Existence of Invariant Series
Consecutive-k-out-of-n:G Systems

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Key Words — Consecutive-k-out-of-n:G system, Series system, Invariant assignment.

Summary & Conclusions — A consecutive-k-out-of-n:G line has an invariant optimal assignment iff \( k \leq n \leq 2k \). This paper —
- studies a system consisting of consecutive-k-out-of-n:G lines with \( k_1 \leq n_1 \leq 2k \);
- completely characterizes the existence of an invariant optimal assignment.

1. INTRODUCTION

Notation
- \( n \): number of components/positions in a system
- \( p_i \): reliability of component \( i \)
- \( P = \{p_1, p_2, \ldots, p_n\} \)
- \( p_{[i]} \): the \( i \)th smallest reliability in \( P \)
- \( R \): reliability function
- \( \phi \): any item
- \( \text{con}(k/n : G) \): consecutive-k-out-of-n:G line
- \( \text{con}(k/n : F) \): consecutive-k-out-of-n:F system
- \( S(k_1^{a_1}, k_2^{a_2}, \ldots, k_m^{a_m}) \): a series system with \( \sum_{i=1}^{m} a_i \) subsystems consisting of \( a_i \) reliability of \( k_i/2k_i \)
- \( S(k_1/n_1, \ldots, k_m/n_m) \): a series system with \( m \) subsystems; \( k_i/n_i \equiv \text{con}(k_i/n_i : G) \)

Definitions
- \( \text{con}(k/n : G) \): a system of \( n \) components arranged into a line such that the system works iff some consecutive \( k \) components of its \( n \) positions all work.
- \( \text{con}(k/n : F) \): the system fails iff some consecutive \( k \) components of its \( n \) positions all fail.

The \( \text{con}(k/n : G) \) is the counterpart of the more well-known \( \text{con}(k/n : F) \), but has its own applications. The \( \text{con}(k/n : G) \) was first mentioned in [4], but has attracted more attention since the publication of [2].

Consider a set of \( n \) components with reliabilities:
- \( p_{[1]} \leq p_{[2]} \leq \ldots \leq p_{[n]} \).

The problem is to assign the \( n \) components to the \( n \) positions in a \( \text{con}(k/n : G) \) to maximize its reliability. An optimal assignment is invariant if it depends only on the ranks of the \( p_i \) but not on their values. Thus an invariant assignment can be applied to a wide range of component reliabilities, even when the available information on these values is inexact. A system is invariant if there exists an invariant assignment.

Unfortunately, [5] proved the nonexistence of an invariant assignment for \( 2 \leq k < n/2 \). Therefore our attention is focused on the range \( n/2 \leq k \leq n \). Ref [2] proved that for \( n = 2k \), the invariant assignment is
- \( p_{[1]}, p_{[3]}, \ldots, p_{[2k-1]}, p_{[2k]}, p_{[2k-2]}, \ldots, p_{[2]} \).

For \( n < 2k \), simply delete those \( p_{[i]} \) with \( i > n \). However, [1] observed that the proof in [5] was not rigorous, and then gave a completely different proof.

Let,
- \( S(k_1^{a_1}, k_2^{a_2}, \ldots, k_m^{a_m}) \), \( k_1 < k_2 < \ldots < k_m \), \( a_i \geq 1 \) for \( 1 \leq i \leq m \)
- \( \alpha_i \geq 1 \) for \( 1 \leq i \leq m \)

denote a series system with \( \sum_{i=1}^{m} \alpha_i \) subsystems consisting of \( \alpha_i \text{con}(k_i/2k_i : G) \), \( i = 1, \ldots, m \). It is a series system since \( S \) works iff all its subsystems work. Ref [3] proved that \( S(k^{a_1}, (k+1)^{a_2}) \) is invariant for any \( k \geq 1 \), \( a \geq 0 \), and \( \alpha \geq 0 \). A reasonable question arises about the existence of other invariant \( S(k_1^{a_1}, \ldots, k_m^{a_m}) \); this paper shows that there is none. \( S \) is allowed to contain \( \text{con}(k/n : G) \) with \( k \leq n \leq 2k \); then the existence of invariant assignment is completely characterized.

