Group Divisible Designs with Two Associate Classes: 

\[ n = 2 \text{ or } m = 2 \]

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In this paper we find necessary and sufficient conditions for the existence of a group divisible design \( GDD(n, m) \) of index \((\lambda_1, \lambda_2)\) in which \( n = 2 \) or \( m = 2 \), thereby completing the solution of the existence problem for all \( n, m, \lambda_1, \) and \( \lambda_2 \). In the process, necessary and sufficient conditions are found for the existence of an \( x \)-regular partial triple system whose complement in \( \ast K_n \) has a 1-factorization.

1. INTRODUCTION

A group divisible design \( GDD(n, m; k; \lambda_1, \lambda_2) \) is an ordered triple \((V, G, B)\) where \( V \) is a set of varieties or symbols, \( G \) is a partition of \( V \) into \( m \) sets of size \( n \), each set being called a group, and \( B \) is a collection of subsets of \( V \), called blocks, each of size \( k \), such that

1. each pair of symbols that occur together in the same group occur together in exactly \( \lambda_1 \) blocks, and

2. each pair of symbols that occur together in no group occur together in exactly \( \lambda_2 \) blocks.

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Elements occurring together in the same group are called first associates, and elements occurring in different groups are called second associates. We say that the GDD is defined on the set $V$.

GDDs with $\lambda_1 = 0$ have been of great use in constructing other designs. But just as interesting to the statisticians are GDDs in which $\lambda_1 > 0$. According to Raghavarao [12] partially balanced designs with two association classes were classified in 1952 by Bose and Shimamoto into five types: group divisible designs, simple, triangular, latin square type and cyclic. We will concentrate here on group divisible designs. For a wealth of information on GDDs see Raghavarao [12]. Clatworthy [3] gives tables for all the classes of GDDs.

In this paper, we consider the existence of GDDs in the case where $k = 3$. To simplify the notation, let a GDD($n, m; 3; \lambda_1, \lambda_2$) be denoted by a GDD($n, m$) of index ($\lambda_1, \lambda_2$), and let a block of size 3 be called a triple. Together with Dinesh Sarvate, we have already completely solved this existence problem in the case where $n, m \geq 3$, proving the following result.

**Theorem 1.1** [7]. Let $n, m \geq 3$ and $\lambda_1, \lambda_2 \geq 1$. There exists a GDD($n, m$) of index ($\lambda_1, \lambda_2$) if and only if

1. $2$ divides $\lambda_1(n - 1) + \lambda_2(m - 1)n$,
2. $3$ divides $\lambda_1mn(n - 1) + \lambda_2m(m - 1)n^2$.

GDDs with $m = 1$ (so $\lambda_2$ is irrelevant) and $k = 3$ are the well known triple systems (TS), so we denote a GDD($n, 1; 3; \lambda_1, \lambda_2$) by a TS($n$) of index $\lambda_1$. Since a TS($n$) has only one group, $V = G$ so it can be simply represented by ($V; B$). We will use the following famous result.

**Theorem 1.2** ([8], or see [10]). Let $n \geq 3$. There exists a TS($n$) of index $\lambda$ iff

(a) $2$ divides $\lambda(n - 1)$, and
(b) $3$ divides $\lambda n(n - 1)$.

So the existence of a GDD($n, m$) of order ($\lambda_1, \lambda_2$) has been completely settled if $m = 1$ or $n = 1$ (and so also if $\lambda_2 = 0$). It has also been settled if $\lambda_1 = 0$ with the following result.

**Theorem 1.3** [9]. There exists a GDD($n, m$) of index (0, $\lambda_2$) iff

(a) $2$ divides $\lambda_2(m - 1)n$,
(b) $3$ divides $\lambda_2m(m - 1)n^2$, and
(c) $m \geq 3$. 
In this paper, the cases where \( m = 2 \) and where \( n = 2 \) are solved (see Theorem 5.1), thus completing the solution of the existence problem for a GDD\((n, m)\) of index \((\lambda_1, \lambda_2)\) (see Theorem 5.2). At first sight, this would seem to be quite simple to handle compared to the myriad of cases that have to be considered to prove Theorem 1.1. However, it turns out to be a very interesting case, requiring different solution techniques and another necessary condition. In particular, one technique developed here requires knowing when there exists a multigraph on \( n \) vertices whose edges can be partitioned into triples, and whose complement in \( \lambda K_n \) has a 1-factorization (see Theorem 2.9). This result is of interest in its own right (see [15], for example).

Graph theoretically, a GDD\((n, m)\) of index \((\lambda_1, \lambda_2)\) is a partition of the edges of a graph \( H \) into copies of \( K_3 \) (each \( K_3 \) is also called a triple), where \( H \) is the multigraph with vertex set \( V = V_0 \cup V_1 \cup \cdots \cup V_{m-1}, |V_i| = n \) for each \( i \in \mathbb{Z}_m \), in which two vertices are joined by \( \lambda_1 \) edges if they both occur in \( V_i \) for some \( i \), and otherwise are joined by \( \lambda_2 \) edges. Edges joining vertices in the same or different groups are called pure or cross edges respectively. This description of a GDD will often be used in this paper.

It is worth remarking that many papers have been written on GDDs; for example considering the case where \( k = 4 \) [2], and the case where not all groups have the same size [4]. See [5] for many references.

2. PRELIMINARY RESULTS: \( m = 2 \)

In this section we obtain several building blocks. In Section 3, these will be put together in various ways to show that the following necessary conditions are sufficient for the existence of a GDD\((n, 2)\) of index \((\lambda_1, \lambda_2)\) (see Theorem 3.7).

**Lemma 2.1.** If there exists a GDD\((n, 2)\) of index \((\lambda_1, \lambda_2)\) then

1. \( 2 \) divides \( \lambda_1(n-1) + \lambda_2 n \),
2. \( 3 \) divides \( \lambda_1 n(n-1) + \lambda_2 n^2 \), and
3. \( \lambda_1 \geq \lambda_2 n/2(n-1) \).

**Proof.** Conditions (1) and (2) follow because each vertex must have even degree, and the number of edges must be divisible by 3. (3) follows since any cross edge must be contained in a triple that contains another cross edge and a pure edge, so the number of pure edges must be at least half the number of cross edges.

We now proceed to produce some building blocks and other useful results.
Lemma 2.2. Let \( n \geq 3 \). There exists a GDD\((n, 2)\) of index \((n, 2n - 2)\).

Proof. Define

\[
B = \{(a, 0), (b, 0), (c, 1), (a, 1), (b, 1), (c, 0) \mid 0 \leq a < b \leq n - 1, c \in \mathbb{Z}_n\}.
\]

Then \((\mathbb{Z}_n \times \mathbb{Z}_2, \{Z_n \times \{i\} \mid i \in \mathbb{Z}_2\}, B)\) is a GDD\((n, 2)\) of index \((n, 2n - 2)\).

The following is a result of Petersen.

Theorem 2.3 [11]. Let \( H \) be a regular multigraph of even degree. Then there exists a 2-factorization of \( H \).

Lemma 2.4 is a special case of a result of Rodger and Stubbbs.

Lemma 2.4 [16]. Let \( \lambda, n \geq 1 \). Suppose that \( 0 \leq x \leq \lambda(n - 1) \), \( x \) is even, and 3 divides \( xn \). Then there exists an \( x \)-regular multigraph of multiplicity at most \( \lambda \) with \( n \) vertices whose edges can be partitioned into triples.

These two results can be combined to obtain Corollary 2.5. Let \( E(H) \) be the set of edges in \( H \).

Corollary 2.5. Suppose that \( \lambda, n \geq 1 \), \( 0 \leq x \leq \lambda(n - 1) \), 3 divides \( xn \), and \( \lambda(n - 1) \) and \( x \) are even. Then there exists an \( x \)-regular multigraph \( H \) of multiplicity at most \( \lambda \) with \( n \) vertices whose edges can be partitioned into triples, such that \( \lambda K_n - E(H) \) has a 2-factorization.

Proof. Choose \( H \) using Lemma 2.4, then apply Theorem 2.3 to \( \lambda K_n - E(H) \).

We will need a companion result to Corollary 2.5 to cope with the situation where \( \lambda(n - 1) \) is odd. Obtaining this result will require the following results, the first by Stern and Lenz, the second by Rees, and the third by Simpson. For any \( D \subseteq \mathbb{Z}_{2^n} \setminus \{0\} \), let \( H[D] \) be the graph with vertex set \( \mathbb{Z}_n \) and edge set \( \{(j, j + d) \mid d \in D, j \in \mathbb{Z}_n\} \), reducing the sum modulo \( n \).

Lemma 2.6 [18]. There exists a 1-factorization of \( H[D] \) if and only if there exists a \( d \in D \) such that \( n/\gcd(n, d) \) is even.

Note that if \( d = n/2 \in D \) then since \( n/\gcd(n, d) \) is even, \( H[D] \) has a 1-factorization.

Theorem 2.7 [15]. For all \( n \equiv 0 \pmod{6} \) and for all even \( x \) with \( 0 \leq x < n \) except \((n, x) \in \{(12, 10), (6, 4)\} \), there exists an \( x \)-regular simple graph \( H \) on \( n \) vertices whose edges can be resolvably partitioned into triples, such that \( K_n - E(H) \) has a 1-factorization.
Theorem 2.8 [17]. For any \( y \geq 1 \) and for some \( x \in \{ 3y, 3y + 1 \} \), the integers in \( \{ y + 1, y + 2, \ldots, 3y + 1 \} \setminus \{ x \} \) can be partitioned into pairs \((a_i, b_i)\) with \( b_i > a_i \) such that \( \{ b_i - a_i \mid 1 \leq i \leq y \} = \{ 1, 2, \ldots, y \} \).

We can now present the companion to Corollary 2.5. It is a result that is of interest in its own right.

Theorem 2.9. Suppose that \( \lambda \geq 1 \) and \( n \geq 3 \). Then

(i) there exists an \( x \)-regular graph \( H \) on \( n \) vertices and of multiplicity at most \( \lambda \) whose edges can be partitioned into triples, such that

(ii) \( \lambda K_n - E(H) \) has a 1-factorization.

if and only if \( 0 \leq x \leq \lambda(n - 1) \), if \( x > 0 \) then 3 divides \( xn \), if \( x < \lambda(n - 1) \) then 2 divides \( n \), and 2 divides \( x \).

Proof. It is obvious that if (i) and (ii) are true then \( 0 \leq x \leq \lambda(n - 1) \), if \( x > 0 \) then 3 divides \( xn \), if \( x < \lambda(n - 1) \) then 2 divides \( n \), and 2 divides \( x \); therefore we will now prove the opposite statement.

