On the arrangement graph

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Received 27 August 1996
Communicated by F. Dehne

Abstract

The arrangement graph was proposed as a generalization of the star graph topology. In this paper we investigate the topological properties of the (n, k)-arrangement graph A n,k. It has been shown that the (n, n - 2)-arrangement graph A n,n-2 is isomorphic to the n-alternating group graph AG n. In addition, the exact value of average distance of A n,k has been derived. © 1998 Published by Elsevier Science B.V. All rights reserved.

Keywords: Interconnection networks; Graph isomorphism; Average distance

1. Introduction

Recently, a widely studied interconnection network topology called the star graph was proposed by Akers et al. [2]. It has been known as an attractive alternative to the hypercube [1]. The star graph is node and edge symmetric, and strongly hierarchical as is the case of the hypercube. The n-star graph S n is regular of degree n - 1, the number of nodes n!, and diameter [3(n - 1)/2]. For a similar number of nodes, the star graph has a lower node degree, a shorter diameter, and a smaller average distance than the comparable hypercube.

In addition, Jow et al. presented another interconnection scheme based on the Cayley graph of the alternating group, called the alternating group graph [8]. The n-alternating group graph AG n is regular of degree 2(n - 2), the number of nodes n!/2, and diameter [3(n - 2)/2]. The alternating group graph is also node and edge symmetric.

A common drawback of S n and AG n is the restriction on the number of nodes: n! for S n and n!/2 for AG n. The set of values of n! (or n!/2) is spread widely over the set of integers; so, one may be faced with the choice of either too few or too many available nodes.

Even before [8] was published, Day and Tripathi [5] proposed a generalized star graph, called the arrangement graph, as an attractive interconnection scheme for massively parallel systems. An arrangement graph is specified by two parameters n and k, satisfying 1 &lt;= k &lt;= n - 1. The (n, k)-arrangement graph A n,k is regular of degree k(n - k), the number of nodes n!(n - k)!, and diameter [3k/2]. The (n, n - 1)-arrangement graph A n,n-1 is isomorphic to the n-star graph S n [6]. The arrangement graph provides more flexibility than the star graph in terms of choosing the major design parameters: degree, diameter, and number of nodes. The arrangement graph has been shown
to be node and edge symmetric, strongly hierarchical, maximally fault tolerant, and strongly resilient [5].

In this paper we further look into the topological properties of the arrangement graph. We first show that the \((n, n-2)\)-arrangement graph \(A_{n,n-2}\) is isomorphic to the \(n\)-alternating group graph \(AG_n\). Then, we derive the exact value for the average distance of \(A_{n,k}\). Due to the isomorphism between \(A_{n,n-2}\) and \(AG_n\), the average distance derived here for the arrangement graph as well as those discussed in the literature [4-7] can be applied directly to the alternating group graph. Therefore, we solve two of four open problems for the alternating group graph listed in [8]:

(1) enumeration of the node disjoint paths; and
(2) exact value of the average distance.

2. Graph definitions and basic properties

In this section we introduce the definitions and notations of the arrangement graph and the alternating group graph, and address their basic topological properties.

2.1. Graph definitions

For simplicity, let \(\langle n \rangle = \{1, 2, \ldots, n\}\) and \(\langle k \rangle = \{1, 2, \ldots, k\}\).

**Definition 1** [5]. The \((n, k)\)-arrangement graph

\(A_{n,k} = (V_1, E_1), \quad 1 \leq k \leq n-1,\)

is defined as follows:

\[ V_1 = \{p_1 p_2 \ldots p_k \mid p_i \in \langle n \rangle \text{ and } p_i \neq p_j \text{ for } i \neq j\}, \]

and

\[ E_1 = \{(p, q) \mid p \text{ and } q \in V_1 \text{ and for some } i \in \langle k\rangle, \]

\[ p_i \neq q_i \text{ and } p_j = q_j \text{ for } j \neq i\}. \]

That is, the nodes of \(A_{n,k}\) are the arrangements of \(k\) elements out of the \(n\) symbols \(\langle n \rangle\), and the edges of \(A_{n,k}\) connect arrangements which differ in exactly one of their \(k\) positions. An edge of \(A_{n,k}\) connecting two arrangements which differ only in position \(i\) is called an \(i\)-edge. An example of \(A_{n,k}\) for \(n = 4\) and \(k = 2\) is given in Fig. 1.