2. MAIN RESULTS

2.1 Two Lines

Lemma 1. If \( \text{con}(k/2k : G) \) contains a component with reliability 1, then it is reduced to \( \text{con}((k-1)/2(k-1) : G) \).

Proof: Without loss of generality, let \( p_k = 1 \). Then the system works iff the sequence
- \( (p_1, p_2, \ldots, p_{k-1}, p_{k+1}, \ldots, p_{2k-1}) \)
contains \( k-1 \) consecutive working components.

Lemma 2. Consider \( S(k, k') \) with \( k < k' \). In an invariant assignment, both the best and the worst components go to the \( 2k' \)-line.
Proof: Let \( P \) be the set of reliabilities
\[
\{p_1[0] = p, p_2[0] = p_2, \ldots = p_{2(2k+2k')-1} = \epsilon, p_{2k+2k'} = 1\},
\]
where \( \epsilon \to 0 \). Compare two assignments:
\( A_1: p_2k+2k' \) goes to the 2k-line,
\( A_2: p_{2k+2k'} \) goes to the 2k'-line.
Since \( \epsilon \gg \epsilon' \), it suffices to compare \( A_1 \) and \( A_2 \), conditional
on the event that exactly \( k + k' - 1 \) \( \epsilon \)-components work.
By lemma 1, \( S(k, k') \) reduces to
\[
S(k-1, k') \quad \text{under} \quad A_1,
\]
\[
S(k, k'-1) \quad \text{under} \quad A_2.
\]
The conditional probabilities that the system works under
\( A_1 \) and \( A_2 \) are, respectively,
\[
k(k'+1)^{k+k'-1},
\]
\[
k+1)^{k+k'-1}.
\]
The latter is larger since \( k < k' \).
The reason that the worst element also goes to the 2k'-line is that the worst component gets lost during the reduction, i.e., the system reliability does not depend on that component. So it must be optimal to put the worst component in that position.

Corollary 1. Consider \( S(k, k') \) with \( k < k' \). In an invariant assignment the \( k' - k \) best and worst components all go to the 2k'-line.

Proof: The proof is obvious.

Theorem 1. \( S(k, k+2) \) is not invariant.

Proof: By corollary 1, it suffices to identify a set \( P \) of 4k+4 reliabilities such that assigning the 3 worst components to the (k+2)-line is not optimal. Let
\[
P = \{0 < p_1[1] = p_2[n] = p_3 = p < p_4[1] = r < 1\}.
\]
\( A \) be the assignment that \( p_1[1], p_2[1], p_3[1] \) all go to the (k+2)-line;
\( A' \) be the assignment that \( p_1[1] \) and \( p_2[1], p_3[1] \) all go to the (k+2)-line.

By adding the probabilities that \( p_1, p_{i+1}, \ldots, p_{i+k-1} \)
is the first set of \( k \) consecutive working components over
\( i = 1, 2, \ldots, k+1 \), then
\[
R(A') - R(A) = r^{2k}. \left[\begin{array}{l}
[p + \bar{p} \cdot r + (k-1) \cdot \bar{r} \cdot r] \cdot p + [p + \bar{p} \cdot r + k \cdot \bar{r} \cdot r + \bar{r} \cdot p] \\
- [1 + k \cdot \bar{r}] \\
\cdot [p^2 + \bar{p} \cdot p \cdot r + \bar{p} \cdot r^2 + (k-1) \cdot \bar{r} \cdot r^2 + \bar{r} \cdot r \cdot p] \\
= r^{2k} \cdot p \cdot \bar{r} \cdot (r - p) \cdot [(k - 1) \cdot \bar{r} + r] > 0.
\end{array}\right.
\]

Corollary 2. \( S(k, k') \) is not invariant for \( k' \geq 2k + 2 \).