For each \( \lambda \geq 1 \) and each even \( n \geq 4 \), let \( S(n, \lambda) \) be the set of integers \( x \) for which (i) and (ii) are true. Let \( \ell = 2 \) if \( n \equiv 0 \) or \( 4 \) (mod 6) and let \( \ell = 6 \) if \( n \equiv 2 \) (mod 6).

Since there exists a 1-factorization of \( K_n \), \( 0 \in S(n, \lambda) \), and for \( x < \lambda(n - 1) \) if \( x \in S(n, \lambda) \) then \( x \in S(n, \lambda') \) for all \( \lambda' \geq \lambda \). Also, by Theorem 1.2 there exists a TS(n) of index \( \ell \); so if \( x = y(n - 1) + x' \) with \( 0 \leq x' < \lambda(n - 1) \) and \( \lambda < \ell \), and if \( x' \in S(n, \lambda) \), then \( x \in S(n, \lambda + y\ell) \). Therefore we need only consider the cases where \( 0 < x < \ell(n - 1) \).

Suppose that \( n \equiv 0 \) (mod 6). We need only consider the cases where \( x < 2(n - 1) \). If \( x < n \) then the result follows from Theorem 2.7 unless \( (n, x) \in \{ (12, 10), (6, 4) \} \). Fortunately, since we do not require the set of triples to be resolvable, we can obtain solutions in these cases too: for each \( m \in \{ 3, 6 \} \) the complement of the edges in the triples of a GDD(2, \( m \)) of index (0, 1) is a 1-factor. If \( n \equiv x \equiv 2n - 4 \) then we can simply combine a solution where \( x' = n - 2 \) and \( \lambda' = 1 \) with a solution where \( x'' = x - (n - 2) \) and \( \lambda'' = 1 \).

If \( n \equiv 2 \) or \( 4 \) (mod 6) then since \( x \) is even and 3 divides \( xn \), we have that \( x \equiv 0 \) (mod 6). So let \( x = 6y \). If \( x = n - 2 \) then \( n \equiv 2 \) (mod 6); since there exists a GDD(2, \( 3y + 1 \)) of index (0, 1) we have that \( n - 2 \in S(n, 1) \). If \( x < n - 2 \) then define \( a_i \) and \( b_i \) as in Theorem 2.8, and let \( T = \{ \{ j, a_i, j, b_i + j \} \mid j \in \mathbb{Z}_n \} \), reducing sums modulo \( n \). Then \( T \) is a set of triples that partition \( H = H[D'] \) where \( D' = \{ 1, 2, \ldots, 3y + 1 \} \setminus \{ s \} \), and \( K_n - E(H) = H[D] \) where \( D = \{ 1, 2, \ldots, n/2 \} \setminus D' \). Since \( x < n - 2 \), \( n/2 \in D \), so \( K_n - E(H) \) has a 1-factorization by Lemma 2.6. So it remains to consider \( x \geq n \).
If \( n \equiv 4 \pmod{6} \) then \( \ell = 2 \) so we can assume that \( x < 2(n-1) \); so \( n + 2 \leq x \leq 2n - 8 \) (since \( x \equiv 0 \pmod{6} \)). We can combine a solution where \( x' = n-4 \) and \( \ell' = 1 \) with a solution where \( x'' = x - (n-4) \leq n-4 \) and \( \ell'' = 1 \).

If \( n \equiv 2 \pmod{6} \) then \( \ell = 6 \), so we can assume that \( x < 6(n-1) \); so \( n + 4 \leq x \leq 6n - 12 \) (since \( x \equiv 0 \pmod{6} \)). Let \( \ell' \) be such that \( \ell'(n-2) < x \leq (\ell' + 1)(n-2) \). Combine \( \ell' \) solutions where \( x' = n-2 \) and \( \ell' = 1 \) with a solution where \( x'' = x - \ell'(n-2) \leq n-2 \) and \( \ell'' = 1 \).

It will be useful to let \( [x, y, z] \) denote the graph with vertex set \( \mathbb{Z}_n \times \mathbb{Z}_2 \) in which two vertices \((u, i)\) and \((v, j)\) are joined by \( x \) edges if \( i = j = 0 \), by \( y \) edges if \( i = j = 1 \), and by \( z \) edges if \( i \neq j \).

The next four results are crucial building blocks in the construction of the GDD’s in Section 3.

**Lemma 2.10.** For each \( i \in \mathbb{Z}_2 \), let \( T_i \) be an \( xn \)-regular multigraph on the vertex set \( \mathbb{Z}_n \times \{1\} \) that has a 1-factorization. Then there exists a set of triples whose edges partition the edges of \([0, x, 0] + T_i\).

**Proof.** Partition the \( xn \)-1-factors in a 1-factorization of \( T_i \) into \( n \) sets \( S_0, S_1, \ldots, S_n-1 \), each of size \( x \). For each \( a \in \mathbb{Z}_n \) and for each edge \( \{(a, i), (v, i)\} \) in a 1-factor in \( S_a \), let \( B \) contain the triple \( \{(a, i), (v, i), (a, i+1)\} \), reducing the sum modulo 2.

A quasigroup \( (\mathbb{Z}_n, \cdot) \) of order \( n \) is an \( n \times n \) array in which each cell contains exactly one symbol, and each symbol in \( \mathbb{Z}_n \) occurs exactly once in each row and exactly once in each column; if cell \((a, b)\) contains \( c \) then we write \( a \cdot b = c \). A quasigroup \( (\mathbb{Z}_n, \cdot) \) is symmetric if \( a \cdot b = b \cdot a \) for all \( a, b \in \mathbb{Z}_n \), and is idempotent if \( a \cdot a = a \) for all \( a \in \mathbb{Z}_n \). It is well known (and easy to see!) that there exists a symmetric idempotent quasigroup of order \( n \) for all odd \( n \geq 1 \).

**Lemma 2.11.** Let \( n \) be odd, and let \( F \) be any 1-factor of \([0, 1, 0]\). Then there exists an edge-disjoint decomposition of \([1, 1, 0] - F \) and of \([0, 1, 1] - F \) into copies of \( K_1 \).

**Proof.** Let \((\mathbb{Z}_n, \cdot)\) be a symmetric idempotent quasigroup of order \( n \).

Let \( i \in \mathbb{Z}_n \) and let \( F' = \{ (a, 0), (a, 1) \mid a \in \mathbb{Z}_n \} \). Let \( B_i = \{ (a, i), (b, i), (a \cdot b, i+1) \mid 0 \leq a < b \leq n-1 \} \), reducing \( i + 1 \) modulo 2. Then clearly the triples in \( B_i \) partition the edges in \([1, 1, 0] - F'\) or \([0, 1, 1] - F'\) if \( i = 0 \) or 1 respectively. The first coordinate of the symbols in the triples in \( B_i \) whose second coordinate is \( i+1 \) can easily be renamed to produce a set of triples \( B_i \) that partition the edges of \([1, 1, 0] - F \) or \([0, 1, 1] - F \) as required.
Lemmas 2.12. Let \( i \in \mathbb{Z}_2 \), and let \( H_i \) be a \( 2x \)-regular graph on the vertex set \( \mathbb{Z}_n \times \{i\} \). Then there exists a \( 2x \)-regular multigraph \( T \) consisting of \( 2x \)
1-factors, each being in \([0, 1, 0] \), such that there exists an edge-disjoint decomposition of \( H_i + T \) into copies of \( K_x \).

Proof. By Theorem 2.3, \( H_i \) has a 2-factorization into \( x \) 2-factors \( T_0, T_1, \ldots, T_{x-1} \). For each \( j \in \mathbb{Z}_x \), \( T_j \) consists of vertex disjoint cycles which we can arbitrarily orient to form directed cycles; call the resulting directed graph \( T_j \). Let \( H_i \) be the corresponding directed graph. For each directed edge \( (a, b) \in T_j \), let \( \{(a, i), (b, i + 1)\} \in F_{2j} \) and \( \{(a, i), (a, i + 1)\} \in F_{2j + 1} \). Let \( T \) be the \( 2x \)-regular multigraph formed by the sum of \( F_{0}, \ldots, F_{2x-1} \). Then \( B = \{(a, i), (b, i), (b, i + 1)\} \in E(H_i) \) is a set of triples whose edges partition the edges of \( H_i + T \).

Lemmas 2.13. Let \( n \geq 4 \) be even. Let \( \epsilon = 0 \) if \( n \equiv 0 \pmod{4} \), \( \epsilon = 1 \) if \( n \equiv 6 \pmod{12} \), and \( \epsilon = 3 \) if \( n \equiv 2 \) or \( 10 \pmod{12} \). For each \( i \in \mathbb{Z}_2 \) there exists a simple graph \( H_i \) on the vertex set \( \mathbb{Z}_n \times \{i\} \) such that:

(i) \( H_0 \) is \((n/2 + \epsilon)\)-regular and \( H_1 \) is \((n/2 - \epsilon)\)-regular,

(ii) the edges of \([0, 1, 0] + H_0 + H_1\), can be partitioned into triples,

(iii) there exists a 1-factorization of \( K_n - E(H_i) \), \( i \in \mathbb{Z}_2 \).

Proof. Let \( D = \{2k - 1 \mid 1 \leq k \leq n/4\} \). Define

\[
D_0 = \begin{cases} 
D & \text{if } \epsilon = 0, \\
D \cup \{2\} & \text{if } \epsilon = 1, \\
D \cup \{2, 4\} & \text{if } \epsilon = 3,
\end{cases}
\]

and define

\[
D_1 = \begin{cases} 
D & \text{if } \epsilon = 0, \\
(D \cup \{2\}) \setminus \{n/2 - 2\} & \text{if } \epsilon = 1, \\
D \cup \{n/2 - 4\} & \text{if } \epsilon = 3.
\end{cases}
\]

In any case, define \( H_i = H[D_i] \) on the vertex set \( \mathbb{Z}_n \times \{i\} \), for each \( i \in \mathbb{Z}_2 \). Then clearly \( H_i \) satisfies (i), and since \( n/2 \notin D \), it follows from Lemma 2 that (iii) is satisfied.

If \( \epsilon = 0 \) then let \( B = \{(j, 0), (j + 2k - 1, 0), (j + k + n/4, 1) \mid j \in \mathbb{Z}_n, 1 \leq k \leq n/4\} \).