To define the alternating group graph, we describe some terminologies and notations for ease of exposition. In a permutation \(p = p_1 p_2 \ldots p_i \ldots p_j \ldots p_n\), the pair \((i, j), i < j,\) is said to constitute an inversion if \(p_i > p_j\). A permutation \(p\) is called even or odd depending on the number of inversions in \(p\) being even or odd. The alternating group \(A_n\) is defined as the set of all even permutations of \(n\) elements [3].

Let

\[ g_i^+ = (1 \ 2 \ i), \quad g_i^- = (1 \ i \ 2), \]

\[ \Omega = \{g_i^+ \mid 3 \leq i \leq n\} \cup \{g_i^- \mid 3 \leq i \leq n\}, \]

where \(\Omega\) is known to be a generator set for \(A_n\) [8]. Let \(T_{ij}\) denote the transposition that swaps the elements at positions \(i\) and \(j\), then \(g_i^+ = T_{12} \cdot T_{2i} \text{ and } g_i^- = T_{2i} \cdot T_{i2}\) for \(3 \leq i \leq n\).

**Definition 2** [8]. The \(n\)-alternating group graph \(AG_n = (V_2, E_2)\) is defined as follows:

\[ V_2 = A_n, \quad \text{and} \]

\[ E_2 = \{(p, q) \mid p, q \in V_2, \text{ and} \]

\[ q = p \circ g \text{ for some } g \in \Omega\}, \]

where \(\circ\) is the composition operator. The nodes of \(AG_n\) are even permutations of the \(n\) symbols \(\langle n \rangle\). An edge of \(AG_n\) connecting two even permutations \(p\) and \(q\) is called an \(i\)-edge if \(q = p \circ g_i^+\) and \(a = +\) or \(-\). Note that \(q = p \circ g_i^+\) if and only if \(p = q \circ g_i^-\). An example of \(AG_n\) for \(n = 4\) is given in Fig. 2.
2.2. Basic properties

There are \((n - 1)!/(n - k)\) nodes in \(A_{n,k}\) which have element \(i\) in position \(j\), for any fixed \(i\) and \(j\) (\(1 \leq i < n\), \(1 \leq j < k\)). These nodes are interconnected in a manner identical to an \(A_{n-1,k-1}\) graph. For a fixed position \(j\), an \(A_{n,k}\) can be partitioned into \(n\) node-disjoint copies of \(A_{n-1,k-1}\). This partitioning of \(A_{n,k}\) into \(n\) copies of \(A_{n-1,k-1}\) can be done in \(k\) different ways corresponding to the \(k\) possible values of \(j\) (\(1 \leq j < k\)) and can be carried out recursively. Fig. 1 shows that \(A_{4,2}\) can be viewed as an interconnection of four \(A_{3,1}'\)s by fixing the symbol in position 2.

Due to the node symmetry of the \((n, k)\)-arrangement graph, the problem of routing between two arbitrary nodes in \(A_{n,k}\) is reduced to the problem of routing between an arbitrary node and the identity node \(I_k\). In order to solve the problem of the routing, a cycle representation for the label of each node in \(A_{n,k}\) was introduced in [5]. The cycle representation of a node \(p\) with \(c\) cycles including \(e\) external cycles can be denoted as

\[ C(p) = C_1, C_2, \ldots, C_c, C'_{c+1}, C'_{c+2}, \ldots, C_c' \]

where \(C_1, C_2, \ldots, C_c\) are external cycles and \(C'_{c+1}, C'_{c+2}, \ldots, C_c'\) are internal cycles. Let \(m\) denote the total number of elements in these cycles. It has been shown in [5] that the distance \(d(p)\) from node \(p\) to the identity node \(I_k\) in \(A_{n,k}\) is given by:

\[ d(p) = c + m - 2e. \quad (1) \]

