An $O(kn)$ Algorithm for a Circular Consecutive-$k$-out-of-$n$:F System

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Key Words — Circular consecutive-$k$-out-of-$n$:F system, System reliability, Algorithm.

Notation

- $n$: number of components in a system
- $k$: minimum number of consecutive failed components which causes system failure
- $i$: index for a component; $i = 1, 2, \ldots, n$
- $p_i, q_i$: probability that component $i$ functions, fails; $p_i + q_i = 1$

- $s_{ys}$: circular consecutive-$k$-out-of-$n$:F system
- $sys-i$: linear consecutive-$k$-out-of-$n$:F system
- $sys-i$: linear consecutive-$k$-out-of-$(n+i)$:F systems, for $i = 1, 2, \ldots, k-1$; $q_j = 1$ for $j = 1, 2, \ldots, i, n+1, \ldots, n+i$

- $F_{sys}$: Pr{sys is failed}
- $F_{sys-i}$: Pr{sys-i is failed}, for $i = 0, 1, \ldots, k-1$
- $F_{sys-i}$: Pr{sys-i is failed}, for $i = 1, \ldots, k-1$, wherein the first $(n+i-1)$ components are considered

Abstract — An $O(kn)$ algorithm is described for evaluating the reliability of a circular consecutive-$k$-out-of-$n$:F system.

1. INTRODUCTION

A consecutive-$k$-out-of-$n$:F system is $n$ ordered components such that the system fails if and only if at least $k$ consecutive components fail. Such a system is relevant to telecommunication. There are two topologies for this system: a straight line and a circle. The reliability of this system was first studied by Chiang & Niu [2], and later by Derman, Lieberman, Ross [3], Hwang [4], Shanthikumar [5], and Antonopoulou & Papastavridis [11].

Hwang [4] proposed two recursive algorithms to evaluate the reliabilities of linear and circular consecutive-$k$-out-of-$n$:F systems. These two algorithms require $O(n)$ and $O(n \cdot k^2)$ computing time respectively. Antonopoulou & Papastavridis [1] announced an $O(n \cdot k)$ recursive algorithm for computing the reliability of a circular such system. This paper demonstrates an algorithm $(Sys)$ to evaluate the reliability of a circular system which needs only $O(n \cdot k)$ computing time.

2. MODEL

Assumptions

1. Each component, subsystem and system either functions or fails.
2. All $n$ component states are mutually $s$-independent.
3. Components 1, 2, \ldots, $n$ are arranged on a circle in that order.
4. The system or subsystem fails if and only if at least $k$ consecutive components all fail.

3. COMPUTATION OF RELIABILITY

We express $Pr\{\text{sys is failed}\}$ by using the $Pr\{\text{sys-i is failed}\}$ formulas. Thus, $(3-1)$ is $Sys$-algorithm.

$$F_{sys} = F_{sys-0} + q_1 \cdot (F_{sys-1} - F'_{sys-1}) + q_1 \cdot q_2 \cdot (F_{sys-2} - F'_{sys-2}) + \ldots + q_1 \cdot q_2 \cdot \ldots \cdot q_{k-1} \cdot (F_{sys-(k-1)} - F'_{sys-(k-1)}).$$ (3-1)

Hwang [4] proved that the time complexity of computing the reliability of a linear consecutive-$k$-out-of-$n$:F system is $O(n)$. Because $n > k$, the computing time of each $F_{sys-i}$ or $F'_{sys-i}$ is $O(n+k) = O(n)$, for $i = 0, 1, \ldots, k-1$. Furthermore, the time complexity for calculating $q_0, q_1, q_2, \ldots, q_i$ is $O(k)$. So the time complexity for $(3-1)$ in the Sys-algorithm is $O(n) + O(k \cdot n) + O(k) = O(k \cdot n)$. Intuitively, the time complexity for the formula in [1] is $O(n^2 \cdot k)$.

4. PROOF

4.1 Lemma

Before proving $Sys$-algorithm we need the lemma.

