Finding Inheritance Hierarchies in Fuzzy-Valued Concept-Networks

Yih-Jen Horng and Shyi-Ming Chen

Abstract—In this paper, we extend the works of [2] and [4] to present a new method for finding the inheritance hierarchies in fuzzy-valued concept-networks, where the relevant values (degrees of generalization or degrees of similarity) between concepts in a fuzzy-valued concept network are represented by fuzzy numbers. The proposed method is more flexible than the ones presented in [2] and [4] due to the fact that it allows the grades of similarity and the grades of generalization between concepts to be represented by fuzzy numbers rather than crisp real values between zero and one. In [2], we also presented an algorithm for finding the collection of inheritance hierarchies in interval-valued fuzzy concept-networks. However, if we can allow the relevant values (degrees of generalization or degrees of similarity) between concepts to be represented by fuzzy numbers, then there is room for more flexibility.

In this paper, we extend the works of [2] and [4] to present a new method for finding the collection of inheritance hierarchies in fuzzy-valued concept-networks, where the relevant values (degrees of generalization or degrees of similarity) between concepts in a fuzzy-valued concept network are represented by fuzzy numbers. The proposed method is more flexible than the ones presented in [2] and [4] due to the fact that it allows the grades of similarity and the grades of generalization between concepts to be represented by fuzzy numbers rather than crisp real values between zero and one.

II. FUZZY-VALUED CONCEPT-NETWORKS

In [4], Itzkovich and Hawkes presented a fuzzy extension of inheritance hierarchies to provide a more refined construction that facilitates the representation of relations among concepts under uncertain conditions. The extension is done in the following two steps:

1. Incorporate the synonymy relation in the inheritance hierarchy, resulting in a new construction denoted as a concept-network.

2. The relations on the concept-network are fuzzified to yield a new construction denoted as a fuzzy concept-network, where the relevant values between concepts are represented by real values between zero and one.

The definitions of fuzzy concept-networks are reviewed from [4] as follows.

Definition 2.1: The similarity relation \( R_{sim} \) over a finite set of concepts \( C, C = \{ c_1, c_2, \ldots, c_n \} \), is a binary fuzzy relation which satisfies all of the following properties:

1. Reflexive: \( \mu_{sim}(c_i, c_i) = 1 \).
2. Symmetric: \( \mu_{sim}(c_i, c_j) = \mu_{sim}(c_j, c_i) \).
3. Transitive: \( \mu_{sim}(c_i, c_k) \geq \mu_{sim}(c_i, c_j) \wedge \mu_{sim}(c_j, c_k) \).

Definition 2.2: The graded generalization relation \( R_g \) over a finite set of concepts \( C, C = \{ c_1, c_2, \ldots, c_n \} \), is a binary fuzzy relation which satisfies all of the following properties:

1. Reflexive: \( \mu_g(c_i, c_i) = 1 \).
2. Anti-symmetric: If \( \mu_g(c_i, c_j) > 0 \) and \( \mu_g(c_j, c_i) > 0 \), then \( c_i = c_j \).
3. Transitive: \( \mu_g(c_i, c_k) \geq \mu_g(c_i, c_j) \wedge \mu_g(c_j, c_k) \).
Definition 2.3: A fuzzy concept-network is denoted by $FCN(C, R)$, where $C$ is a finite set of concepts and $R$ consists of two relations $R_{sim}$ and $R_{d}$ over $C$ as defined in Definitions 2.1 and 2.2.

For example, Fig. 1 shows an example of a fuzzy concept network.

In [2], we presented the concepts of interval-valued fuzzy concept networks. In an interval-valued fuzzy concept-network, the degrees of similarity and the degrees of generalization between concepts are represented by real intervals in $[0, 1]$. Two intervals $[a, b]$ and $[c, d]$ are called equal if $a = c$ and $b = d$. If $[a, b] > [c, d]$ then it implies $a > c$ and $b > d$ or $a = c$ and $b = d$. The definitions of interval-valued fuzzy concept networks are reviewed from [2] as follows:

Definition 2.4: The interval-valued similarity relation $R_{sim}$, over a finite set of concepts $C$, is given by $R_{sim} = \{ (c_1, c_2) \mid a \leq c_1 \cap c_2 \leq b \}$. Two intervals $[a, b]$ and $[c, d]$ are called equal (i.e., $[a, b] = [c, d]$) if and only if $a = c$ and $b = d$.

Definition 2.5: The interval-valued generalization relation $R_{d}$, over a finite set of concepts $C$, is defined as $R_{d} = \{ (c_1, c_2) \mid a \leq c_1 \leq b \cap c_2 \}$. Two intervals $[a, b]$ and $[c, d]$ are called unequal (i.e., $[a, b] \neq [c, d]$) if and only if $a > c$ and $b > d$ or $a = c$ and $b = d$.

Fig. 1. A fuzzy concept network.

Fig. 2. An interval-valued fuzzy concept network.

Fig. 3. A triangular fuzzy number.

Fig. 4. A trapezoidal fuzzy number.

In Fig. 3, where $t_1 \leq t_2 \leq t_3$, or by a trapezoidal distribution parametrized by a quadruple $(q_1, q_2, q_3, q_4)$ shown in Fig. 4, where $q_1 \leq q_2 \leq q_3 \leq q_4$.

In the following, we introduce two kinds of fuzzy-valued concept networks. The first one allows the degrees of similarity and the degrees of generalization between concepts to be represented by triangular fuzzy numbers, whereas the second one allows the degrees of similarity and the degrees of similarity to be represented by trapezoidal fuzzy numbers.

Definition 2.7: Let $A$ and $B$ be two triangular fuzzy numbers, where

$A = (a_1, a_2, a_3)$,

$B = (b_1, b_2, b_3)$

$0 \leq a_1 \leq a_2 \leq a_3 \leq 1$, and $0 \leq b_1 \leq b_2 \leq b_3 \leq 1$. The triangular fuzzy numbers $A$ and $B$ are called equal (i.e., $A = B$) if and only if $a_1 = b_1$, $a_2 = b_2$, and $a_3 = b_3$. Otherwise, the triangular fuzzy numbers $A$ and $B$ are called unequal (i.e., $A \neq B$).