In addition, it has been presented in [5] that the connectivity \(\kappa(A_{n,k})\) of \(A_{n,k}\) is \(k(n-k)\), and the fault-diameter \(D_f(A_{n,k})\) of \(A_{n,k}\) is at most \(\lfloor 3k/2 \rfloor + 4\). In [6], it has been known that the arrangement graph \(A_{n,k}\), for \(1 \leq k \leq n-2\), contains cycles of any arbitrary length \(L\), \(3 \leq L \leq |A_{n,k}|\), where \(|A_{n,k}|\) is the number of nodes in \(A_{n,k}\). Furthermore, it has been shown in [7] that an \((n-k+1) \times (n-k+2) \times \cdots \times (n-1) \times n\) grid can be embedded into \(A_{n,k}\) with unit expansion and dilation three.

3. Isomorphism between \(A_{n,n-2}\) and \(AG_n\)

In this section, we show the alternating group graph is a spatial arrangement graph. Before proving it, we give a formal definition of isomorphism between two graphs in the following.

**Definition 3.** Two graphs \(G_1\) and \(G_2\) are isomorphic if there is a one-to-one function \(f\) from \(V(G_1)\) onto \(V(G_2)\) such that \((p, q) \in E(G_1)\) if and only if \((f(p), f(q)) \in E(G_2)\), where \(f(p)\) is the image of \(p\).

**Theorem 4.** The \((n, n-2)\)-arrangement graph \(A_{n,n-2}\) is isomorphic to the \(n\)-alternating group graph \(AG_n\).

**Proof.** To prove that \(AG_n\) and \(A_{n,n-2}\) are isomorphic, we define a one-to-one function \(f_1\) from the nodes of \(AG_n\) to those of \(A_{n,n-2}\) by:

\[ f_1(p_1p_2p_3 \ldots p_{n-1}p_n) = p_3 \ldots p_{n-1}p_n. \]

Note that \(f_1^{-1}(q_3 \ldots q_{n-1}q_n) = q_1q_2q_3 \ldots q_{n-1}q_n\) or \(q_2q_1q_3 \ldots q_{n-1}q_n\) depending on which one is an even permutation.

Now, we prove \(f_1\) preserves adjacency. Let \(p\) and \(q\) be two nodes linked with an \(i\)-edge in \(AG_n\). Then \(f_1(p)\) and \(f_1(q)\) are linked with an \((i-2)\)-edge in \(A_{n,n-2}\). Conversely, let \(s\) and \(t\) be two nodes linked with a \((j-2)\)-edge in \(A_{n,n-2}\), i.e.,

\[ s = s_3 \ldots s_{j-1}s_js_{j+1} \ldots s_n \quad \text{and} \quad t = s_3 \ldots s_{j-1}s_js_{j+1} \ldots s_n. \]

Without loss of generality, assume \(s\) is an even permutation, then

\[ f_1^{-1}(s) = s_1s_3 \ldots s_{j-1}s_js_{j+1} \ldots s_n \quad \text{and} \quad f_1^{-1}(t) = s_2s_3 \ldots s_{j-1}t_js_{j+1} \ldots s_n. \]
where $s_1 < s_2$ and $t_1 = s_1$, and thus $f_1^{-1}(t) = f_1^{-1}(s) \circ g_j^+$. That is, $f_1^{-1}(t)$ and $f_1^{-1}(s)$ are linked with a $j$-edge in $AG_n$. □

The example for $A_{4,2}$ in Fig. 1 is the resulting graph when $f_1$ is applied to $AG_4$ in Fig. 2.

