which implies that quadratic stability of the unforced system of (1) because the last two terms in the inequality are positive-semi-definite. In order to establish the $H_2$ disturbance attenuation property, we assume $x(0) = 0$ and need to show that

$$J := \sum_{k=0}^{\infty} \left[ z^T(k)z(k) - \gamma^2 w^T(k)w(k) \right] < 0$$

whenever $w(k) \neq 0$. (25)

The existence of the sum in (25) is guaranteed by the boundedness of $w(k)$ and the quadratic stability of the unforced system of (1). It is obvious that (25) holds if $x(k) = 0$ for all $k \geq 0$. Therefore, we assume $x(k) \neq 0$ in the sequel.

Abbreviating $A + \Delta A(k)$ by $A_\Delta$ and defining

$$\Gamma = \left[ P^{-1} + \gamma^2 B_1^T \right]^{-1} > 0$$

it is straightforward to show by using the matrix inversion lemma that (26) and (22) imply

$$B_1^T \Gamma B_1 < \gamma^2 I$$

and

$$U(k) := A_\Delta^T \Gamma A_\Delta - \Gamma + \gamma^2 A_\Delta^T \Gamma B_1 [I - \gamma^2 B_1^T \Gamma B_1]^{-1} B_1^T \Gamma A_\Delta + C_\Delta^T C_1 < 0.$$ (28)

Using $x(0) = 0$, we have

$$_{N-1} \sum_{k=0}^{N} [ x^T(k+1) \Gamma x(k+1) - x^T(k) \Gamma x(k) ] = x^T(N + 1) \Gamma x(N + 1) \geq 0.$$ (26)

Let

$$J(k) := z^T(k)z(k) - \gamma^2 w^T(k)w(k)$$

$$+ x^T(k+1) \Gamma x(k+1) - x^T(k) \Gamma x(k)$$

$$J_N := \sum_{k=0}^{N} [ z^T(k)z(k) - \gamma^2 w^T(k)w(k) ].$$

Then, we have

$$J_N = \sum_{k=0}^{N} J(k) - x^T(N + 1) \Gamma x(N + 1).$$

Using (27) and (28), it can be verified that

$$J(k) = x^T(k)U(k)x(k) - \gamma V^T(k)[I - \gamma^2 B_1^T \Gamma B_1]V(k) \leq 0$$

where

$$V(k) := w(k) - \gamma^2 [I - \gamma^2 B_1^T \Gamma B_1]^{-1} B_1^T \Gamma A_\Delta x(k).$$

Since we assumed that $x(k) \neq 0$, we must have $J(k) < 0$ for some $k \geq 0$. Hence, $J_N < 0$ for sufficiently large $N$, which implies $J < 0$.

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Minimal Periodic Realizations of Transfer Matrices

Ching-An Lin and Chwan-Wen King

Abstract—Periodic controllers designed based on the so-called lifting technique are usually represented by transfer matrices. Real-time operations require that the controllers be implemented as periodic systems. We study the problem of realizing an $N_1 \times N_2$ proper rational transfer matrix as an $n_1$-input $n_2$-output $N$-periodic discrete-time system. We propose an algorithm to obtain periodic realizations which have a minimal number of states. The result can also be used to remove any redundant states that exist in a periodic system.

I. INTRODUCTION

It is reported in the literature that linear periodic controllers may be superior to the linear time-invariant ones for a large class of control problems [5], [2], [1], [3]. For discrete-time systems, Khargonekar, Poolla and Tannenbaum [5] proposed a framework for the design of linear periodic controllers for linear time-invariant plants. They show that to an $n_1$-input $n_2$-output linear $N$-periodic system there corresponds an $n_1$-input $n_2$-output linear time-invariant system and conversely to an $N_1$-input $N_2$-output linear time-invariant system there corresponds an $n_1$-input $n_2$-output linear $N$-periodic system. It is asserted [5] that from an input-output point of view, this correspondence is isomorphic in that both algebraic and analytic properties of systems are preserved and hence, the design of periodic controllers can be done using various LTI design techniques. However, the $n_1$-input $n_2$-output $N$-periodic controller so designed is required that the controller be realized as a periodic system. There are straightforward realizations of such a transfer matrix as an $N$-periodic system but usually with a large number of states. The result can also be used to remove any redundant states that exist in a periodic system.