If \( \epsilon = 1 \) then let \( B = \{(j, 0), (j + 2k - 1, 0), (j + k + (n + 2)/4, 1) \mid j \in \mathbb{Z}_n, 1 \leq k \leq (n - 2)/4\} \cup \{(j, 1), (j + 2k - 1, 1), (j + k + (n + 2)/4, 0) \mid j \in \mathbb{Z}_n, 1 \leq k \leq (n - 6)/4\} \).

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If \( k = 3 \) then let \( B = \{ \{( j, 0), (j + 2k - 1, 0), (j + k + (n + 6)/4, 1) \mid j \in \mathbb{Z}_n, 1 \leq k \leq (n - 2)/4 \} \cup \{(j, 0), (j + 2, 0), (j + 1, 1)\}, \{(j, 0), (j + 4, 0), (j + 2, 1)\}, \{(j, 1), (j + (n - 4)/2, 1), (j + (n - 4)/2, 0) \mid j \in \mathbb{Z}_n\} \cup \{(j, 1), (j + 2k - 1, 1), (j + k + (n - 2)/4, 0) \mid j \in \mathbb{Z}_n, 1 \leq k \leq (n - 10)/4\} \}.

Then in each case, \( B \) is a set of triples which partition the edges of \([0, 1, 0]+H_0+H_1\).

The following structure will be needed in Section 3.

Let \( n \) be even, and let \( \mathcal{H} \) be a partition of \( \mathbb{Z}_n \) into sets of size 2. A symmetric quasigroup \((\mathbb{Z}_n, \cdot)\) with holes \( \mathcal{H} \) and of order \( n \) is an \( n \times n \) array in which: cell \((a, b)\) contains exactly one symbol in \( \mathbb{Z}_n \) if \( \{a, b\} \notin \mathcal{H} \) and no symbols if \( \{a, b\} \in \mathcal{H} \); for each \( a \in \mathbb{Z}_n \) row and column \( a \) contain each symbol in \( \mathbb{Z}_n \) exactly once except for symbols \( a \) and \( b \), where \( \{a, b\} \in \mathcal{H} \); and cells \((a, b)\) and \((b, a)\) either contain the same symbol or are both empty, for \( 0 \leq a < b \leq n - 1 \); if cell \((a, b)\) contains \( c \) then we write \( a \cdot b = c \).

The following is well known (see [10], for example).

**Lemma 2.14.** For all even \( n \geq 6 \), there exists a symmetric quasigroup with holes \( \mathcal{H} \) and of order \( n \), where \( \mathcal{H} \) is a partition of \( \mathbb{Z}_n \) into sets of size 2.

Since maximum packings and minimum coverings of triple systems have been completely determined, we have the following result.

**Lemma 2.15 [6, 9].** Let \( n \equiv 2 \pmod{6} \), \( n \geq 8 \) and let \( L \) be a set of 2 independent edges in \( K_n \). Then there exists an edge-disjoint decomposition of \((6y + 2)K_n + 2L\) and of \((6y + 4)K_n - 2L\) into copies of \( K_3 \), for all \( y \geq 0 \).

Finally, it will probably help enormously to list the values of \( n \) that satisfy conditions (1) and (2) of Lemma 2.1 for all values of \( \lambda_1 \) and \( \lambda_2 \). This is done in Table I.

**Table I**
The Values of \( n \pmod{6} \) for Each Value of \( \lambda_1 \pmod{6} \) and \( \lambda_2 \pmod{6} \) that Satisfy Conditions (1) and (2) of Lemma 2.1

<table>
<thead>
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<th>( \lambda_2 )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
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<td>0</td>
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<td>0, 3</td>
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<td>3</td>
<td>—</td>
<td>3, 5</td>
<td>—</td>
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<tr>
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<td>3</td>
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</tr>
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</tr>
</tbody>
</table>
3. EXISTENCE WHEN $m = 2$

We begin with a result that helps us deal with condition (3) of Lemma 2.1. It allows us to focus on large values of $n$, so then this lower bound on $\lambda_1$ will no longer be a moving target (that is, a function of $n$).

**Proposition 3.1.** If conditions (1)-(3) of Lemma 2.1 are sufficient for the existence of a GDD$(n, 2)$ of index $(\lambda_1, \lambda_2)$ whenever $\lambda_2 \leq 2(n-1)$, then they are sufficient for all $\lambda_2 \geq 1$.

**Proof.** Suppose that $n$, $\lambda_1$, and $\lambda_2$ satisfy Conditions (1)-(3) of Lemma 2.1, that $2x(n-1) < \lambda_2 \leq (2x+2)(n-1)$, and that $x \geq 1$. Then by (3),

$$\lambda_1 \geq \frac{\lambda_2}{2} + x + \varepsilon$$

where

$$\varepsilon = \begin{cases} \frac{1}{2} & \text{if } \lambda_2 \text{ is odd and } \lambda_2 < (2x+1)(n-1), \\ 1 & \text{if } \lambda_2 \text{ is even}, \\ \frac{1}{2} & \text{if } \lambda_2 \text{ is odd and } \lambda_2 \geq (2x+1)(n-1). \end{cases}$$

Let $\lambda'_1 = \lambda_1 - xn$ and $\lambda'_2 = \lambda_2 + 2x(n-1)$. Then $\lambda'_2 \leq 2(n-1)$, and since $\lambda'_1 = \lambda_1 - xn \geq \lambda_2/2 + x + x - xn = (\lambda_2 - 2x(n-1))/2 + \varepsilon = \lambda'_2/2 + \varepsilon$, so $\lambda'_1 \geq \lambda'_2/2(n-1)$, so (3) is satisfied by $n$, $\lambda'_1$ and $\lambda'_2$. (1) and (2) are easily seen to be satisfied too. Therefore, by our assumption there exists a GDD$(n, 2)$ of index $(\lambda_1 - xn, \lambda_2 - 2x(n-1))$. Also, by Lemma 2.2 there exists a GDD$(n, 2)$ of index $(xn, x(2n-2))$ for any $x \geq 1$. So together these two GDDs form a GDD$(n, 2)$ of index $(\lambda_1, \lambda_2)$.

Therefore, it remains to consider the case where $\lambda_2 \leq 2(n-1)$; or $n \geq \lambda_2/2 + 1$. Under this condition, (3) simply becomes $\lambda_1 \geq (\lambda_2 + 1)/2$. So throughout the rest of this section we will assume that $n$ and $\lambda_1$ satisfy these lower bounds imposed by $\lambda_2$.

**Proposition 3.2.** Suppose that $n$ is odd, $\lambda_1 \geq \lambda_2/2 + 1$ and $n \geq \lambda_2/2 + 1$. Let $n$, $\lambda_1$ and $\lambda_2$ satisfy conditions (1) and (2) of Lemma 2.1. Then there exists a GDD$(n, 2)$ of index $(\lambda_1, \lambda_2)$.

**Proof.** Since $n$ is odd, $\lambda_2$ is even (see Table 1). Let $\lambda = \lambda_1 - \lambda_2/2$. So $\lambda \geq 1$. The result will follow if we can find an integer $t$ that satisfies the following conditions:

(i) $0 \leq 2t \leq \lambda(n-1)$ and $3$ divides $(\lambda(n-1) - 2t) \ n$, and

(ii) $\lambda_2 - \lambda(n-1) \leq 2t \leq \lambda_2$, and $3$ divides $(\lambda(n-1) - \lambda_2 + 2t) \ n$.

For, once these conditions are met, we can proceed as follows. Condition (i) ensures that the conditions of Corollary 2.5 are met when $x = \lambda(n-1) - 2t$, so there exists a $(\lambda(n-1) - 2t)$-regular graph $H_0$ on the

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vertex set $Z_n \times \{0\}$ such that there exists a set $B_0$ of triples which partition the edges of $H_0$; so $\lambda K_n - E(H_0)$ is a $2t$-regular graph. Similarly, condition (ii) ensures that the conditions of Corollary 2.5 are met with $x = \lambda(n - 1) - \lambda_2 + 2t$, so there exists a $(\lambda(n - 1) - \lambda_2 + 2t)$-regular graph $H_1$ on the vertex set $Z_n \times \{1\}$ such that there exists a set $B_1$ of triples which partition the edges of $H_1$; so $\lambda K_n - E(H_1)$ is a $(\lambda_2 - 2t)$-regular graph. Since $\lambda_2$ is even, by Lemma 2.12 there exists a set $F_0$ of $2t$ 1-factors and a set $F_1$ of $\lambda_2 - 2t$ 1-factors, each 1-factor being in $[0, 1, 0]$, such that for each $i \in Z_2$ there exists a collection $B'_i$ of triples which partition the edges of $\lambda K_n - E(H_i)$ and the edges in the 1-factors in $F_i$. Finally, if $F$ is the $\lambda_2$-regular multigraph consisting of all the edges in $F_0$ and $F_1$, then by Lemma 2.11 there exists a collection $B$ of triples that partition the edges of $[\lambda_2/2, \lambda_2, \lambda_2/2] - E(F)$. Then each edge $\{(u, i), (v, i)\}$ with $i \in Z_2$ is contained in $\lambda$ triples in $B$, and $B'$, and is in $\lambda_2/2$ triples in $B$, and clearly each edge $\{(u, 0), (v, 1)\}$ is in $\lambda_2$ triples, so the result will follow. So it remains to find an appropriate integer $t$. Recall that $\lambda \geq 1$.

If $\lambda_2 = 6x + 2$ and $n \equiv 3 \pmod{6}$ then $\lambda_1 \geq 3x + 2$ (since $\lambda_1 \geq \lambda_2/2 + 1$) and $n \geq 3x + 3$ (since $n \geq \lambda_2/2 + 1$). Choose $t = \lceil (3x + 1)/2 \rceil$. Then $2t \leq n - 1$, $3$ divides $n$, and $\lambda_2 - (n - 1) \leq 2t$.

If $\lambda_2 = 6x + 2$ and $n \equiv 5 \pmod{6}$ then $\lambda_1 \equiv 2 \pmod{3}$ (see Table 1), so $\lambda \equiv 1 \pmod{3}$. If $x$ is odd then $n \geq 3x + 2$, so choose $t = (3x + 1)/2$. If $x$ is even then $n \geq 3x + 5$ (since $n \equiv 5 \pmod{6}$), so choose $t = (3x + 4)/2$.

If $\lambda_2 = 6x + 4$ and $n \equiv 3 \pmod{6}$ then $n \geq 3x + 3$, so choose $t = \lceil (3x + 1)/2 \rceil$.

If $\lambda_2 = 6x + 4$ and $n \equiv 5 \pmod{6}$ then $\lambda_1 \equiv 1 \pmod{3}$ (see Table 1). If $x$ is even then $n \geq 3x + 5$, so choose $t = (3x + 2)/2$. If $x$ is odd then $n \geq 3x + 8$, so choose $t = (3x + 5)/2$.