Lemma: In $sys-i$, for $i = 1, \ldots, k-1$: 

\[ \square \]
Proof: By the sum-of-disjoint method, the failed probability of a linear sys-\(i\), for \(i = 1, \ldots, k-1\), is:

\[
F_{\text{sys-}i} = \Pr\{S_1 \cup S_2 \cup \ldots \cup S_{i-1} \cap S_{i+k+1}\}
\]

\[
= \Pr\{S_1\} + \Pr\{S_1 \cap S_2\} + \ldots + \Pr\{S_1 \cap S_2 \cap \ldots \cap S_{i-k-1} \cap S_{i-k}\}
\]

\[
+ \Pr\{S_1 \cap S_2 \cap \ldots \cap S_{i-k} \cap S_{i-k+1}\}.
\]

(4-2)

In the subsystem of the linear sys-\(i\) wherein the first \(n+1\) components are considered. Similarly, the failed probability of this subsystem is

\[
F_{\text{sys-}i} = \Pr\{S_1 \cup S_2 \cup \ldots \cup S_{i-k}\}
\]

\[
= \Pr\{S_1\} + \Pr\{S_1 \cap S_2\} + \ldots + \Pr\{S_1 \cap S_2 \cap \ldots \cap S_{i-k}\}.
\]

(4-3)

Eq (4-1) is obtained by subtracting (4-3) from (4-2). Q.E.D.

4.2 Sys-Algorithm

In the circular system,

\[
F_{\text{sys}} = F_{\text{sys-0}} + q_1[F_{\text{sys-1}} - F'_{\text{sys-1}}]
\]

\[
+ q_1 q_2[F_{\text{sys-2}} - F'_{\text{sys-2}}]
\]

\[
+ \ldots + q_1 q_2 \ldots q_{k-1}[F_{\text{sys-}k-1} - F'_{\text{sys-}k-1}].
\]

(4-4)

Proof: By the sum-of-disjoint method, the failed probability of the circular system is:

\[
F_{\text{sys}} = \Pr\{T_1 \cup T_2 \cup \ldots \cup T_n\}
\]

\[
= \Pr\{T_1\} + \Pr\{T_1 \cap T_2\} + \ldots + \Pr\{T_1 \cap T_2 \cap \ldots \cap \overline{T_{n-k}} \cap T_{n-k+1}\}
\]

\[
+ \Pr\{T_1 \cap T_2 \cap \ldots \cap \overline{T_{n-k}} \cap \overline{T_{n-k+1}} \cap T_{n-k+2}\}
\]

\[
+ \Pr\{T_1 \cap T_2 \cap \ldots \cap \overline{T_{n-k}} \cap \overline{T_{n-k+1}} \cap \overline{T_{n-k+2}}\}
\]

\[
+ \ldots + \Pr\{T_1 \cap T_2 \cap \ldots \cap \overline{T_{n-k}} \cap T_{n-k+1}\}.
\]

(4-5)

By (4-2), we can express the failed probability of sys-0 as:

\[
F_{\text{sys-0}} = \Pr\{T_1\} + \Pr\{T_1 \cap T_2\} + \ldots + \Pr\{T_1 \cap T_2 \cap \ldots \cap \overline{T_{n-k}} \cap T_{n-k+1}\}.
\]

(4-6)

Now we consider,

\[
\Pr\{\overline{T_1} \cap \overline{T_2} \cap \ldots \cap \overline{T_{n-k}} \cap \overline{T_{n-k+1}} \cap T_{n-k+2}\}
\]

in (4-5).

Notation

\(E\) event: \(T_1 \cap T_2 \cap \ldots \cap T_{n-k} \cap T_{n-k+1}\) and \(T_{n-k+2}\) fails.

\(F\) event: component 1 fails.