Definition 2.8: Let $X$ and $Y$ be two trapezoidal fuzzy numbers, where

$X = (x_1, x_2, x_3, x_4)$

$Y = (y_1, y_2, y_3, y_4)$

$0 \leq x_1 \leq x_2 \leq x_3 \leq x_4 \leq 1$, and $0 \leq y_1 \leq y_2 \leq y_3 \leq y_4 \leq 1$. The trapezoidal fuzzy numbers $X$ and $Y$ are called equal (i.e., $X = Y$) if and only if $x_1 = y_1$, $x_2 = y_2$, $x_3 = y_3$, and $x_4 = y_4$. Otherwise, the triangular fuzzy numbers $X$ and $Y$ are called unequal (i.e., $X \neq Y$).

The definitions of fuzzy-valued concept-networks using triangular fuzzy numbers to represent the grades of generalization and grades...
of similarity between concepts are presented in Definitions 2.9–2.11. The definitions of fuzzy-valued concept-networks using trapezoidal fuzzy numbers to represent the degrees of generalization and grades of similarity between concepts are presented in Definitions 2.12–2.14.

**Definition 2.9:** The similarity relation \( R_{\text{simtr}} \) represented by triangular fuzzy numbers over a finite set of concepts \( C, C = \{c_1, c_2, \ldots, c_n\} \), is a binary fuzzy relation which satisfies all of the following properties.

1) **Reflexive:** \( \mu_{\text{simtr}}(c_i, c_i) = (1, 1, 1) \).
2) **Symmetric:** If \( \mu_{\text{simtr}}(c_i, c_j) = \mu_{\text{simtr}}(c_j, c_i) \).
3) **Transitive:** Let the degree of similarity between any concepts \( c_x \) and \( c_y \) be represented by \( \mu_{\text{simtr}}(c_x, c_y) \), where \( \mu_{\text{simtr}}(c_x, c_y) = (S_1(c_x, c_y), S_2(c_x, c_y), S_3(c_x, c_y)) \), and \( 0 \leq S_1(c_x, c_y) \leq S_2(c_x, c_y) \leq S_3(c_x, c_y) \leq 1 \). Then

\[
S_1(c_x, c_y) \geq \bigvee_{c_j} (S_1(c_i, c_j) \land S_1(c_j, c_k))
\]
\[
S_2(c_x, c_y) \geq \bigvee_{c_j} (S_2(c_i, c_j) \land S_2(c_j, c_k))
\]
\[
S_3(c_x, c_y) \geq \bigvee_{c_j} (S_3(c_i, c_j) \land S_3(c_j, c_k))
\]

**Definition 2.10:** The graded generalization relation \( R_{\text{grti}} \) represented by triangular fuzzy numbers over a finite set of concepts \( C, C = \{c_1, c_2, \ldots, c_n\} \), is a binary fuzzy relation which satisfies all of the following properties.

1) **Reflexive:** \( \mu_{\text{grti}}(c_i, c_i) = (1, 1, 1) \).
2) **Anti-symmetric:** If \( \mu_{\text{grti}}(c_i, c_j) \neq (0, 0, 0) \) and \( \mu_{\text{grti}}(c_j, c_i) \neq (0, 0, 0) \), then \( c_i = c_j \).
3) **Transitive:** Let the degree of generalization between any concepts \( c_x \) and \( c_y \) be represented by \( \mu_{\text{grti}}(c_x, c_y) \), where \( \mu_{\text{grti}}(c_x, c_y) = (g_1(c_x, c_y), g_2(c_x, c_y), g_3(c_x, c_y)) \), and \( 0 \leq g_1(c_x, c_y) \leq g_2(c_x, c_y) \leq g_3(c_x, c_y) \leq 1 \). Then

\[
g_1(c_x, c_y) \geq \bigvee_{c_j} (g_1(c_i, c_j) \land g_1(c_j, c_k))
\]
\[
g_2(c_x, c_y) \geq \bigvee_{c_j} (g_2(c_i, c_j) \land g_2(c_j, c_k))
\]
\[
g_3(c_x, c_y) \geq \bigvee_{c_j} (g_3(c_i, c_j) \land g_3(c_j, c_k))
\]

**Definition 2.11:** A fuzzy-valued concept-network using triangular fuzzy numbers to represent the degrees of generalization and the degrees of similarity between concepts is denoted by \( \text{FVCNTAt}(C, R) \), where \( C \) is a finite set of concepts and \( R \) consists of two relations \( R_{\text{simtr}} \) and \( R_{\text{grti}} \) over \( C \) as defined in Definitions 2.9 and 2.10.

For example, Fig. 5 shows a fuzzy-valued concept-network using triangular fuzzy numbers to represent the degrees of generalization and degrees of similarity between concepts.

**Definition 2.12:** The similarity relation \( R_{\text{simtr}} \) represented by trapezoidal fuzzy numbers over a finite set of concepts \( C, C = \{c_1, c_2, \ldots, c_n\} \), is a binary fuzzy relation which satisfies all of the following properties.

1) **Reflexive:** \( \mu_{\text{simtr}}(c_i, c_i) = (1, 1, 1, 1) \).
2) **Symmetric:** \( \mu_{\text{simtr}}(c_i, c_j) = \mu_{\text{simtr}}(c_j, c_i) \).
3) **Transitive:** Let the degree of similarity between any concepts \( c_x \) and \( c_y \) be represented by \( \mu_{\text{simtr}}(c_x, c_y) \), where \( \mu_{\text{simtr}}(c_x, c_y) = (S_1(c_x, c_y), S_2(c_x, c_y), S_3(c_x, c_y)) \), and \( 0 \leq S_1(c_x, c_y) \leq S_2(c_x, c_y) \leq S_3(c_x, c_y) \leq 1 \). Then

\[
S_1(c_x, c_y) \geq \bigvee_{c_j} (S_1(c_i, c_j) \land S_1(c_j, c_k))
\]
\[
S_2(c_x, c_y) \geq \bigvee_{c_j} (S_2(c_i, c_j) \land S_2(c_j, c_k))
\]
\[
S_3(c_x, c_y) \geq \bigvee_{c_j} (S_3(c_i, c_j) \land S_3(c_j, c_k))
\]

**Definition 2.13:** The graded generalization relation \( R_{\text{grti}} \) represented by trapezoidal fuzzy numbers over a finite set of concepts \( C, C = \{c_1, c_2, \ldots, c_n\} \), is a binary fuzzy relation which satisfies all of the following properties.