The next corollary solves one of four open problems listed in [8]. We get the result by using Theorem 4 and the fact that for any two nodes of $A_{n,k}$, there exists $k(n - k)$ node-disjoint paths of length at most four plus the distance between the two nodes [4].

**Corollary 5.** Given any distinct nodes $s$ and $t$ in an $n$-alternating group graph $AG_n$, there are $2(n - 2)$ node-disjoint paths of length at most four plus the distance between the two nodes. That is, the connectivity $\kappa(AG_n)$ of $AG_n$ is $2(n - 2)$ and the fault-diameter $D_f(AG_n)$ of $AG_n$ is at most $\lfloor 3(n - 2)/2 \rfloor + 4$.

The following corollary describes the routing distance in $AG_n$. Although the result was discussed in [8], here we get it directly by applying Eq. (1) with Theorem 4.

**Corollary 6.** The distance $d(p)$ from node $p$ to the identity node $I_n$ in $AG_n$ is:

$$d(p) = \begin{cases} 
  n + k - 1, & \text{if } p_1 = 1 \text{ and } p_2 = 2, \\
  n + k - l - 2, & \text{if } p_1 \neq 1 \text{ and } p_2 = 2, \\
  n + k - l - 2, & \text{if } p_1 = 1 \text{ and } p_2 \neq 2, \\
  n + k - l - 3, & \text{if } 1 \in C_i \text{ and } 2 \in C_j, \\
  n + k - l - 4, & \text{if } 1 \in C_i \text{ and } 2 \in C_j,
\end{cases}$$

where $k$ and $l$ denote the numbers of cycles and invariants in $C(p)$.

**Proof.** The distance from $p = p_1 p_2 p_3 \ldots p_n$ to $I_n = 123 \ldots n$ in $AG_n$ is equivalent to the distance from $p'' = p_3 \ldots p_n$ to $I''_n = 3 \ldots n$ in $A_{n,n-2}$. Let $c''$, $m''$, and $e''$ denote the number of cycles, the total number of elements in these cycles, and the number of external cycles in $C(p'')$ with respect to $I''_n$; that is, $d(p) = d(p'') = c'' + m'' - 2e''$. Note that symbols 1 and 2 with respect to $I''_n$ are foreign symbols.

(a) If $p_1 = 1$ and $p_2 = 2$, then $c'' = k$, $m'' = n - l$, $e'' = 0$, and $d(p'') = n + k - l$.

(b) If $p_1 \neq 1$ and $p_2 = 2$, then $c'' = k$, $m'' = n - l$, $e'' = 1$, and $d(p'') = n + k - l - 2$.

(c) If $p_1 = 1$ and $p_2 \neq 2$, then $c'' = k$, $m'' = n - l$, $e'' = 1$, and $d(p'') = n + k - l - 2$.

(d) For $1, 2 \in C_i$, if either $p_1 = 2$, or $p_2 = 1$, $c'' = k$, $m'' = n - l - 1$, $e'' = 1$; otherwise, $c'' = k + 1$, $m'' = n - l$, $e'' = 2$. Therefore, $d(p'') = n + k - l - 3$.

(e) If $1 \in C_i$ and $2 \in C_j$, $c'' = k$, $m'' = n - l$, and $e'' = 2$; so, $d(p'') = n + k - l - 4$. □

4. Average distance

In this section, we derive the exact values for the average distances of the $(n, k)$-arrangement graph $A_{n,k}$ and the $n$-alternating group graph $AG_n$. The average distance of a symmetric interconnection network is determined by the summation of distances of all nodes from a given node over the total number of nodes. Average distance is a better indicator of the average message delay in an interconnection network than its diameter. Since $A_{n,k}$ is node-symmetric, its average distance among all pairs of nodes $p$ and $q$ (possibly $p = q$) equals the average distance from the identity node $I_n$ to all nodes.