If $\lambda_2 = 6x$ and $n \equiv 1 \pmod{6}$ then: if $x$ is odd then $n \geq 3x + 4$, so choose $t = (3x + 3)/2$; if $x$ is even then $n \geq 3x + 1$, so choose $t = 3x/2$.

If $\lambda_2 = 6x$ and $n \equiv 3 \pmod{6}$ then $n \geq 3x + 3$, so choose $t = \lceil 3x/2 \rceil$.

If $\lambda_2 = 6x$ and $n \equiv 5 \pmod{6}$ then $\lambda_1 \equiv 0 \pmod{3}$ (see Table 1) and so $\lambda \geq 3$, and $n \geq 3x + 2$. If $x$ is even then choose $t = 3x/2$, and if $x$ is odd then choose $t = (3x + 3)/2$. 

It turns out that if $\lambda_2$ is odd then we need to consider the smallest value of $\lambda_1$ by itself.

**Proposition 3.3.** Suppose that $\lambda_2$ is odd and $\lambda_1 = (\lambda_2 + 1)/2$. Let $n$, $\lambda_1$ and $\lambda_2$ satisfy conditions (1)–(3) of Lemma 2.1. Then there exists a GDD$(n, 2)$ of index $(\lambda_1, \lambda_2)$.
Proof. By (3) of Lemma 2.1, \( n \geq \lambda_2 + 1 \). Since \( \lambda_2 \) is odd, \( n \) and \( \lambda_1 \) are even (see Table 1), so we can write \( \lambda_2 = 6x + 2y \), \( \lambda_1 = 12x + 4y - 1 \), and \( n \geq 12x + 4y \), where \( y \in \mathbb{Z}_2 \). So Table 1 shows that \( \lambda_1, \lambda_2 \) and \( n \) are restricted even more: if \( \lambda_1 \equiv 0 \pmod{6} \) then \( \lambda_2 \equiv 5 \pmod{6} \) so \( n \equiv 0 \pmod{6} \); if \( \lambda_1 \equiv 2 \pmod{6} \) then \( \lambda_2 \equiv 3 \pmod{6} \) so \( n \equiv 0 \) or \( 4 \pmod{6} \); and if \( \lambda_1 \equiv 4 \pmod{6} \) then \( \lambda_2 \equiv 1 \pmod{6} \) so \( n \equiv 0 \) or \( 2 \pmod{6} \). Notice that in every case

(a) either \( n \equiv 0 \pmod{6} \) or \( n/2 - \lambda_1 \equiv 0 \pmod{3} \).

It will also be useful later to notice that if \( n \equiv 2 \) or \( 10 \pmod{12} \) then \( \lambda_1 \equiv 4 \) or \( 2 \pmod{6} \) respectively, and so since \( n/2 \geq (\lambda_2 + 1)/2 = \lambda_1 \) we have:

(b) if \( n \equiv 2 \) or \( 10 \pmod{12} \) then \( n/2 \geq \lambda_1 + 3 \);

and if \( n \equiv 6 \pmod{12} \) then \( n/2 \) is odd, so we have:

(c) if \( n \equiv 6 \pmod{12} \) then \( n/2 \geq \lambda_1 + 1 \).

Let \( \varepsilon \) be defined as in Lemma 2.13. By Lemma 2.13, for each \( i \in \mathbb{Z}_2 \), there exists a simple graph \( H_i \) on the vertex set \( \mathbb{Z}_n \times \{i\} \) satisfying (i)–(iii). Let \( B_0 \) be a set of triples that partitions the edges of [0, 1, 0] + \( H_0 + H_1 \) (see (ii)). By (iii), \( K_n - E(H_i) \) can be partitioned into \( n - 1 - (n/2 + (-1)^i \varepsilon) = n/2 - 1 - (-1)^i \varepsilon \) 1-factors.

We want to apply Theorem 2.9 with \( x = n/2 - \lambda_1 - (-1)^i \varepsilon \) and \( \lambda = 1 \), so we have some things to check. If \( n \equiv 2 \) or \( 4 \pmod{6} \) then \( \varepsilon \in \{0, 3\} \), so by (a) we have that 3 divides \( xn \). In each case \( n/2 - (-1)^i \varepsilon \) is even, so \( x \) is even because \( \lambda_1 \) is even. Clearly \( x \equiv n - 1 \), and by (b) and (c) we have that \( x \geq 0 \).

Therefore, by Theorem 2.9, for each \( i \in \mathbb{Z}_2 \) there exists a set of triples \( B'_i \) and there exists an \( (n/2 - \lambda_1 - (-1)^i \varepsilon) \)–regular graph \( H'_i \) with vertex set \( \mathbb{Z}_n \times \{i\} \) whose edges are partitioned by the triples in \( B'_i \) such that \( K_n - E(H'_i) \) has a 1-factorization into \( n - 1 - (n/2 - \lambda_1 - (-1)^i \varepsilon) = n/2 + \lambda_1 - 1 + (-1)^i \varepsilon \) 1-factors.

Finally, for each \( i \in \mathbb{Z}_2 \), since \( \lambda_1 \geq 2 \) we can take the \((\lambda_1 - 2)(n - 1)\) 1-factors in a 1-factorization of \((\lambda_1 - 2) K_n \) on the vertex set \( \mathbb{Z}_n \times \{i\} \). So for each \( i \in \mathbb{Z}_2 \), altogether on the vertex set \( \mathbb{Z}_n \times \{i\} \) we have defined \((n/2 - 1 - (-1)^i \varepsilon) + (n/2 + \lambda_1 - 1 + (-1)^i \varepsilon) + (\lambda_1 - 2)(n - 1) = n(\lambda_1 - 1) = n(\lambda_2 - 1)/2 \) 1-factors. By Lemma 2.10, there exists a set \( B_i \) of triples that partition the edges in these 1-factors together with the edges in [0, \( \lambda_2 - 1, 0 \)].

Then clearly the triples in \( B_0, B_1, B'_0 \) and \( B'_1 \) form a GDD\((n, 2)\) of index \((\lambda_1, \lambda_2)\). 

Before presenting our last proposition, we need to deal with two exceptional cases.

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**Lemma 3.4.** Let \( n \equiv 2 \text{ or } 4 \mod 6 \), \( \lambda_1 = 6y + 6 \), \( \lambda_2 = 12y + 9 \) and \( n \equiv 6y + 6 \). Then there exists a GDD\((n, 2)\) of index \((\lambda_1, \lambda_2)\).

**Proof.** If \( n \equiv 2 \mod 6 \) then there exists a TS\((2n)\) of index 2, and by Proposition 3.3 there exists a GDD\((n, 2)\) of index \((6y + 4, 12y + 7)\), which together produce a GDD\((n, 2)\) of index \((6y + 6, 12y + 9)\).

If \( n \equiv 4 \mod 6 \) then define \( \varepsilon \) as in Lemma 2.13. By Lemma 2.13, for each \( i \in \mathbb{Z}_2 \) there exists a simple graph \( H_i \) on the vertex set \( \mathbb{Z}_n \times \{i\} \) that is \((n/2 + (-1)^i \varepsilon)\)-regular, such that there exists a set \( B \) of triples that partition the edges of \( [0, 1, 0] + H_0 + H_1 \), and such that \( K_n - E(H_i) \) has a 1-factorization into a set \( F_i(i) \) of \( n/2 - 1 - (-1)^i \varepsilon \) 1-factors. Since 6 divides \( x = 3n/2 - 6y - 6 - (-1)^i \varepsilon \) and \( 0 \leq x \leq n - 1 \), by Theorem 2.9, for each \( i \in \mathbb{Z}_2 \) there exists a set \( B_i \) of triples and an \( x \)-regular graph \( H_i \) in \((6y + 5)\) \( K_n \) defined on the vertex set \( \mathbb{Z}_n \times \{i\} \) whose edges are partitioned by the triples in \( B_i \), such that \((6y + 5)\) \( K_n - E(H_i) \) has a 1-factorization into a set \( F_i(i) \) of \((6y + 5)(n - 1) - x \) 1-factors. In \( F_i(i) \) and \( F_j(i), i \neq j \in \mathbb{Z}_2 \) there are a total of \((6y + 4)\) \( n \) 1-factors, which altogether with the edges in \([0, 12y + 8, 0]\) can be partitioned into a set \( B' \) of triples (by Lemma 2.10).

Clearly the triples in \( B, B', B_0 \) and \( B_1 \) together form a GDD\((n, 2)\) of index \((6y + 6, 12y + 9)\).

**Lemma 3.5.** Let \( \lambda_1 \equiv 4 \mod 6 \), \( \lambda_2 = 1 \) and \( n \equiv 2 \mod 6 \). Let \( n, \lambda_1 \) and \( \lambda_2 \) satisfy conditions (1)-(3) of Lemma 2.1. Then there exists a GDD\((n, 2)\) of index \((\lambda_1, \lambda_2)\).

**Proof.** Let \( \lambda_1 = 6y + 4 \). Let \( F = \{\{2a, 2a + 1\} \mid a \in \mathbb{Z}_{n/2}\} \) and \( F_0 = \{\{(a, 0), (b, 0)\} \mid \{a, b\} \in F\} \). Let \( L = \{\{0, 1\}, \{2, 3\}\} \), and for each \( i \in \mathbb{Z}_2 \) let \( L_i = \{\{(a, i), (b, i)\} \mid \{a, b\} \in L\} \).

Let \((Z_{n, \varepsilon})\) be a symmetric quasigroup with holes \( F \) and of order \( n \) (see Lemma 2.14). Define

\[
B = \{\{(a, 0), (b, 0), (a \cdot b, 1)\} \mid 0 \leq a < b \leq n - 1, \{a, b\} \notin F\}
\]

\[
\cup \{\{(2a, 0), (2a + 1, 0), (2a, 1)\}, \{(2a, 0),
(2a + 1, 0), (2a + 1, 1)\} \mid 2 \leq a < n/2\}
\]

\[
\cup \{(2a, 0), (2a, 1), (2a + 1, 1)\}, \{(2a + 1, 0), (2a, 1), (2a + 1, 1)\} \mid 0 \leq a \leq 1\}.
\]

Then the triples in \( B \) contain: each edge \( \{(a, 0), (b, 0)\} \) exactly once if \( \{a, b\} \notin F \), exactly twice if \( \{a, b\} \in F \setminus L \), and not at all if \( \{a, b\} \in L \); each edge \( \{(a, 0), (b, 1)\} \) exactly once; and each edge \( \{(a, 1), (b, 1)\} \) exactly twice if \( \{a, b\} \in L \), and otherwise not at all.
Using Lemma 2.15, let \( B_0 \) be a collection of triples that partition the edges of \((6y+2)K_n+2L_0\) on the vertex set \( Z_n \times \{0\} \), and let \( B_1 \) be a collection of triples that partition the edges of \((6y+4)K_n-2L_1\) on the vertex set \( Z_n \times \{1\} \).