1) **Reflexive:** \( \mu_{\text{grti}}(c_i, c_i) = (1, 1, 1, 1) \).
2) **Anti-symmetric:** If \( \mu_{\text{grti}}(c_i, c_j) \neq (0, 0, 0, 0) \) and \( \mu_{\text{grti}}(c_j, c_i) \neq (0, 0, 0, 0) \), then \( c_i = c_j \).
3) **Transitive:** Let the degree of similarity between any concepts \( c_x \) and \( c_y \) be represented by \( \mu_{\text{grti}}(c_x, c_y) \), where \( \mu_{\text{grti}}(c_x, c_y) = (g_1(c_x, c_y), g_2(c_x, c_y), g_3(c_x, c_y), g_4(c_x, c_y)) \), and \( 0 \leq g_1(c_x, c_y) \leq g_2(c_x, c_y) \leq g_3(c_x, c_y) \leq g_4(c_x, c_y) \leq 1 \). Then

\[
g_1(c_x, c_y) \geq \bigvee_{c_j} (g_1(c_i, c_j) \land g_1(c_j, c_k))
\]
\[
g_2(c_x, c_y) \geq \bigvee_{c_j} (g_2(c_i, c_j) \land g_2(c_j, c_k))
\]
\[
g_3(c_x, c_y) \geq \bigvee_{c_j} (g_3(c_i, c_j) \land g_3(c_j, c_k))
\]
\[
g_4(c_x, c_y) \geq \bigvee_{c_j} (g_4(c_i, c_j) \land g_4(c_j, c_k))
\]

**Definition 2.14:** A fuzzy-valued concept-network using trapezoidal fuzzy numbers to represent the degrees of generalization and the degrees of similarity between concepts is denoted by \( \text{FVCNTAt}(C, R) \), where \( C \) is a finite set of concepts and \( R \) consists of two relations \( R_{\text{simtr}} \) and \( R_{\text{grti}} \) over \( C \) as defined in Definitions 2.9 and 2.13.

For example, Fig. 6 shows a fuzzy-valued concept-network using trapezoidal fuzzy numbers to represent the degrees of generalization and the degrees of similarity between concepts.

III. AN ALGORITHM FOR FINDING THE INHERITANCE HIERARCHIES IN FUZZY-VALUED CONCEPT-NETWORKS

In [4], Itzkovich and Hawkes present an algorithm for finding the collection of inheritance hierarchies in fuzzy concept-networks, where the degrees of generalization and the degrees of similarity between concepts are represented by real values between zero and one. In [2], we have presented an algorithm for finding the collection...
of inheritance hierarchies in interval-valued fuzzy concept networks, where the degrees of generalization and the degrees of similarity between concepts are represented by interval values in $[0, 1]$.

In this section, we extend the works of [2] and [4] to present an algorithm for finding the inheritance hierarchies in fuzzy-valued concept networks, where the degrees of generalization and the degrees of similarity between concepts are represented by fuzzy numbers. Firstly, we present a method to model the fuzzy-valued concept-networks by means of concept matrices. If there are $n$ concepts in a fuzzy concept-network, then an $n \times n$ concept matrix will be used to model the fuzzy-valued concept-network.

**Case 1:** If a fuzzy-valued concept-network uses triangular fuzzy numbers to represent the degrees of generalization and the degrees of similarity between concepts.

If $\mu_{\text{sim}}(c_i, c_j) = \mu_{ij}$, where $\mu_{ij} \in [0, 1]$, then let $M(i, j) = \begin{pmatrix} \mu_{ij} & \mu_{ij} & \mu_{ij} \\ \mu_{ij} & \mu_{ij} & \mu_{ij} \\ \mu_{ij} & \mu_{ij} & \mu_{ij} \end{pmatrix}$.

If $\mu_{\text{gen}}(c_i, c_j) = \mu_{ij}$, where $\mu_{ij} \in [0, 1]$, then let $M(i, j) = (\mu_{ij}, \mu_{ij}, \mu_{ij})$ and $M(j, i) = (0, 0, 0)$.

If $\beta_{\text{sim}}(c_i, c_j) = (\mu_{ij}^1, \mu_{ij}^2, \mu_{ij}^3)$, where $0 \leq \mu_{ij}^1 \leq \mu_{ij}^2 \leq \mu_{ij}^3 \leq 1$, then let $M(i, j) = (\mu_{ij}^1, \mu_{ij}^2, \mu_{ij}^3)$ and $M(j, i) = (0, 0, 0)$.

If there are no relationships between the concepts $c_i$ and $c_j$, then let $M(i, j) = M(j, i) = (0, 0, 0)$.

Furthermore, we let $M(i, i) = (1, 1, 1)$, where $1 \leq i \leq n$, due to the fact that each concept $c_i$ is reflexive to itself.

**Example 3.1:** Given a fuzzy-valued concept-network shown in Fig. 5, where $c_1$ is a generalization of $c_2$ with $\mu_{\text{gen}}(c_2, c_1) = (0.6, 0.8, 1.0)$, $c_2$ is also a generalization of $c_3$ with $\mu_{\text{gen}}(c_2, c_3) = (0.9, 0.95, 1)$, and $c_3$ is similar to $c_4$ with $\mu_{\text{sim}}(c_3, c_4) = (0.8, 0.9, 1)$. Then, we can use a $4 \times 4$ concept matrix $M$ to model the fuzzy-valued concept-network

$$M = \begin{bmatrix} (1, 1, 1) & (0, 0, 0) & (0, 0, 0) & (0, 0, 0) \\ (0, 0, 0) & (1, 1, 1) & (0, 0, 0) & (0, 0, 0) \\ (0, 0, 0) & (0, 0, 0) & (0.8, 0.9, 1) & (1, 1, 1) \end{bmatrix}$$

**Case 2:** If a fuzzy-valued concept-network uses trapezoidal fuzzy numbers to represent the degrees of generalization and the degrees of similarity between concepts.