By applying

\[
\Pr[E] = \Pr[E|F] + \Pr[E|\overline{F}] = \Pr[E|F] \Pr[F]
\]

\[
+ \Pr[E|\overline{F}] \Pr[\overline{F}].
\]

(4-7)

We have

\[
\Pr\{T_1 \cap T_2 \cap \ldots \cap T_{n-k+1} \cap T_{n-k+2}\}
\]

\[
= \Pr\{1 \text{ fails}\} \Pr\{T_1 \cap T_2 \cap \ldots \cap T_{n-k+1} \cap T_{n-k+2}|1 \text{ fails}\}
\]

\[
+ \Pr\{1 \text{ functions}\} \Pr\{T_1 \cap T_2 \cap \ldots \cap T_{n-k+1}
\]

\[
\cap T_{n-k+2}|1 \text{ functions}\}
\]

\[
= \Pr\{1 \text{ fails}\} \Pr\{T_1 \cap T_2 \cap \ldots \cap T_{n-k+1}
\]

\[
\cap T_{n-k+2}|1 \text{ fails}\} + 0
\]

\[
= q_1(F_{\text{sys-1}} - F'_{\text{sys-1}}).
\]

(4-8)

In (4-8),

\[
\Pr\{T_1 \cap T_2 \cap \ldots \cap T_{n-k+1} \cap T_{n-k+2}|\text{component-1 functions}\}
\]

\[
= 0,
\]

because \(T_{n-k+2}\) is the event that all the components from \((n-k+2)\) to \(n\) and component-1 all fail, which is against the condition that component-1 functions. And

\[
\Pr\{T_1 \cap T_2 \cap \ldots \cap T_{n-k+1} \cap T_{n-k+2}|\text{component-1 fails}\}
\]

\[
= F_{\text{sys-1}} - F'_{\text{sys-1}},
\]

due to the lemma (4-1).

Notation

\(E\) event: \(T_1 \cap T_2 \cap \ldots \cap T_{n-k+1} \cap T_{n-k+3}\)

\(F\) event: components 1 & 2 both fail

By (4-7) we have:

\[
\Pr\{T_1 \cap T_2 \cap \ldots \cap T_{n-k+2} \cap T_{n-k+3}\}
\]
WU: AN O(kn) ALGORITHM FOR A CIRCULAR CONSECUTIVE-k-out-of-n:F SYSTEM

\[ q_1 q_2 (F_{sys-2} - F'_{sys-2}). \]  

(4-9)

By (4-8) & (4-9) & the lemma, we get the general formula for other terms in (4-4):

\[ \Pr \{ T_1 \cap T_2 \cap \ldots \cap T_{n-k+1} \cap T_{n-k+2} \} \]

\[ = q_1 q_2 \cdots q_k (F_{sys} - F'_{sys}), \]  

for \( i = 1, 2, \ldots, k - 1. \)  

(4-10)

Apply (4-5) & (4-10).

Q.E.D

REFERENCES


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Mean Time to Failure for a Consecutive-\( k \)-out-of-\( n \):F System

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Formulas for \( \mu, \mu_c, \mu_w \) in [1], contain some typographical errors. They must be the following formulas.

**Notation**

- \( n \): number of components in the system
- \( k \): minimum number of consecutive failed components that cause system failure
- \( p(t) \): reliability of component at time \( t \)
- \( \Gamma(x) \): Gamma function
- \( c, w \): as a subscript, implies the circular, Weibull case

**Formulas**

\[
\mu = \sum_{m=0}^{(k+2)n} (-1)^m \int_0^\infty p^m(t) \, dt \sum_{i=m-n}^m \left( \binom{n-ik}{i} \binom{ik}{m-i} \right) - \sum_{i=m-n}^m \left( \binom{n-ik-k}{i} \binom{ik+k}{m-i} \right)
\]

\[
\mu_w = a \Gamma \left( \frac{b+1}{b} \right) \sum_{m=0}^{(k+2)n} (-1)^m m^{-1/b} \sum_{i=m-n}^m \left( \binom{n-ik}{i} \binom{ik}{m-i} \right) - \sum_{i=m-n}^m \left( \binom{n-ik-k}{i} \binom{ik+k}{m-i} \right)
\]

\[
\mu_c = a \Gamma \left( \frac{b+1}{b} \right) \sum_{m=0}^{(k+2)n} (-1)^m \sum_{i=m-n}^m \left( \binom{n-ik}{i} \binom{ik}{m-i} \right)
\]

**REFERENCES**


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