Finally, let \( (Z_n \times \{0\}, F_0, B') \) be a GDD\((n/2, 2)\) of index \((0, 1)\).

Then the triples in \( B, B', B_0 \) and \( B_1 \) together form a GDD\((n, 2)\) of index \((6y+4, 1)\).

**Proposition 3.6.** Suppose that \( n \) is even, \( \lambda_1 \geq \lambda_2/2+1 \) and \( n \geq \lambda_2/2+1 \).

Let \( n, \lambda_1 \) and \( \lambda_2 \) satisfy conditions (1) and (2) of Lemma 2.1. Then there exists a GDD\((n, 2)\) of index \((\lambda_1, \lambda_2)\).

**Proof.** The result will follow if we can find an integer \( t \) that satisfies the following conditions:

(i) \( 0 \leq t, nt \leq \lambda_1(n-1) \), and 3 divides \( \lambda_1(n-1)-tn \), and

(ii) \( t \leq \lambda_2, (\lambda_2-t)n \leq \lambda_1(n-1) \), and 3 divides \( \lambda_1(n-1) -(\lambda_2-t)n \).

For, once these conditions are met, we proceed as follows.

Since \( n \) is even \( \lambda_1 \) is even, so \( (\lambda_1(n-1)-tn) \) is even. Therefore, by Theorem 2.9 and using (i), there exists a \((\lambda_1(n-1)-tn)\)-regular graph \( H_0 \) on the vertex set \( Z_n \times \{0\} \) of multiplicity at most \( \lambda_1 \) and there exists a set \( B_0 \) of triples such that: these triples partition the edges of \( H_0 \); and \( T_0 = \lambda_1K_n - E(H_0) \) has a 1-factorization into \( tn \) 1-factors. Similarly, by Theorem 2.9 and (ii), there exists a \((\lambda_1(n-1) -(\lambda_2-t)n)\)-regular graph \( H_1 \) on the vertex set \( Z_n \times \{1\} \) and there exists a set \( B_1 \) of triples such that: these triples partition the edges of \( H_1 \); and \( T_1 = \lambda_1K_n - E(H_1) \) has a 1-factorization into \( (\lambda_2-t)n \) 1-factors. Finally, by Lemma 2.10, there exists a set \( B \) of triples which partition the edges of \([0, \lambda_2, 0] + T_0 + T_1 \). Then clearly the triples in \( B_0, B_1 \) and \( B \) together form a GDD\((n, 2)\) of index \((\lambda_1, \lambda_2)\). So it remains to find a suitable value of \( t \) in each case.

In the following, to check that \( tn \leq \lambda_1(n-1) \) it is easier to check that \( t \leq (\lambda_1 -t)(n-1) \). Also, we will choose \( t \) so that \( t \geq \lambda_2/2 \), in which case \( tn \leq \lambda_1(n-1) \) implies that \( (\lambda_2-t)n \leq \lambda_1(n-1) \).

If \( \lambda_2 = 6x \) then \( \lambda_2 \geq 3x +1 \) and \( n \geq 3x +1 \). Choose \( t = 3x \). From Table 1, 3 divides \( \lambda_1, n \) or \( n-1 \), and since 3 divides \( t \), the divisibility by 3 conditions in (i)–(ii) are met.

If \( \lambda_2 = 6x +1 \) and \( n \equiv 0 \) (mod 6) then \( \lambda_1 \geq 3x +2 \) and \( n \geq 3x +2 \). Choose \( t = 3x +1 \).

If \( \lambda_2 = 6x +1 \) and \( n \equiv 2 \) (mod 6), then \( \lambda_1 \equiv 4 \) (mod 6) (see Table 1), so \( \lambda_1 \geq 3x +4 \) and \( n \geq 3x +2 \). Choose \( t = 3x +2 \). Then all conditions in (i)–(ii)
are met except that if $x = 0$ then $\lambda_2 < t$; but then we seek a $\text{GDD}(n, 2)$ of index $(6y + 4, 1)$ which was constructed in Lemma 3.5.

If $\lambda_2 = 6x + 2$ then $\lambda_1 \geq 3x + 2$ and $n \geq 3x + 2$. Choose $t = 3x + 1$.

If $\lambda_2 = 6x + 3$ and $n \equiv 0 \pmod{6}$ then $\lambda_1 \geq 3x + 3$ and $n \geq 3x + 3$. Choose $t = 3x + 2$.

If $\lambda_2 = 6x + 3$ and $n \equiv 2 \pmod{6}$ then $\lambda_1 \equiv 0 \pmod{6}$ (see Table 1), so $\lambda_1 \geq 3x + 3$ and $n \geq 3x + 5$. Choose $t = 3x + 3$. Then all conditions in (i)-(ii) are met except that if $\lambda_1 = 3x + 3$ then $nt > \lambda_1(n-1)$. However, if $\lambda_1 = 3x + 3$ then we can write $\lambda_1 = 6y + 6$, $\lambda_2 = 12y + 9$ and $n \equiv 2 \pmod{6}$, so we can use Lemma 3.4.

If $\lambda_2 = 6x + 3$ and $n \equiv 4 \pmod{6}$ then $\lambda_1 \geq 3x + 3$ and $n \geq 3x + 4$. Choose $t = 3x + 3$. Then all conditions in (i)-(ii) are satisfied unless $\lambda_1 = 3x + 3$, for then $nt > \lambda_1(n-1)$. If $\lambda_1 = 3x + 3$ then again the GDD can be obtained from Lemma 3.4.

Finally, we can present the main result.

**Theorem 3.7.** Let $n \geq 3$ and $\lambda_1, \lambda_2 \geq 1$. There exists a $\text{GDD}(n, 2)$ of index $(\lambda_1, \lambda_2)$ if and only if

1. $2$ divides $\lambda_1(n-1) + \lambda_2n$,
2. $3$ divides $\lambda_1n(n-1) + \lambda_2n^2$, and
3. $\lambda_1 \geq \lambda_2n/2(n-1)$.

**Proof.** By Proposition 3.1, it suffices to consider the case where $\lambda_2 \leq 2(n-1)$, so $n \geq \lambda_2/2 + 1$ and therefore by (3) $\lambda_1 \geq (\lambda_2 + 1)/2$. If $n$ is odd (so $\lambda_2$ is even) the result follows from Proposition 3.2. If $\lambda_1 = (\lambda_2 + 1)/2$ then the result follows from Proposition 3.3. If $n$ is even and $\lambda_1 \geq \lambda_2/2 + 1$ then the result follows from Proposition 3.6.

4. **EXISTENCE WHEN $n = 2$**

In this section we prove that the following necessary conditions for the existence of a $\text{GDD}(2, m)$ of index $(\lambda_1, \lambda_2)$ are sufficient (see Theorem 4.10).
**Lemma 4.1.** If there exists a GDD(2, m) of index \( \lambda_1, \lambda_2 \) then

1. \( 2 \) divides \( \lambda_1 + 2\lambda_2(m - 1) \),
2. \( 3 \) divides \( \lambda_1 m + 2\lambda_2 m(m - 1) \), and
3. \( \lambda_1 \leq (m - 1)\lambda_2 \).

**Remark.** Condition (1) implies that \( \lambda_1 \) is even.

**Proof.** Both (1) and (2) follow since each vertex must have even degree and the number of edges must be divisible by 3. (3) follows since each pure edge is contained in a triple containing two cross edges, so the number of pure edges is at most half the number of cross edges.

We can easily handle the case where \( n = m = 2 \) now.

**Lemma 4.2.** There exists a GDD(2, 2) of index \( \lambda_1, \lambda_2 \) if conditions (1)–(3) of Lemma 4.1 and condition (3) of Lemma 2.1 hold.

**Proof.** The conditions (3) of Lemma 2.1 and 4.1 imply that \( \lambda_1 = \lambda_2 \), so the GDD(2, 2) of index \( \lambda_1, \lambda_2 \) must be a TS(4) of index \( \lambda = \lambda_1 = \lambda_2 \). By Theorem 1.2 conditions (1) and (2) of Lemma 4.1 ensure that a TS(4) of index \( \lambda \) exists.

In view of this result, throughout the rest of this section we can assume that \( m \geq 3 \).

Our proof that conditions (1)–(3) of Lemma 4.1 are sufficient for the existence of a GDD(2, m) of index \( \lambda_1, \lambda_2 \) relies heavily on the following lemma.

**Lemma 4.3.** If there exists an edge-disjoint decomposition of \( \lambda_2 K_m \) into a collection \( B \) of copies of \( K_3 \) and a spanning subgraph \( H \) such that the edges of \( H \) can be directed to form \( H^+ \) so that in \( H^+ \) each vertex has out-degree \( \lambda_1/2 \), then there exists a GDD(2, m) of index \( \lambda_1, \lambda_2 \).

**Proof.** Suppose \( \lambda_2 K_m \) on the vertex set \( Z_m \) has been decomposed into a directed graph \( H^+ \) and a collection \( B \) of \( K_3 \)'s as described. Let

\[
B_1 = \{ \{ (0, a), (1, a), (0, b) \}, \{ (0, a), (1, a), (1, b) \} \mid (a, b) \text{ is a directed edge in } H^+ \}
\]

and

\[
B_2 = \{ \{ (0, a), (0, b), (0, c) \}, \{ (1, a), (1, b), (0, c) \}, \{ (1, a), (0, b), (1, c) \}, \{ (0, a), (1, b), (1, c) \} \mid \{ a, b, c \} \text{ is a triple in } B' \}.
\]
Then since each vertex \( a \in \mathbb{Z}_m \) has out-degree \( \lambda_1/2 \) in \( H^+ \), the edge \( \{0, a\} \) is in \( \lambda_1 \) triples defined in \( B_1 \). Also, for each \( a \neq b \), the edge \( \{a, b\} \) occurs \( x \) times in \( H \) and \( \lambda_2 - x \) times in copies of \( K_3 \) in \( B \), so the edges \( \{(i, a), (j, b)\}, i, j \in \mathbb{Z}_2 \) occur in \( x \) triples in \( B_1 \) and \( \lambda_2 - x \) triples in \( B_2 \). Therefore \( \langle \mathbb{Z}_2 \times \mathbb{Z}_m, \{ \mathbb{Z}_2 \times \{ i \} \mid i \in \mathbb{Z}_m \}, B_1 \cup B_2 \rangle \) is a GDD(2, \( m \)) of index \( (\lambda_1, \lambda_2) \).