If $\beta_{\text{sim}}(c_i, c_j) = \mu_{ij}$, where $\mu_{ij} \in [0, 1]$, then let $N(i, j) = N(j, i) = (\mu_{ij}, \mu_{ij}, \mu_{ij}, \mu_{ij})$.

If $\mu_{\text{gen}}(c_i, c_j) = \mu_{ij}$, where $\mu_{ij} \in [0, 1]$, then let $N(i, j) = (\mu_{ij}, \mu_{ij}, \mu_{ij}, \mu_{ij})$ and $N(j, i) = (0, 0, 0, 0)$.

If $\alpha_{\text{sim}}(c_i, c_j) = (\mu_{ij}^1, \mu_{ij}^2, \mu_{ij}^3, \mu_{ij}^4)$, where $0 \leq \mu_{ij}^1 \leq \mu_{ij}^2 \leq \mu_{ij}^3 \leq \mu_{ij}^4 \leq 1$, then let $N(i, j) = (\mu_{ij}^1, \mu_{ij}^2, \mu_{ij}^3, \mu_{ij}^4)$ and $N(j, i) = (0, 0, 0, 0)$.

If there are no relationships between the concepts $c_i$ and $c_j$, then let $N(i, j) = N(j, i) = (0, 0, 0, 0)$.

Furthermore, we let $N(i, i) = (1, 1, 1, 1)$, where $1 \leq i \leq n$, due to the fact that each concept $c_i$ is reflexive to itself.

**Example 3.2:** Given a fuzzy-valued concept-network shown in Fig. 6, where $c_1$ is a generalization of $c_3$ with $\mu_{\text{gen}}(c_3, c_1) = (0.6, 0.7, 0.8, 0.9)$, $c_2$ is similar to $c_3$ with $\mu_{\text{sim}}(c_2, c_3) = (0.7, 0.8, 0.9, 1)$, and $c_4$ is similar to $c_5$ with $\mu_{\text{sim}}(c_4, c_5) = (0.8, 0.9, 0.1)$. Then, we can use a $5 \times 5$ concept matrix $N$ to model the fuzzy-valued concept-network shown in the equation at the bottom of the page.

In the following, we present a method for performing the $\alpha$-cuts operations on a fuzzy-valued concept-network. Assume that a fuzzy-valued concept-network using triangular fuzzy numbers to represent the degrees of generalization and the degrees of similarity between concepts has been modeled by a concept matrix $M$. Let $P$ be a probability matrix derived from performing the $\alpha$-cut operation on the concept matrix $M$, where $\alpha_1$ is a threshold value between zero and one. Then, in a probability matrix $P$, the element $P(i, j)$ indicates the degree of probability that $M(i, j)$ is larger than or equal to $\alpha_1$, where $M(i, j)$ is represented by a triangular fuzzy number $(a_{ij}, b_{ij}, c_{ij})$. The larger the value of $P(i, j)$, the more the degree of the probability that the degree of relationship (generalization relationship or similarity relationship) between the concepts $c_i$ and $c_j$ is larger than $\alpha_1$. The value of $P(i, j)$ is decided by the following cases, where $M(i, j)$ is represented by a triangular fuzzy number parametrized by $(a_{ij}, b_{ij}, c_{ij})$ and $0 \leq a_{ij} \leq b_{ij} \leq c_{ij} \leq 1$.

If $a_{ij} \leq b_{ij} \leq c_{ij}$ then

**Case A1:** If $\alpha_1 > a_{ij}$ (see Fig. 7), then we let $P(i, j) = 0$.

**Case A2:** If $\alpha_1 \leq a_{ij}$ (see Fig. 8), then we let $P(i, j) = 1$.

else

**Case A3:** If $\alpha_1 > c_{ij}$ (see Fig. 9), then we let $P(i, j) = 0$.

**Case A4:** If $b_{ij} \leq \alpha_1 \leq c_{ij}$ and $b_{ij} \neq c_{ij}$ (see Fig. 10), then we let $P(i, j) = (c_{ij} - \alpha_1)^2/(c_{ij} - b_{ij})(c_{ij} - a_{ij})$. (Since the area of the shadow triangle is $(1/2)((c_{ij} - \alpha_1)^2/(c_{ij} - b_{ij}))$ and the area of the whole triangle parametrized by $(a_{ij}, b_{ij}, c_{ij})$ is $(c_{ij} - \alpha_1)^2/(c_{ij} - b_{ij})(c_{ij} - a_{ij}))$, the $P(i, j)$ is equal to the proportion of the shadow triangle to the whole triangle parametrized by $(a_{ij}, b_{ij}, c_{ij})$ which is $(c_{ij} - \alpha_1)^2/(c_{ij} - b_{ij})(c_{ij} - a_{ij}))$.

$$N = \begin{bmatrix} (1, 1, 1, 1) & (0, 0, 0, 0) & (0, 0, 0, 0) & (0, 0, 0, 0) & (0, 0, 0, 0) \\ (0, 0, 0, 0) & (1, 1, 1, 1) & (0, 0, 0, 0) & (0, 0, 0, 0) & (0, 0, 0, 0) \\ (0, 0, 0, 0) & (0, 0, 0, 0) & (0, 0, 0, 0) & (0, 0, 0, 0) & (0, 0, 0, 0) \\ (0.6, 0.7, 0.8, 0.9) & (0.7, 0.8, 0.9, 1) & (1.1, 1.1, 1.1) & (0.0, 0.0, 0.0) & (0.0, 0.0, 0.0) \\ (0.8, 0.9, 0.9, 1) & (0.0, 0.0, 0.0) & (0.0, 0.0, 0.0) & (0.8, 0.9, 0.9, 1) & (1.1, 1.1, 1.1) \end{bmatrix}$$
**Case A5:** If \( b_{ij} \leq \alpha_1 \leq c_{ij} \) and \( b_{ij} = c_{ij} \) (see Fig. 11), then we let \( P(i,j) = 0 \).