Let $\bar{D}(A_{n,k})$ denote the average distance of $A_{n,k}$. The value of this measure for the $(n, k)$-arrangement graph is

$$\bar{D}(A_{n,k}) = \frac{\sum_{p \in A_{n,k}} d(p)}{N(n, k)},$$

where $N(n, k) = n!/(n - k)!$.

It is known that the average number of cycles including invariants in a permutation of $n$ symbols is $H_n$, where $H_n = \sum_{i=1}^{n} 1/i$ denotes the $n$th Harmonic number [9]. Here, we show the average number of cycles including invariants in a permutation of $k$ elements out of $n$ symbols is $H_k$.

**Lemma 7.** The average number of cycles including invariants in a permutation of $k$ elements out of $n$ symbols is $H_k$.

**Proof.** We consider a permutation $p$ formed by choosing arbitrarily $k$ elements out of the $n$ symbols $(n)$. Let $p'$ be the permutation obtained from replacing each foreign element of $p$ with its corresponding nonforeign element. Then the total number of cycles of $C(p')$ is equal to that of $C(p)$. The number of cycles in all permutations of the $k$ symbols $(k)$ is $k! \times H_k$. Therefore,
fore, the total number of cycles in all permutations of $k$ elements out of the $n$ symbols $(n)$ is

$$\binom{n}{k} \times k! \times H_k.$$  

Dividing it by the total number of permutations of $k$ elements out of the $n$ symbols $(n)$, we derive the average number of cycles in a permutation as $H_k$. 

Given a node $p$, let $u$ be the total number of invariants, $c^*$ be the total number of cycles including invariants, and $m^*$ be the total number of misplaced symbols, i.e., $m^* = k - u$. In Eq. (1), $c$ denotes the total number of cycles of $C(p)$ excluding invariants, and $m$ denotes the total number of elements in these cycles including the corresponding nonforeign elements for the external cycles. It then follows that for $c$ and $m$ as defined above, $c = c^* - u$ and $m = m^* + e$. Similar to the argument of Theorem 2 in [1], we rewrite Eq. (1) as

$$d(p) = k + c^* - e - 2u. \tag{2}$$

The following theorem gives the exact value for the average distance of $A_{n,k}$, which is obtained by computing the total value of Eq. (2) for each node in $A_{n,k}$ and then dividing by $n!/(n-k)!$.

**Theorem 8.** The average distance $\overline{D}(A_{n,k})$ of the $(n,k)$-arrangement graph is given by:

$$\overline{D}(A_{n,k}) = H_k + \frac{k(k - 2)}{n}.$$  

**Proof.** The first term in the summation of Eq. (2) over all nodes of $A_{n,k}$ is $k \times n!/(n-k)!$, and the second term is $H_k \times n!/(n-k)!$. The third term is the total number of foreign symbols in all nodes of $A_{n,k}$, i.e., $k \times (n-k) \times (n-1)!/(n-k)!$. The final term is the total number of symbols in the correct position for all nodes multiplied by $-2$, i.e., $-2 \times k \times (n-1)!/(n-k)!$. Therefore,

$$\overline{D}(A_{n,k}) = k + H_k - k \times (n-k)/n - 2k/n
\begin{align*}
&= H_k + k \times (k-2)/n. \quad \Box \end{align*}$$

The next corollary solves another open problem listed in [8]. That is, the exact value for the average distance of $AG_n$ is found while the result for the average distance of $AG_n$ proposed in [8] is just an upper bound.

**Corollary 9.** The average distance $\overline{D}(AG_n)$ of the $n$-alternating group graphs is

$$n + H_n + \frac{7}{n} - \frac{1}{n-1} - 6.$$  

**Proof.** By applying Eq. (3) with Theorem 4,

$$\overline{D}(AG_n) = \overline{D}(A_{n,n-2}). \quad \Box$$

**References**