**Proposition 4.4.** Suppose that \( \lambda_2(m-1) \) is even and \( m \geq 3 \). Then conditions (1)-(3) of Lemma 4.1 are sufficient for the existence of a GDD(2, \( m \)) of index \( (\lambda_1, \lambda_2) \).

**Proof.** Recall that (1) of Lemma 4.1 implies that \( \lambda_1 \) is even. Condition (2) implies that 3 divides \( \lambda_2(m-1) - \lambda_1 m \). Condition (3) implies that \( \lambda_2(m-1) - \lambda_1 \geq 0 \). Therefore we can apply Corollary 2.5 with \( x = \lambda_2(m-1) - \lambda_1 \) and \( \lambda = \lambda_2 \) (and replacing \( n \) with \( m \)) to produce an \( x \)-regular multigraph \( G \) of multiplicity at most \( \lambda_2 \) whose edges can be partitioned into triples, so that \( H = \lambda_2 K_m - E(G) \) has a 2-factorization into \( \lambda_1 \) 2-factors. Each 2-factor consists of edge-disjoint cycles that can be oriented to form directed cycles. The resulting directed graph \( H^+ \) has out-degree \( \lambda_1/2 \) at each vertex, so the proposition follows from Lemma 4.3.

It remains to consider the case where \( \lambda_2(m-1) \) is odd, so we know that \( \lambda_2 \) is odd, and \( \lambda_1 \) and \( m \) are even. We begin by showing that it essentially suffices to consider the case where \( \lambda_2 = 1 \). It may help to consult Table II which lists the values of \( m \) (mod 6) that satisfy conditions (1)-(2) of Lemma 4.1.

A Kirkman triple system KTS(\( n \)) is a TS(\( n \)) (\( V, B \)) of index 1 and order \( n \) in which \( B \) can be partitioned into sets of size \( n/3 \) so that each such set is a partition of \( V \). We will use the following theorem in the proof of Proposition 4.7.

**Theorem 4.5** [13]. For all \( n \equiv 3 \) (mod 6) there exists a KTS(\( n \)).

**TABLE II**

| \( \lambda_2 \) (mod 6) for Each Value of \( \lambda_1 \) (mod 6) and \( \lambda_2 \) (mod 6) that Satisfy Conditions 1-2 of Lemma 4.1 when \( \lambda_2(m-1) \) is Odd | \( \lambda_2 \) | 1 | 3 | 5 |
|---|---|---|---|
| \( \lambda_1 \) | 0 | 0, 4 | 0, 2, 4 | 0, 4 |
| 2 | 0 | 0 | 0 | 2 |
| 4 | 0 | 2 | 0 | 0 |
Similarly a GDD$(n, m) \ (V, G, B)$ of index $(0, \lambda_2)$ is resolvable if $B$ can be partitioned into sets of size $|V|/3$ so that each such set is a partition of $V$.

We will use the following special case of a result of Assaf, Hartman, Rees and Stinson.

**Theorem 4.6** [1, 14]. For all $m \geq 4$, there exists a resolvable GDD$(6, m)$ of index $(0, 1)$.

**Proposition 4.7.** Suppose $\lambda_2(m-1)$ is odd and $m \geq 4$. If conditions (1)--(3) of Lemma 4.1 are sufficient for the existence of a GDD$(2, m)$ of index $(\lambda_1, \lambda_2)$ when $\lambda_2=1$, then they are sufficient for all $\lambda_2 \geq 1$, except possibly for the case where $\lambda_2 = (6 \ (mod \ 6), m \equiv 2 \ (mod \ 6)$ and $\lambda_1 = 2$.

**Proof.** Suppose $\lambda_2(m-1)$ is odd, and suppose conditions (1)--(3) of Lemma 4.1 are sufficient for the existence of a GDD$(2, m^*)$ of index $(\lambda_1^*, \lambda_2^*)$ for all $m^* \geq 3$ and $\lambda_2^* \geq 1$.

Suppose that: $m \equiv 4 \ (mod \ 6)$ and $\lambda_2 \equiv 1 \ (mod \ 6)$ then $\lambda_1 \geq 2$, and that $m, \lambda_1$ and $\lambda_2$ satisfy the conditions (1)--(3) of Lemma 4.1. Let $\lambda_2^* = \lambda_2 - 1$ and $\lambda_2 = 1$. We consider the case $m \equiv 0, 2$ and $4 \ (mod \ 6)$ in turn.

**Case 1:** $m \equiv 0 \ (mod \ 6)$. Let $\lambda_1^* = \min\{\lambda_2^*(m-1), \lambda_1\}$ and $\lambda_2^* = \lambda_1 - \lambda_1^*$. Then since $\lambda_1^*$ and $\lambda_2^*$ are even, and since 3 divides $m$, we have that $m, \lambda_1^*$ and $\lambda_2^*$ satisfy conditions (1)--(3) of Lemma 4.1. So by Proposition 4.4, there exists a GDD$(2, m)$ of index $(\lambda_1^*, \lambda_2^*)$. Also, $\lambda_2^*$ is even, and since $\lambda_1^* \leq m-1$, we have that $\lambda_2^* \leq \lambda_2$, so $m, \lambda_1^*$ and $\lambda_2^* = 1$ satisfy conditions (1)--(3) of Lemma 4.1. So by our assumption there exists a GDD$(2, m)$ of index $(\lambda_1^*, \lambda_2^*)$. Together these two GDDs form a GDD$(2, m)$ of index $(\lambda_1, \lambda_2)$ as required.

**Case 2:** $m \equiv 2 \ (mod \ 6)$. Let $m = 6x + 2$ where $x \geq 1$, and let $\lambda_2 = 6y + \varepsilon$ where $\varepsilon \in \{1, 3, 5\}$ and $y \geq 0$. Then from Table II, $\lambda_2 = 6z + \varepsilon - 3$ where $z \geq 1$ (recall that this proposition does not consider the case where $\lambda_2 \equiv 5 \ (mod \ 6), m \equiv 2 \ (mod \ 6)$ and $\lambda_1 = 2$, so $\varepsilon \neq 0$). Then by (3), $\lambda_1 \leq (m-1) \lambda_2 = (6x+1)(6y+\varepsilon) = 6(6xy + \varepsilon y + y) + \varepsilon$, so since $\lambda_1 = 6z + \varepsilon - 3$ it must be that in this case

$$\lambda_1 \leq (m-1) \lambda_2 - 3.$$

Define $\lambda_1^*$ to be the largest integer congruent to 4 (mod 6) such that $\lambda_1^* \leq \min\{m-4, \lambda_1\}$, and write $\lambda_1^* = 6z^* + 4$. Notice that $z^* \geq 0$ since $x \geq 1$ and $z \geq 1$ (so $\lambda_1 \geq 4$). Define $\lambda_1^* = \lambda_1 - \lambda_1^*$, so $\lambda_2 = 6z + \varepsilon - 3 - 6z^* - 4 = 6(z - z^*) - 1 + \varepsilon - 1$.

Since $\lambda_2 \leq m-4$, $\lambda_1^* \equiv 4 \ (mod \ 6)$ and $\lambda_2^* = 1$, we have that $m, \lambda_1^*$ and $\lambda_2^*$ satisfy conditions (1)--(3) of Lemma 4.1, so by assumption there exists a GDD$(2, m)$ of index $(\lambda_1^*, \lambda_2^*)$. 

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If \( x \geq m - 4 \) then \( \lambda_2 = m - 4 \) so \( \lambda_1 = \lambda_1 - (m - 4) \leq (m - 1) \lambda_2 - 3 - (m - 4) = (m - 1) \). If \( \lambda_1 < (m - 4) \) then \( \lambda_1 \leq \lambda_1 < m - 4 < (m - 1) \). So in any case \( m \), \( \lambda_1 \) and \( \lambda_2 \) satisfy (3) of Lemma 4.1.

If \( \lambda_1 \geq m - 4 \) then \( \lambda_1 = (m - 4) \equiv -1 \pmod{6} \), and \( \lambda_2 = \lambda_2 + 1 \equiv 0 \pmod{6} \), so \( \lambda_1 \equiv \lambda_2 \equiv 0 \pmod{6} \). Also, if \( \lambda_1 < m - 4 \) then \( \lambda_1 = 0 \), \( 2 \), or \( 4 \pmod{6} \) if \( \lambda_2 = 1 \), \( 3 \), or \( 5 \pmod{6} \) respectively (see Table II); so \( \lambda_2 = \lambda_2 - 1 \equiv \lambda_1 \pmod{6} \). Therefore, in any case, we have \( m \), \( \lambda_1 \) and \( \lambda_2 \) satisfy (2) of Lemma 4.1, and clearly (1) is satisfied. So, since \( \lambda_2 \) is even, by Proposition 4.4 there exists a GDD(2, \( m \)) of index \((\lambda_1, \lambda_2)\).

Together, the GDD(2, \( m \)) of index \((\lambda_1, \lambda_2)\) and that of index \((\lambda_1', \lambda_2')\) form a GDD(2, \( m \)) of index \((\lambda_1, \lambda_2)\) as required.

Case 3: \( m \equiv 4 \pmod{6} \). Let \( m = 6x + 4 \), where \( x \geq 0 \), and let \( \lambda_2 = 6y + \varepsilon \) where \( \varepsilon \in \{1, 3, 5\} \) and \( y \geq 0 \). From Table II, we can write \( \lambda_1 = 6z \), where \( z \geq 1 \). Then by (3), \( \lambda_1 \equiv (m - 1) \lambda_2 = (6x + 3)(6y + \varepsilon) = 6(6xy + 3y + \varepsilon x) + 3z \), so since \( \varepsilon \) is odd and \( \lambda_1 = 6z \) it must that

\[
\lambda_1 \leq (m - 1) \lambda_2 - 3.
\]

Define \( \lambda_1' \) to be the largest integer congruent to 0 \pmod{6} such that \( \lambda_1' \leq \min\{m - 4, \lambda_1\} \), and write \( \lambda_1'' = 6z' \) where \( z' \geq 0 \). Define \( \lambda_1 = \lambda_1 - \lambda_1'' = 6(z - z') \).