**Case A6:** If \( a_{ij} \leq \alpha_1 \leq b_{ij} \) and \( a_{ij} \neq b_{ij} \) (see Fig. 12), then we let \( P(i,j) = 1 - ((\alpha_1 - a_{ij})^2/(b_{ij} - a_{ij})(c_{ij} - a_{ij})) \). (Since the area of the empty triangle is \((1/2)\((\alpha_1 - a_{ij})^2/(b_{ij} - a_{ij})\)) and the area of the whole trapezoidal parametrized by \((a_{ij}, b_{ij}, c_{ij})\) is \(\frac{1}{2}(c_{ij} - a_{ij})\), the \(P(i,j)\) is equal to one minus the proportion of the empty triangle to the whole trapezoidal parametrized by \((a_{ij}, b_{ij}, c_{ij})\) which is \(1 - ((\alpha_1 - a_{ij})^2/(b_{ij} - a_{ij})(c_{ij} - a_{ij}))\).

**Case A7:** If \( a_{ij} \leq \alpha_1 \leq b_{ij} \) and \( a_{ij} = b_{ij} \) (see Fig. 13), then we let \( P(i,j) = 1 \).

**Case A8:** If \( \alpha_1 < a_{ij} \) (see Fig. 14), then we let \( P(i,j) = 1 \).

Assume that a fuzzy-valued concept-network using trapezoidal fuzzy numbers to represent the degrees of generalization and the degrees of similarity between concepts has been modeled by a concept matrix \(N\). Let \(Q\) be a probability matrix derived from performing the \(\alpha_1\)-cut operation on the concept matrix \(N\), where \(\alpha_1\) is a threshold value between zero and one. Then, in a probability matrix \(Q\), the element \(Q(i,j)\) indicates the degree of probability that \(N(i,j)\) is larger than or equal to \(\alpha_1\), where \(N(i,j)\) is represented by a trapezoidal fuzzy number \((a_{ij}, b_{ij}, c_{ij}, d_{ij})\).

The larger the value of \(Q(i,j)\), the more the degree of the probability that the degree of relationship (generalization relationship or similarity relationship) between the concept \(c_i\) and \(c_j\) is larger than \(\alpha_1\). The value of \(Q(i,j)\) is decided by the following cases, where \(N(i,j)\) is represented by a fuzzy number parametrized by \((a_{ij}, b_{ij}, c_{ij}, d_{ij})\) and \(0 \leq a_{ij} \leq b_{ij} \leq c_{ij} \leq d_{ij} \leq 1\).

- **Case B1:** If \( \alpha_1 > a_{ij} \) (see Fig. 15), then we let \(Q(i,j) = 0\).
- **Case B2:** If \( \alpha_1 \leq a_{ij} \) (see Fig. 16), then we let \(Q(i,j) = 1\).
- **Case B3:** If \( \alpha_1 > d_{ij} \) (see Fig. 17), then we let \(Q(i,j) = 0\).
- **Case B4:** If \( c_{ij} \leq \alpha_1 \leq d_{ij} \) and \( c_{ij} \neq d_{ij} \) (see Fig. 18), then we let \(Q(i,j) = (d_{ij} - \alpha_1)^2/(d_{ij} - c_{ij})(d_{ij} + c_{ij} - b_{ij} - a_{ij})\). (Since the area of the shadow triangle is \((1/2)\((d_{ij} - \alpha_1)^2/(d_{ij} - c_{ij})\)) and the area of the whole trapezoid parametrized by \((a_{ij}, b_{ij}, c_{ij}, d_{ij})\) is \(\frac{1}{2}(d_{ij} + c_{ij} - b_{ij} - a_{ij})\), the \(Q(i,j)\) is equal to the proportion...
of the shadow triangle to the whole triangle parametrized by 
\((a_{ij}, b_{ij}, c_{ij}, d_{ij})\) which is 
\((d_{ij} - c_{ij})^2/(d_{ij} + c_{ij})(d_{ij} + c_{ij} - b_{ij} - a_{ij})\).

**Case B5:** If \(c_{ij} \leq a_{ij} \leq d_{ij}\) and \(c_{ij} = d_{ij}\) (see Fig. 19), then we let \(Q(i,j) = 0\).

**Case B6:** If \(b_{ij} \leq a_{ij} \leq c_{ij}\) (see Fig. 20), then we let \(Q(i,j) = (d_{ij} + c_{ij} - 2a_{ij})/(d_{ij} + c_{ij} - b_{ij} - a_{ij})\). (Since the area of the shadow trapezoidal is \(\frac{1}{2}(d_{ij} + c_{ij} - 2a_{ij})\) and the area of the whole trapezoidal parametrized by \((a_{ij}, b_{ij}, c_{ij}, d_{ij})\) is \(\frac{1}{2}(d_{ij} + c_{ij} - b_{ij} - a_{ij})\), the \(Q(i,j)\) is equal to the proportion of the shadow trapezoidal to the whole triangle parametrized by \((a_{ij}, b_{ij}, c_{ij}, d_{ij})\) which is \((d_{ij} + c_{ij} - 2a_{ij})/(d_{ij} + c_{ij} - b_{ij} - a_{ij})\).

