential should be defined as

$$V_{\text{ellip}}(d) = -\lim_{T \rightarrow \infty} \frac{1}{T} \ln W_{\text{ellip}}(T/2,2d/\pi) = -\lim_{b \rightarrow \infty} \frac{1}{2b} \ln W_{\text{ellip}}(a,b) .$$

(4.21)

Note that the area enclosed by the elliptical contour is set to be $\pi ab = Ta$ in order to make $d$ equal to the average separation. Therefore,

$$V_{\text{ellip}}(d) = -\frac{a^2Q^2}{4} \sum_{n \in \mathbb{Z}} \int dk \frac{k_1^2}{k_1^2 + \mu^2} \frac{J_i^2(\sqrt{k_1^2 + k_2^2 + a^2})}{k_1^2 + k_2^2 + a^2} .$$

(4.22)

After some algebra, one can then derive

$$V_{\text{ellip}}(d) = \frac{Q^2}{4\mu} \left[ I_1 \left( \frac{4ud}{\pi} \right) - \coth\mu\pi L_1 \left( \frac{4ud}{\pi} \right) \right] .$$

(4.23)

Here $I_1$ and $L_1$ are, respectively, the associated Bessel function and associated Struve's function of order 1. Note that in deriving Eq. (4.23) we have made use of the integral representation of $I_m$ and $L_m$: i.e.,

$$I_m(x) = \frac{2(x/2)^m}{\Gamma(m + \frac{1}{2})\Gamma(\frac{1}{2})} \int_0^1 (1-y^2)^{m-1/2} \cosh y dy ,$$

(4.24)

$$L_m(x) = \frac{2(x/2)^m}{\Gamma(m + \frac{1}{2})\Gamma(\frac{1}{2})} \int_0^1 (1-y^2)^{m-1/2} \sinh y dy .$$

(4.25)

V. CONCLUSION

In summary, we have

$$V_{\Box}(d) = \frac{Q^2}{2\mu} \left( \coth\mu\pi - \coth\mu\pi \cos \mu d + \sin \mu d \right) ,$$

(5.1)

$$V_{\Theta}(d) = V_{\Box}(d)$$

$$= \frac{Q^2}{2\mu} \left[ \coth\mu\pi + \frac{\sinh(d\pi - 2d) - \sinh\mu\pi}{2\mu d\sinh\mu\pi} \right] ,$$

(5.2)

$$V_{\text{ellip}}(d) = \frac{Q^2}{4\mu} \left[ I_1 \left( \frac{4ud}{\pi} \right) - \coth\mu\pi L_1 \left( \frac{4ud}{\pi} \right) \right] .$$

(5.3)

Note that $\theta = d/\rho \in [0,\pi]$ for rectangular contour and rhombic contour, $\theta \in [0,\pi/2]$ for triangular contour, and $\theta \in [0,\pi/4]$ for elliptical contour since we are living in a finite-size system, namely, $\theta \in [0,2\pi]$. Note also that the equivalence between rhombic contour and triangular contour is not quite a surprise. For instance, imagine $\theta = \pi$ in the case of triangular contour: then the triangular contour becomes the $\theta = \pi/2$ rhombic contour except for some differences in the order and direction in the contour.

It is also straightforward to show that

$$V_{\Box}(d) \rightarrow \frac{Q^2}{2} \left[ 1 - \frac{\mu d}{2} \coth\mu\pi \right] ,$$

(5.4)

$$V_{\Theta}(d) \rightarrow \frac{Q^2}{2} \left[ 1 - \frac{2\mu d}{3} \coth\mu\pi \right] ,$$

(5.5)

$$V_{\text{ellip}}(d) \rightarrow \frac{Q^2}{2} \left[ 1 - \frac{16\mu d}{3\pi^2} \coth\mu\pi \right] \text{ as } \mu d << 1 .$$

(5.6)