Since \( \lambda_1' \leq m - 4 \), \( \lambda_1' \equiv 0 \pmod{6} \) and \( \lambda_2 = 1 \), we have that \( m \), \( \lambda_1' \) and \( \lambda_2' \) satisfy conditions (1)-(3) of Lemma 4.1, so by assumption there exists a GDD(n, \( m \)) of index \((\lambda_1', \lambda_2')\).

If \( \lambda_2' \geq m - 4 \) then \( \lambda_2' = m - 4 \), so \( \lambda_1' = \lambda_1' - (m - 4) \leq (m - 1) \lambda_2' - 3 - (m - 4) = (m - 1) \). If \( \lambda_1' < (m - 4) \) then clearly \( \lambda_1' \leq (m - 1) \lambda_2' \). So \( m \), \( \lambda_1' \) and \( \lambda_2' \) satisfy (3) of Lemma 4.1. Clearly 3 divides \( \lambda_1' \) and \( (m - 1) \), so (2) is satisfied, and 2 divides \( \lambda_1' \) and \( \lambda_2' \) so (1) is satisfied. Therefore by Proposition 4.4 there exists a GDD(2, \( m \)) of index \((\lambda_1', \lambda_2')\) which together with the GDD(2, \( m \)) of index \((\lambda_1', \lambda_2')\) forms a GDD(n, \( m \)) of index \((\lambda_1, \lambda_2)\).
the vertex set \( \{ \infty \} \cup Z_{6x+3} \). Let \( (Z_{6x+3}, T) \) be a KTS(6x + 3) with parallel classes \( \pi_0, \ldots, \pi_{3x} \) (see Theorem 4.5). Clearly \( \pi_1 \) has \( 2x + 1 \) triples. Partition the triples in \( \pi_0 \) into 2 sets \( T_0 \) and \( R_0 \) so that \( |T_0| = y \). Direct some of the edges of \( K_m \) as follows to form \( H^+ \).

(i) Join \( \infty \) with an edge directed to each vertex in each triple in \( T_0 \).

(ii) Join each vertex in each triple in \( R_0 \) with an edge directed to \( \infty \).

(iii) Direct the edges in each triple in \( T_0 \) to form directed 3-cycles.

(iv) Direct the edges in each triple in \( \bigcup_{i=1}^{3x-1} \pi_i \) to form directed 3-cycles (\( \pi_{3y-1} \) exists since \( 3y - 1 < 3x \)).

Then \( \infty \) is incident with \( 3y = \lambda_1 / 2 \) edges directed out in (i). Each \( v \in Z_{6x+3} \) is incident with 1 edge directed out in (ii)-(iii) and \( 3y - 1 \) edges directed out in (iv), so also has outdegree \( \lambda_1 / 2 \) in \( H^+ \). The edges in \( K_m \) remaining undirected are partitioned by the triples in \( R_0 \cup (\bigcup_{i=3x}^{3y} \pi_i) \).

**Case 2:** \( m = 6x + 2 \). Since \( m = 6x + 2 \) and \( \lambda_2 = 1 \), we have that \( \lambda_1 = 6y + 4 \). Since \( \lambda_1 \leq m - 1 \), we have \( 0 \leq y < x \) and \( x \geq 2 \).

If \( m = 8 \) then \( \lambda_1 = 4 \). Define \( K_4 \) on the vertex set \( Z_6 \times Z_2 \). Let \( H^+ \) contain the directed edges in \( \{(i, 0), (i, 1), (i, 2), (i, 3), (i, 4), (i, 5)\} \cup \{(i, 0), (i, 2, 0), (i + 1, 1), (i + 3, 1), (i, 0), (i + 1, 1), (i + 3, 1), (i + 2, 1), (i + 3, 1), (i, 1), (i, 1, 1), (i + 3, 1)\} \) \( i \in Z_6 \) reducing sums modulo 4. Then each vertex has outdegree 2 = \( \lambda_1 / 2 \), and the edges remaining undirected are partitioned by the triples in \( \{(i, 0), (i + 1, 0), (i + 2, 1)\} \) \( i \in Z_6 \).

So we can now assume that \( x \geq 2 \). We define \( K_m \) on the vertex set \( \{ \infty, 1 \} \cup Z_{6x-3} \). Let \( (Z_{6x-3}, T) \) be a KTS(6x - 3) with parallel classes \( \pi_0, \ldots, \pi_{3x-3} \). Partition the triples in: \( \pi_0 \) into 3 sets \( T_0, T_1 \) and \( R_0 \); \( \pi_1 \) into 3 sets \( T_2 \) and \( R_1 \); \( \pi_2 \) into 2 sets \( T_3 \) and \( R_2 \); so that \( |T_i| = y \) for \( i \in Z_2 \) (\( \pi_2 \) exists since \( x \geq 2 \)). Then \( |R_0| = |R_1| = 2x - 1 - 2y > 0 \), and \( |R_2| > 0 \). Direct some of the edges of \( K_m \) as follows (to form \( H^+ \)).

(i) \( H^+ \) contains the directed edges \( \{(\infty, 1), (\infty, 2), (\infty, 3), (\infty, 4)\} \) \( i \in Z_3 \), reducing the sum in the subscript modulo 5.

(ii) For each \( i \in Z_3 \) direct the edge from \( \infty, i \) to each vertex in a triple in \( T_i \).

(iii) For each vertex \( v \) in a triple in \( T_0 \cup R_0 \), \( T_1 \cup R_0 \), \( T_3 \cup R_1 \), \( T_3 \cup R_2 \), and \( R_2 \) direct the edge from \( v \) to \( \infty_0, \infty_3, \infty_2 \), and \( \infty_4 \) respectively.

(iv) Direct the edges in each triple in \( \bigcup_{i=3x}^{3y} T_i \) to form directed 3-cycles.

(v) Direct the edges in each triple in \( \bigcup_{i=3x}^{3y-1} \pi_i \) to form directed 3-cycles (\( \pi_{3y-1} \) exists since \( 3y - 1 < 3x - 3 \)).
The edges directed in (i)–(v) form $H^+$. For each $i \in \mathbb{Z}_3$, $\infty_i$ is incident with 2 edges directed out in (i) and 3 edges directed out in (ii) so has outdegree $3y + 2 = \lambda_1/2$ in $H^+$. Each vertex $v \in \mathbb{Z}_{6x - 3}$ is incident with 5 edges directed out in (iii)–(iv), and $3y - 3$ directed out in (v), so also has degree $\lambda_1/2$. The edges in $K_m$ remaining undirected are partitioned by the triples in $((\bigcup_{i \in \mathbb{Z}_9} R_i) \cup (\bigcup_{i \in \mathbb{Z}_3} \pi_i))$.

Case 3: $m = 6x$. In this case we have to consider 3 further cases, since $\lambda_1 \equiv 0, 2$ or 4 (mod 6) (see Table 11), so we consider the cases $\lambda_1 \equiv 0$ or 2 (mod 6) and $\lambda_1 \equiv 4$ (mod 6) in turn.

Suppose that $\lambda_1 = 6y + 2e$, where $e \in \{0, 1\}$, and suppose $\lambda_1 \neq 2$. Since $2 < \lambda_1 \leq m - 1$, we have that $1 \leq y < x$, and so $x \geq 2$. Let $K_m$ be defined on the vertex set $\{0, x, 2\} \cup \mathbb{Z}_{6x - 3}$, and let $(\mathbb{Z}_{6x - 3}, T)$ be a KTS$(6x - 3)$ with parallel classes $\pi_0, \pi_1, ..., \pi_{3x - 3}$. Of course, $\pi_1(i \in \mathbb{Z}_{3x - 3})$ contains $2x - 1$ triples. Partition the triples in $\pi_0$ into 3 sets $T_0, T_1$, and let $R_0$ so that $|T_0| = |T_1| = y$ (so $|R_0| = 2x - 1 - 2y \geq 1$), and partition $\pi_1$ into 2 sets $T_2$ and $R_1$ so that $|T_2| = y$ (so $\pi_1$ exists since $x \geq 2$). Direct some of the edges of $K_m$ as follows (to form $H^+$).

(i) For each $i \in \mathbb{Z}_3$, direct the edge from $\infty_i$ to each vertex in a triple in $T_i$.

(ii) For each vertex $v$ in a triple in $T_0 \cup R_0$, $T_1 \cup R_0$, and $R_1$, direct the edge from $v$ to $\infty_1, \infty_0$, and $\infty_2$, respectively.

(iii) Direct the edges in each triple in $T_0 \cup T_1 \cup T_2$ to form directed 3-cycles.

(iv) Direct the edges in each triple in $((\bigcup_{i \in \mathbb{Z}_3} \pi_i)$ to form directed 3-cycles $(\pi_m - 3 + e)$ exists since $3x - 3 > 3y - 2 + e$.

(v) If $e = 1$ then direct the edges directed in $\mathbb{Z}_9$, $\infty_1$ to $\infty_1, \infty_2$, and $\infty_3$ to $\infty_0$.

The edges directed in (i)–(v) form $H^+$. For each $i \in \mathbb{Z}_3$, $\infty_i$ has outdegree $3 |T_i| + e = 3y + e = \lambda_1/2$ (from (i) and (v)). For each $v \in \mathbb{Z}_{6x - 3}$, $v$ has 3 edges directed out defined in (ii) and (iii), and has $3y - 3 + e$ edges directed out defined in (iv), so has outdegree $\lambda_1/2$. The edges in $K_m$ remaining undirected are partitioned by the triples in $R_0 \cup R_1 \cup (\{0, x, 2\} \cup ((\bigcup_{i \in \mathbb{Z}_3} \pi_i)$ if $e = 0$, and by $R_0 \cup R_1 \cup (\bigcup_{i \in \mathbb{Z}_3} \pi_i)$ if $e = 1$.

Suppose that $\lambda_1 = 6y + 4$ or $\lambda_1 = 2$, and suppose that $m \notin \{12, 18\}$. Since $1 \leq \lambda_1 \leq m - 1$, we have that $0 \leq y < x$ and $x = 1$ or $x = 2$. We define $K_m$ on the vertex set $\mathbb{Z}_9 \times \mathbb{Z}_6$. Let $(\mathbb{Z}_9 \times \mathbb{Z}_6, \{[i] \times Z_6 | i \in \mathbb{Z}_9, T\})$ be a resolvable GD$(6, x)$ with parallel classes $\pi_0, \pi_1, ..., \pi_{3x - 2}$ (see Theorem 4.6). Direct some of the edges of $K_m$ as follows (to form $H^+$).