**Case B7:** If \(a_{ij} \leq a_{ij} \leq b_{ij}\) and \(a_{ij} \neq b_{ij}\) (see Fig. 21), then we let \(Q(i,j) = 1 - ((a_{ij} - c_{ij})^2/(b_{ij} - a_{ij})(d_{ij} + c_{ij} - b_{ij} - a_{ij}))\). (Since the area of the empty triangle is \((1/2)((a_{ij} - c_{ij})^2/(b_{ij} - a_{ij}))\) and the area of the whole trapezoidal parametrized by \((a_{ij}, b_{ij}, c_{ij}, d_{ij})\) is \(\frac{1}{2}(d_{ij} + c_{ij} - b_{ij} - a_{ij})\), the \(Q(i,j)\) is equal to one minus the proportion of the empty triangle to the whole triangle parametrized by \((a_{ij}, b_{ij}, c_{ij}, d_{ij})\) which is \(1 - ((a_{ij} - c_{ij})^2/(b_{ij} - a_{ij})(d_{ij} + c_{ij} - b_{ij} - a_{ij}))\).

**Case B8:** If \(a_{ij} \leq a_{ij} \leq b_{ij}\) and \(a_{ij} = b_{ij}\) (see Fig. 22), then we let \(Q(i,j) = 1\).

**Case B9:** If \(a_{ij} < a_{ij}\) (see Fig. 23), then we let \(Q(i,j) = 1\).

Let \(S\) be a confidence matrix derived from \(P\), and let \(a_{ij}\) be a threshold value between zero and one. If \(P(i,j) \geq a_{ij}\), then we let \(S(i,j) = 1\). Otherwise, we let \(S(i,j) = 0\). \(S(i,j) = 1\) indicates that the degree of probability \(\beta\) in which the degree of relationship between the concepts \(c_i\) and \(c_j\) is larger than or equal to \(\alpha\) is larger than or equal to \(\alpha_2\), where \(\alpha_2 \in [0, 1]\).

Let \(S\) be a confidence matrix derived from \(Q\), and let \(a_{ij}\) be a threshold value between zero and one. If \(Q(i,j) \geq a_{ij}\), then we let \(S(i,j) = 1\). Otherwise, we let \(S(i,j) = 0\). \(S(i,j) = 1\) indicates that the degree of probability \(\beta\) in which the degree of relationship between the concepts \(c_i\) and \(c_j\) is larger than or equal to \(\alpha\) is larger than or equal to \(\alpha_2\), where \(\alpha_2 \in [0, 1]\).

In the following, we assume that a fuzzy-valued concept-network which consists of \(n\) concepts using triangular fuzzy numbers to represent the degrees of generalization and the degrees of similarity between concepts has been modeled by an \(n \times n\) concept matrix \(M\), where \(M(i,j) = (\mu_{ij}, \mu_{ij}^2, \mu_{ij}^3)\). \(0 \leq \mu_{ij}^1 \leq \mu_{ij}^2 \leq \mu_{ij}^3 \leq 1\), \(i \leq j \leq n\). The algorithm for performing \(\alpha\)-cuts operations on the fuzzy-valued concept-network to obtain the probability matrix.
Example 3.3: Given a fuzzy-valued concept-network shown in Fig. 24. Fuzzy-valued concept-network of Example 3.3.

$P$ and the confidence matrix $S$ is now presented as follows.

**$\alpha$-Cuts Operations Algorithm (Algorithm A) for Fuzzy-Valued Concept-Networks Using Triangular Fuzzy Numbers:**

for $i \leftarrow 1$ to $n$
for $j \leftarrow 1$ to $n$
begin
if $(\mu^1_{ij} = \mu^2_{ij} = \mu^3_{ij})$
if $(\alpha_1 \geq \mu^1_{ij})$ then $P(i,j) = 0$
else $P(i,j) = 1$
else
begin
if $(\mu^2_{ij} \leq \alpha_1 \leq \mu^2_{ij})$ and $(\mu^2_{ij} \neq \mu^3_{ij})$
then $P(i,j) = 0$
if $(\mu^2_{ij} \leq \alpha_1 \leq \mu^3_{ij})$
and $(\mu^2_{ij} = \mu^3_{ij})$
then $P(i,j) = 0$
if $(\mu^1_{ij} \leq \alpha_1 \leq \mu^2_{ij})$ and $(\mu^1_{ij} \neq \mu^2_{ij})$
then $P(i,j) = 1$
end.
if $P(i,j) \geq \alpha_2$ then $S(i,j) = 1$
else $S(i,j) = 0$
end.

Example 3.3: Given a fuzzy-valued concept-network shown in Fig. 24. Assume that $\alpha_1 = 0.6$ and $\alpha_2 = 0.7$. Then we can use the concept matrix $M$ to model the fuzzy-valued concept-network shown at the bottom of the page. By performing the $\alpha$-cuts operations, the probability matrix $P$ and confidence matrix $S$ can be obtained as follows:

$$P = \begin{bmatrix}
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 1/8 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1/2 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1
\end{bmatrix}$$

$$S = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1/18 & 13/14 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}$$

In the following, we assume that a fuzzy-valued concept-network which consists of $n$ concepts using trapezoidal fuzzy numbers to represent the degrees of generalization and the degrees of similarity between concepts has been modeled by an $n \times n$ concept matrix $M$, where $N(i,j) = (\mu^1_{ij}, \mu^2_{ij}, \mu^3_{ij}, \mu^4_{ij})$, $0 \leq \mu^1_{ij} \leq \mu^2_{ij} \leq \mu^3_{ij} \leq \mu^4_{ij} \leq 1$, $1 \leq i \leq n$, and $1 \leq j \leq n$. The algorithm for performing $\alpha$-cuts operations on the fuzzy-valued concept-network to obtain the probability matrix $Q$ and the confidence matrix $S$ is shown above the matrix at the bottom of the next page.

Example 3.4: Given a fuzzy-valued concept-network shown in Fig. 25. Assume that $\alpha_1 = 0.6$ and $\alpha_2 = 0.7$. Then we can use the concept matrix $M$ to model the fuzzy-valued concept-network shown at the bottom of the page. By performing the $\alpha$-cuts operations, the probability matrix $Q$ and confidence matrix $S$ can be obtained as follows:

$$Q = \begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 1/18 \\
0 & 0 & 0 & 1 & 0 & 0 & 13/14 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}$$

$$S = \begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}$$

In the following, we present the definition of concept classes in a fuzzy-valued concept-network based on [4]. A concept class $P_i$ in a fuzzy-valued concept-network using triangular fuzzy numbers to
Fig. 25. Fuzzy-valued concept-network of Example 3.4.

represent the degrees of generalization and the degrees of similarity between concepts is a set of concepts, such that the set of concepts \( C \) in the fuzzy-valued concept-network is the union of each concept class, i.e., \( C = \bigcup_i P_i \). Furthermore, after performing the \( \alpha \)-cuts operations in the fuzzy-valued concept-network, we can define the set of synonymous concepts in each concept class.