By keeping $d = \rho \theta$ fixed and letting $\rho \rightarrow \infty$, one reproduces immediately the $R^1 \times R^1$ flat-space static potentials of the forms

$$V_{\Box}(d) = \frac{Q^2}{2\mu} \left( 1 - e^{-\mu d} \right) ,$$

(5.7)

$$V_{\Theta}(d) = V_{\Box}(d) = \frac{Q^2}{2\mu} \left[ 1 - \frac{1 - e^{-2\mu d}}{2\mu d} \right] ,$$

(5.8)

$$V_{\text{ellip}}(d) = \frac{Q^2}{4\mu} \left[ I_1 \left( \frac{4ud}{\pi} \right) - L_1 \left( \frac{4ud}{\pi} \right) \right] ,$$

(5.9)

which is expected to agree with Ref. 2. Also,

$$V_{\Box}(d) \rightarrow \frac{Q^2}{2} \left[ 1 - \frac{\mu d}{2} \right] ,$$

(5.10)

$$V_{\Theta}(d) \rightarrow \frac{Q^2}{2} \left[ 1 - \frac{2\mu d}{3} \right] ,$$

(5.11)
in the flat-space limit. Note that the higher-order contributions of the binding energy differ from one another and depend on both the contour shapes and the radial function $\rho$ in the model. This indicates that the model on $R^1 \times S^1$ deserves more study.

Furthermore, in the long-distance limit $\mu d \gg 1$, one has

$$V_\Box(d) \rightarrow \frac{Q^2}{2\mu} \left[ 1 - \frac{1}{2\mu d} \right], \quad (5.13)$$

$$V_\phi(d) \rightarrow \frac{Q^2}{2\mu} \left[ 1 - \frac{\pi^2}{16} \right], \quad (5.14)$$

$$V_{\text{ellip}}(d) \rightarrow \frac{Q^2}{2\mu} \left[ 1 - \frac{1}{16 (\mu d)^2} \right]. \quad (5.15)$$

In particular, we discover the van der Wall force for $V_\phi$, $V_\Box$, and $V_{\text{ellip}}$ on $R^1 \times R^1$.

It is, however, not easy to discuss similar long-distance behavior on $R^1 \times S^1$ since $\theta$ space is compact. Nonetheless, we can show that $V_\Box(d)$ and $V_\phi(d)$ are both monotonically increasing functions in $d$. For example, we can show that

$$\frac{2\mu}{Q^2} \frac{\partial V_\Box}{\partial \theta} = \sinh \mu d (\coth \mu d - \coth \mu \rho \pi) > 0.$$  

Hence, $V_\Box(d)$ is a monotonically increasing function in $d$. Moreover,

$$\frac{2\mu}{Q^2} \frac{\partial V_\phi}{\partial \theta} = \frac{Y}{2\mu^2 d^2 \sinh \mu \rho \pi}$$

with

$$Y = \sinh \mu \rho \pi - \sinh (\rho \pi - 2d) - 2\mu \cosh (\rho \pi - 2d).$$

Note that $Y(\theta=0) = 0$, $Y(\theta=\pi/2) > 0$, and $\partial Y / \partial \theta > 0$ for all $\theta \in [0, \pi/2]$. Therefore, $Y$ (hence $\partial V_\phi / \partial \theta > 0$ for all $\theta \in [0, \pi/2]$. Hence $V_\phi(d)$ is also a monotonically increasing function in $d$. This agrees with the large-distance limit on $R^1 \times R^1$ shown above.

In this paper, we study the two-dimensional Schwinger model and its dependence on different contour shapes on $R^1 \times S^1$. By letting $\rho \rightarrow \infty$, we reproduce immediately the results on $R^1 \times R^1$. In this limit, we also discover the long-range van der Wall force form. We consider a rhombic shape contour instead of the ambiguous circular contour considered in Ref. 2. It is hard to imagine separating a $q\bar{q}$ pair by an infinite distance on the one hand, it is also difficult to analyze the circular contour within a presumed finite-sized $R^1 \times S^1$ model on the other hand. Hence we consider rhombic and triangular shape contours instead of the ambiguous circular contour considered in Ref. 2.

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