(i) If $\lambda_1 \neq 2$ then for each $i \in \mathbb{Z}_9$, let $H^+$ contain the directed edges $((i, 1), (i, 4)), ((i, 1), (i, 5)), ((i, 2), (i, 5)), ((i, 2), (i, 0)), ((i, 3), (i, 4)),$
((i, 3), (i, 0)), ((i, 0), (i, 1)), ((i, 5), (i, 3)), ((i, 4), (i, 2)), ((i, 4), (i, 5)),
((i, 5), (i, 0)), and ((i, 0), (i, 4)).

(ii) If $\lambda_1 = 2$ then let $H^+$ contain the directed edges \(((i, 1), (i, 4)), \((i, 2), (i, 4)), \((i, 3), (i, 0)), \((i, 4), (i, 0)), \((i, 5), (i, 2)), \text{ and } ((i, 0), (i, 2)).$

(iii) Direct the edges in each triple in $\bigcup_{i\in Z_p} \pi_i$ to form directed 3-cycles $(\pi_{3y-1}$ exists since $3y-1 < 3x-2)$, and let $H^+$ contain these directed edges.

For each vertex $v \in Z_3 \times Z_{6g}$ there are 2 edges directed out of $v$ defined in (i) if $\lambda_1 = 6y + 4$, there is 1 edge directed out of $v$ in (ii) when $\lambda_1 = 2$, and in either case there are 3 edges directed out of $v$ in (iii), so $v$ has outdegree $\lambda_1 / 2$ in $H^+$. The edges of $K_m$ that have not been directed are partitioned by the triples in $(\bigcup_{i \in Z_3} \pi_i) \cup \{(i, 1), (i, 2), (i, 3) \mid i \in Z_{6g}\}$ if $\lambda_1 = 6y + 4$, and $(\bigcup_{i \in Z_3} \pi_i) \cup \{(i, 1), (i, 2), (i, 3), (i, 3), (i, 4), (i, 5), (i, 5), (i, 0), (i, 1) \mid i \in Z_{6g}\}$ if $\lambda_1 = 2$.

Suppose that $\lambda_1 \in \{2, 4, 10\}$ and $m = 12$. If $\lambda_1 = 10$ then define $K_{12}$ on the vertex set $Z_3 \times Z_{6g}$, let $H^+$ contain the directed edges defined in (i) above for each $i \in Z_3$, and add the directed edges in $\{(0, i), (1, i+j), (1, i), (0, i-j+1) \mid i \in Z_{6g}, j \in Z_3\}$. Then each vertex has outdegree $5 = \lambda_1 / 2$, and the edges remaining undirected are partitioned by the triples $\{(i, 1), (i, 2), (i, 3) \mid i \in Z_3\}$ if $\lambda_1 \in \{2, 4\}$ then define $K_{12}$ on the vertex set $\{\infty, i \in Z_3\} \cup (Z_3 \times Z_3)$. Let $(Z_3 \times Z_3, T)$ be a $KTS(9)$ (see Theorem 4.5) with parallel classes $\pi_i, i \in Z_3$ such that $\pi_i = \{(i, 0), (i, 1), (i, 2) \mid i \in Z_{6g}\}$ and $\{(0, 0), (1, 1), (2, 2)\} \in \pi_3$ (clearly this is possible by renaming the symbols). Let $T_i = \{|(i, j), (i, j+1), \infty_{j+1}\} \mid i \in Z_3, j \in Z_3\}$, reducing sums modulo 3. For $\lambda_1 = 2$ let $H^+$ contain the directed edges in $\{(i, j), \infty_{j} \mid i, j \in Z_3, i \neq j\}$ together with the directed edges in the directed 3-cycles formed from the triples in $\pi_3 \cup \{(0, 0), (1, 1), (2, 2)\}$; then the edges remaining undirected in $K_{12}$ are partitioned by the triples in $\{(0, 0), (1, 1), (2, 2)\}$.

Suppose that $\lambda_1 \in \{2, 4, 10, 16\}$ and $m = 18$, and define $K_{18}$ on the vertex set $Z_3 \times Z_{6g}$. If $\lambda_1 = 16$ then for each $i \in Z_3$ let $H^+$ contain the directed edges defined in (i) above together with the directed edges in $\{(i, j), (i+1, k) \mid j, k \in Z_{6g}, i \in Z_3\}$; the edges remaining undirected are partitioned by the triples in $\{(i, 1), (i, 2), (i, 3) \mid i \in Z_{6g}\}$. If $\lambda_1 = 4$ or 2 then let $H^+$ contain the directed edges defined in (i) or (ii) above respectively; the edges remaining undirected are partitioned by the triples in $\{\text{index } (0, 1) \text{ (see Theorem 1.3) together with the triples in } \{(i, 1), (i, 2), (i, 3) \mid i \in Z_{6g}\}$ if $\lambda_1 = 4$ and in $\{(i, 1), (i, 2), (i, 3), (i, 3), (i, 4), (i, 5), (i, 5), (i, 0), (i, 1) \mid i \in Z_{6g}\}$ if $\lambda_1 = 2$. If $\lambda_1 = 10$ then let $H^+$ contain the
directed edges in \( \{(i, 2j), (i, 2j + 1) \mid i, j \in \mathbb{Z}_3 \} \cup \{(i, j), (i + 1, k) \mid j, k \in \mathbb{Z}_4, j \neq k, i \in \mathbb{Z}_3 \} \setminus \{(i, 2j), (i + 1, 2j + 2) \mid i, j \in \mathbb{Z}_3 \} \); then the edges remaining undirected are partitioned by the triples in \( \{(0, j), (1, j), (2, j), (0, 2k), (1, 2k + 2), (2, 2k + 4) \mid j \in \mathbb{Z}_6, k \in \mathbb{Z}_3 \} \cup (\bigcup_{z \in \mathbb{Z}_3} B_z) \), where \( (i, j) \times \mathbb{Z}_3, \{(z, 2j), (z, 2j + 1) \mid j \in \mathbb{Z}_3 \}, B_z \) is a GDD(2, 3) of index (0, 1) (see Theorem 1.3).

**Proposition 4.9.** Let \( m \equiv 2 \mod (6), m \geq 3, \lambda_1 = 2 \) and \( \lambda_2 \equiv 5 \mod (6) \). There exists a GDD(2, m) of index \((\lambda_1, \lambda_2)\).

**Proof.** Let \( m = 6x + 2 \). Clearly it suffices to consider the case where \( \lambda_2 = 5 \) since by Theorem 1.3 there exists a GDD(2, m) of index (0, 6). Also by Theorem 1.3, there exists a GDD(2, 3x + 1) of index (0, 1), so by taking 5 copies of this GDD, possibly with different groups, it is possible to define a collection \( B \) of triples that partition all the edges of \( 5K_m \), except for 5 1-factors (corresponding to the 5 sets of groups). Therefore it remains to find a set of 5 1-factors of \( 5K_{6x+2} \) whose edges are partitioned by a set \( B_1 \) of triples and a set of edges forming a spanning subgraph \( H \) that can be directed so that each vertex in the resulting directed graph \( H^+ \) has out-degree \( \lambda_1/2 = 1 \).

We define \( K_{6x+2} \) on the vertex set \( \mathbb{Z}_{3x+1} \times \mathbb{Z}_2 \). Let \( B_1 = \{ (i, 0), (i + 1, 0), (i + 2, 1) \mid i \in \mathbb{Z}_{3x+1} \} \) and let \( H^+ \) contain the directed edges in \( \{(0, i), (i, 1), (i, i + 1, 1) \mid i \in \mathbb{Z}_{3x+1} \} \), reducing the sum modulo \( 3x + 1 \). Then clearly each vertex in \( H^+ \) has out-degree 1 as required. \( F_j = \{ (i, j + 1), (i + 1, j + 1) \mid i \in \mathbb{Z}_{3x+1} \} \) forms a 1-factor for each \( j \in \mathbb{Z}_2 \), and it is easy to see that the edges in \( \{(i, 0), (i + 1, 0), (i, 1), (i, 1), (i + 1, 1) \mid i \in \mathbb{Z}_{3x+1} \} \) can be partitioned into 3 1-factors, so the result follows.

We have now settled the case \( n = 2 \) as the following Theorem shows.

**Theorem 4.10.** Let \( m \geq 2 \) and \( \lambda_1, \lambda_2 \geq 1 \). There exists a GDD(2, m) of index \((\lambda_1, \lambda_2)\) if and only if

1. \( 2 \) divides \( \lambda_1 + 2\lambda_2(m - 1) \),
2. \( 3 \) divides \( \lambda_1m + 2\lambda_2m(m - 1) \), and
3. \( \lambda_1 \leq (m - 1) \lambda_2 \).

**Proof.** The necessity follows from Lemma 4.1. The sufficiency follows from Proposition 4.4 if \( \lambda_2(m - 1) \) is even, from Proposition 4.7 if \( \lambda_2(m - 1) \) is odd and \( \lambda_2 = 1 \), from Proposition 4.9 if \( m \equiv 2 \mod (6) \), \( \lambda_1 = 2 \) and \( \lambda_2 \equiv 5 \mod (6) \), and therefore from Proposition 4.7 in all other cases where \( \lambda_2 \geq 2 \).
5. FINAL COMMENTS

We can now summarise the results on this paper with the following result (see Theorems 3.7 and 4.10).

**Theorem 5.1.** Let \( n = 2 \) or \( m = 2 \), and \( \lambda_1, \lambda_2 \geq 1 \). There exists a GDD(\( n, m \)) of index \((\lambda_1, \lambda_2)\) if and only if

1. \( 2 \) divides \( \lambda_1(n-1) + \lambda_2(m-1)n \),
2. \( 3 \) divides \( \lambda_1mn(n-1) + \lambda_2m(m-1)n^2 \),
3. if \( m = 2 \) then \( \lambda_1 \geq \lambda_2n/(n-1) \), and
4. if \( n = 2 \) then \( \lambda_1 \leq (m-1)\lambda_2 \).

This can now be incorporated with Theorems 1.2, 1.3, and 1.1 to prove the following encompassing result.

**Theorem 5.2.** Let \( n, m, \lambda_2 \geq 1 \) and \( \lambda_1 \geq 0 \). There exists a GDD(\( n, m \)) of index \((\lambda_1, \lambda_2)\) if and only if

1. \( 2 \) divides \( \lambda_1(n-1) + \lambda_2(m-1)n \),
2. \( 3 \) divides \( \lambda_1mn(n-1) + \lambda_2m(m-1)n^2 \),
3. if \( m = 2 \) then \( \lambda_1 \geq \lambda_2n/(n-1) \), and
4. if \( n = 2 \) then \( \lambda_1 \leq (m-1)\lambda_2 \).

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REFERENCES

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