**Definition 3.1:** In a concept class \( P_i, \forall c_i, c_j \in P_i \), if \( \mu_{\text{syn}}(c_i, c_j) > 0 \), then we say that \( c_i \) and \( c_j \) are in the same set of synonymous concepts in the fuzzy-valued concept-networks using triangular fuzzy numbers to represent the degrees of generalization and the degrees of similarity between concepts.

A concept class \( X_i \) in a fuzzy-valued concept-network using trapezoidal fuzzy numbers to represent the degrees of generalization and the degrees of similarity between concepts is a set of concepts, such that the set of concepts \( C \) in the fuzzy-valued concept-network is the union of each concept class, i.e., \( C = \bigcup_i X_i \). Furthermore, after performing the \( \alpha \)-cuts operations in the fuzzy-valued concept-network, we can define the set of synonymous concepts in each concept class.

**Definition 3.2:** In a concept class \( X_i, \forall c_i, c_j \in X_i \), if \( \mu_{\text{syn}}(c_i, c_j) > 0 \), then we say that \( c_i \) and \( c_j \) are in the same set of synonymous concepts in the fuzzy-valued concept-networks using trapezoidal fuzzy numbers to represent the degrees of generalization and the degrees of similarity between concepts.

The algorithm for finding the inheritance hierarchies in a fuzzy-valued concept-network is a modification of the one we presented in [2]. The algorithm is shown at the bottom of the next page and continued on the page following that.

**Example 3.5:** We make the same assumptions as in Example 3.3, where the fuzzy-valued concept-network shown in Fig. 24 is modeled by the concept matrix \( M \) and the probability matrix \( P \).

---

**\( \alpha \)-Cuts Operations Algorithm (Algorithm B) for Fuzzy-Valued Concept-Networks Using Trapezoidal Fuzzy Numbers:**

for \( i = 1 \) to \( n \) do
  for \( j = 1 \) to \( n \) do
    begin
      if \( (\mu_{ij}^1 = \mu_{ij}^2 = \mu_{ij}^3 = \mu_{ij}^4) \) then
        if \( (\alpha_1 \geq \mu_{ij}^1) \) then \( Q(i, j) \leftarrow 0 \)
        else \( Q(i, j) \leftarrow 1 \)
      else begin
        if \( (\alpha_1 \geq \mu_{ij}^1) \) then \( Q(i, j) \leftarrow 0 \)
        if \( (\mu_{ij}^2 \leq \alpha_1 \leq \mu_{ij}^3) \) and \( (\mu_{ij}^2 = \mu_{ij}^3) \) then
          \( Q(i, j) = \frac{(\mu_{ij}^3 - \alpha_1)^2}{(\mu_{ij}^3 - \mu_{ij}^2)(\mu_{ij}^3 - \mu_{ij}^4)} \)
        if \( (\mu_{ij}^3 \leq \alpha_1 \leq \mu_{ij}^4) \) and \( (\mu_{ij}^3 \neq \mu_{ij}^4) \) then
          \( Q(i, j) \leftarrow 0 \)
        if \( (\mu_{ij}^2 \leq \alpha_1 \leq \mu_{ij}^3) \) then
          \( Q(i, j) = \frac{\alpha_1 - \mu_{ij}^2}{(\mu_{ij}^3 - \mu_{ij}^2)(\mu_{ij}^4 - \mu_{ij}^2)} \)
        if \( (\mu_{ij}^4 \leq \alpha_1 \leq \mu_{ij}^3) \) and \( (\mu_{ij}^4 \neq \mu_{ij}^3) \) then
          \( Q(i, j) = \frac{(\alpha_1 - \mu_{ij}^4)^2}{(\mu_{ij}^3 - \mu_{ij}^4)(\mu_{ij}^3 - \mu_{ij}^4)} \)
        if \( (\mu_{ij}^4 \leq \alpha_1 \leq \mu_{ij}^4) \) and \( (\mu_{ij}^4 = \mu_{ij}^4) \) then
          \( Q(i, j) \leftarrow 1 \)
      end;
      if \( (Q(i, j) \geq \alpha_2) \) then \( S(i, j) \leftarrow 1 \)
      else \( S(i, j) \leftarrow 0 \)
    end.

---

\[
N = \begin{bmatrix}
(1.1, 1.1) & (0.7, 0.8, 0.9, 1) & (0.8, 0.85, 0.95, 1) & (0.0, 0.0) & (0.0, 0.0) & (0.0, 0.0) \\
(0.7, 0.8, 0.9, 1) & (1.1, 1.1) & (0.0, 0.0) & (0.0, 0.0) & (0.0, 0.0) & (0.0, 0.0) \\
(0.0, 0.0) & (0.0, 0.0) & (1.1, 1.1) & (0.0, 0.0) & (0.3, 0.4, 0.45, 0.7) & (0.5, 0.7, 0.9, 1) \\
(0.0, 0.0) & (0.0, 0.0) & (0.2, 0.4, 0.6) & (1.1, 1.1) & (0.0, 0.0) & (0.0, 0.0) \\
(0.0, 0.0) & (0.0, 0.0) & (0.0, 0.0) & (0.0, 0.0) & (1.1, 1.1) & (0.0, 0.0) \\
(0.0, 0.0) & (0.0, 0.0) & (0.0, 0.0) & (0.0, 0.0) & (0.0, 0.0) & (1.1, 1.1)
\end{bmatrix}
\]
and the confidence matrix $R$ have been obtained. By applying the inheritance hierarchy generation algorithm, we can obtain three sets of concept classes: $\{c_1, c_2, c_3, c_6\}, \{c_1, c_5\}, \{c_2\}$, and three sets of synonymous concepts: $\{c_1, c_4\}, \{c_2, c_5\}, \{c_4, c_6\}$. Assume that we are interested in the concept classes containing $c_2$, then after performing the algorithm, we can find the inheritance hierarchy $\{(c_1, c_2, c_3)\}$ containing $c_2$, graphically as shown in Fig. 26(a). By using replacement among synonymous concepts, we can obtain the other three inheritance hierarchies: $\{(c_5, c_2, c_3)\}, \{(c_1, c_5, c_3)\}, \{(c_5, c_6, c_3)\}$ as shown in Fig. 26(b)–(d), respectively.