Proof: True for \( k' = 2k + 2 \). Prove the general \( k' \) case by induction. By the inductive hypothesis, there exist two sets \( P \) and \( P' \) of \( 2(k + k') \) reliabilities with different optimal assignments. Consider the two sets of \( 2(k + k') \) reliabilities \( P \cup \{0, 1\} \) and \( P' \cup \{0, 1\} \). By lemma 2, the two components with reliabilities 0 and 1 go to the 2k'-line. By lemma 1, the \( S(k, k'+1) \) problem is reduced to the \( S(k, k' - 1) \) problem on \( P \) and \( P' \), which does not have an invariant assignment.

2.2 General Case

Theorem 2. \( S(k_1, \ldots, k_m) \) is not invariant if \( k_m \geq k_1 + 2 \).

Proof: Suppose to the contrary that there exists an invariant assignment \( A \) for \( S(k_1, \ldots, k_m) \). Let
\( L_1 \) denote a \( k_1 \) line,
\( L_2 \) a \( k_m \) line in \( S \). Define:
\( n = 2\Sigma_{i=1}^m k_i \)
\( n' = 2(k_1 + k_m) \).
Then \( S(k_1, \ldots, k_m) \) has \( n \) components and \( L_1 \cup L_2 \) has \( n' \) components. Let
\( r = \{r_1 < r_2 < \ldots < r_{n'}\} \) denote the ranks of the components of \( L_1 \cup L_2 \) among the \( n \) components. Then \( r \) induces a set of ranks \( r' = \{1, \ldots, n'\} \) of components of \( L_1 \cup L_2 \) among themselves. Consequently, \( A \) induces an assignment \( A' \) of \( r' \) to \( L_1 \cup L_2 \). By corollary 2, there exists a set \( P' \) of \( n' \) reliabilities for which \( A' \) is not optimal, say, \( B' \) is optimal. Extend \( P' \) to \( P \) with \( n \) reliabilities by setting
\( p_i \) of \( P = p_i[1] \) of \( P' \)
and filling in \( p_{i+1} \) of \( R, r_i < j < r_{i+1} \), with arbitrary numbers as long as they are consistent with the ranking. Let \( B \) be an assignment of \( R \) to \( S(k_1, \ldots, k_m) \) which differs from \( A \) only by replacing \( A' \) with \( B' \) on the assignment of \( L_1 \cup L_2 \). Then clearly,
\[
R_B(S) > R_A(S),
\]
contradicting the assumption that \( A \) is invariant.

Finally, generalize \( S \) to the case that it can contain con(k/n : G) for \( k \leq n \leq 2k \). Let \( S(k_1/n_1, \ldots, k_m/n_m) \) denote such a system, where \( k_i/n_i \) is con(k/n : G). Ref [3] showed that for \( k \leq n \leq 2k \), con(k/n : G) works if the middle \( 2k - n \) components all work and the remaining \( 2(n - k) \) components constitute a working
\[
con((n - k)/2(n - k) : G).
\]
The \( \sum_{i=1}^{m} (2k_i - n_i) \) largest reliabilities should be assigned
to the middle positions, \( 2k_i - n_i \) of them to line \( i \). The
exact mapping of these reliabilities to the middle positions is irrelevant since the system fails if any one of them fails. Consider two assignments as equivalent if they differ only in positions which must work. Delete these middle positions; then \( S(k_1/n_1, \ldots, k_m/n_m) \) is reduced to \( S(n_1 - k_1, \ldots, n_m - k_m) \). Thus:

**Theorem 3.** \( S(k_1/n_1, \ldots, k_m/n_m) \) is invariant (up to equivalence) iff \( S(n_1 - k_1, \ldots, n_m - k_m) \) is invariant.

**Proof:** The proof is obvious.

2.3 Example

Assign 15 reliabilities, \( p_1 < p_2 < \cdots < p_{15} \), to \( S(1/1, 1/2, 2/3, 2/4, 3/5) \).

Then two equivalent invariant assignments are:

\[
\begin{align*}
p_{13} & \quad p_5 - p_8 & \quad p_6 - p_{15} - p_7 & \quad p_1 - p_9 - p_{12} - p_4 \\
p_{15} & \quad p_6 - p_7 & \quad p_5 - p_{14} - p_8 & \quad p_2 - p_{10} - p_{11} - p_3 \\
& \quad p_1 - p_9 - p_{13} - p_{12} - p_4.
\end{align*}
\]

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