Inheritance Hierarchy Generation Algorithm for Fuzzy-Valued Concept-Networks:
Step 1: Perform the $\alpha$-cuts operations on the fuzzy-valued concept network using the $\alpha$-cuts operations algorithm described previously.
(Notes: (1) If the degrees of generalization and the degree of similarity between concepts in the fuzzy-valued concept networks are represented by triangular fuzzy numbers, then we can choose Algorithm A for performing the $\alpha$-cuts operations on the fuzzy-valued concept network.
(2) If the degrees of generalization and the degree of similarity between concepts in the fuzzy-valued concept networks are represented by trapezoidal fuzzy numbers, then we can choose Algorithm B for performing the $\alpha$-cuts operations on the fuzzy-valued concept network.
(3) Because a triangular fuzzy number $(a, b, c)$ can also be represented by a trapezoidal fuzzy number $(a, b, b, c)$, if the degrees of generalization and the degree of similarity between concepts in the fuzzy-valued concept networks are represented by triangular fuzzy numbers, then we also can firstly translate the triangular fuzzy numbers in the fuzzy-valued concept network into trapezoidal fuzzy numbers, and then we can choose Algorithm B for performing the $\alpha$-cuts operations on the fuzzy-valued concept networks.)

Step 2: for $i = 1$ to $n$ do
  for $j = 1$ to $n$ do
    begin
      if $i = j$ and $c_i$ is not in any concept class then generate a new concept class, and put $c_i$ in the new generated concept class;
      if $i \neq j$ and $S(i, j) = 1$ then
        if $S(j, i) = 1$ then
          begin
            if $c_i$ is not in any concept class then generate a new concept class, and put $c_i$ and $c_j$ in the new generated concept class
            else
              put $c_j$ in the same concept class with $c_i$;
            if $c_i$ is not in any set of synonymous concepts then generate a new set of synonymous concepts, and put $c_i$ and $c_j$ in the new generated set of synonymous concepts
            else
              put $c_j$ in the same set of synonymous concepts with $c_i$;
          end
        else
          begin
            if $c_i$ is not in any concept class and $c_j$ is not in any concept class then generate a new concept class, and put $c_i$ and $c_j$ in the new generated concept class;
            if $c_i$ is in a concept class and $c_j$ is not in any concept class then put $c_j$ in the same concept class with $c_i$;
            if $c_i$ is not in any concept class and $c_j$ is in a concept class then put $c_i$ in the same concept class with $c_j$;
            if $c_i$ is in a concept class and $c_j$ is in a concept class then
              begin
                put all concepts in the concept class containing $c_j$ in the same concept class with $c_i$;
                put all fuzzy-valued generalization in the concept class containing $c_j$ in the concept class containing $c_i$;
              end;
            let $(c_i, c_j)$ be a fuzzy-valued generalization relation in concept class containing $c_i$;
          end;
        end;
      end;
    end;
  end;
find the concept class containing concept $c_k$;
list all fuzzy-valued generalization relations in this concept class which form an inheritance hierarchy;
for all \( c_i \) in this inheritance hierarchy do
begin
find the set of synonymous concepts containing \( c_i \);
for each \( c_j \) in this set of synonymous concepts do
begin
substitute \( c_i \) in the fuzzy-valued generalization relation by \( c_j \);
list all fuzzy-valued generalization relations in this concept class which form a new inheritance hierarchy
end
end.

**Fig. 26.** Inheritance hierarchies of Example 3.5.

**Fig. 27.** Inheritance hierarchies of Example 3.6.

**Example 3.6:** We make the same assumptions as in Example 3.4, where the fuzzy-valued concept-network shown in Fig. 25 is modeled by the concept matrix \( N \) and the probability matrix \( Q \) and the confidence matrix \( R \) have been obtained. By applying the inheritance hierarchy generation algorithm, we can obtain three sets of concept classes: \( \{ c_1, c_2, c_3, c_4 \}, \{ c_5 \}, \{ c_6 \} \), and one set of synonymous concepts: \( \{ c_1, c_2 \} \). Assume that we are interested in the concept classes containing \( c_3 \), then after performing the algorithm, we can find the inheritance hierarchy \( \{ \{ c_1, c_3, c_4 \} \} \) containing \( c_3 \), graphically as shown in Fig. 27(a). By using replacement among synonymous concepts, we can obtain the other inheritance hierarchy: \( \{ \{ c_2, c_3, c_5 \} \} \) as shown in Fig. 27(b).

**IV. CONCLUSIONS**

In this paper, we have extended the works of [2] and [4] to present the concepts of fuzzy-valued concept-networks and to present an algorithm for finding the collection of inheritance hierarchies in fuzzy-valued concept-networks where the degrees of generalization and the degrees of similarity between concepts are represented by triangular fuzzy numbers or trapezoidal fuzzy numbers. The proposed method is more flexible than the ones presented in [2] and [4] due to the fact that it allows the similarity relations and the generalization relations between concepts to be represented by triangular fuzzy numbers or trapezoidal fuzzy numbers rather than crisp real values between zero and one or interval-values in \([0, 1]\).

**ACKNOWLEDGMENT**

The authors would like to thank Mr. K. F. Wang, Department of Computer and Information Science, National Chiao Tung University, for providing very helpful suggestions in this work.

**REFERENCES**