REFERENCES


And it is not clear how to remove these extra states from the realization. In this note, we propose a method to obtain a minimal N-periodic realization of an $N_n \times N_n$ transfer matrix.

Consider the $n_i$-input $n_o$-output N-periodic linear causal discrete-time system described by

$$\begin{align*}
    x(k + 1) &= A_k x(k) + B_k u(k) \\
    y(k) &= C_k x(k) + D_k u(k)
\end{align*}$$

where $A_k \in \mathbb{R}^{n \times n}$, $B_k \in \mathbb{R}^{n \times n}$, $C_k \in \mathbb{R}^{n \times n}$, and $D_k \in \mathbb{R}^{n \times n}$ are N-periodic, i.e., $A_{k+N} = A_k$, $B_{k+N} = B_k$, $C_{k+N} = C_k$, and $D_{k+N} = D_k$ for all $k$. To the system (1.1), we associate an $Nn_i \times Nn_o$-input linear time-invariant system

$$\begin{align*}
    X(k + 1) &= \bar{A}X(k) + \bar{B}U(k) \\
    Y(k) &= \bar{C}X(k) + \bar{D}U(k)
\end{align*}$$

where $\bar{A} \in \mathbb{R}^{Nn_i \times Nn_i}$, $\bar{B} \in \mathbb{R}^{Nn_i \times Nn_o}$, $\bar{C} \in \mathbb{R}^{Nn_o \times Nn_o}$, and $\bar{D} \in \mathbb{R}^{Nn_o \times Nn_o}$ are given by

$$\bar{A} = A_{N-1} \cdots A_1 A_0,$$

$$\bar{B} = \begin{bmatrix} \bar{B}_0 & \bar{B}_1 & \cdots & \bar{B}_{N-1} \end{bmatrix}$$

with $\bar{B}_i = \begin{bmatrix} A_{i-1} & A_i & B_i \end{bmatrix}$, $i = 0, \ldots, N - 2$ \hspace{1cm} (1.3a)

$$\bar{C}' = \begin{bmatrix} \bar{C}_0' & \bar{C}_1' & \cdots & \bar{C}_{N-1}' \end{bmatrix}$$

with $\bar{C}_i = \begin{bmatrix} C_0 & C_i & A_i & \cdots & A_0 \end{bmatrix}$, $i = 0, \ldots, N - 1$ \hspace{1cm} (1.3b)

$$\bar{D} = \begin{bmatrix} \bar{D}_{0,0} & \cdots & \bar{D}_{0,N-1} \\
    \vdots & \ddots & \vdots \\
    \bar{D}_{N-1,0} & \cdots & \bar{D}_{N-1,N-1} \end{bmatrix}$$

with $\bar{D}_{i,j} = \begin{bmatrix} 0 & \bar{D}_{i,j} \\
    \vdots & \ddots \\
    \bar{D}_{i,j} & \cdots \end{bmatrix}$, $i < j$ \hspace{1cm} (1.3c)

$$\bar{D}_{i,j} = \begin{bmatrix} C_i & A_{i-1} & \cdots & A_j & B_j \end{bmatrix}$, $i > j$. \hspace{1cm} (1.3d)

The input-output relations of the systems (1.1) and (1.2) are related as follows. Assume that $x(0) = X(0) = 0$. Suppose $(u(k))_{k=0}^N$ and $(y(k))_{k=0}^N$ are respectively, the input and output of the system (1.1). If the input to the system (1.2) is, for each $k$

$$U(k) = \begin{bmatrix} u(Nk) \\
    u(Nk + 1) \\
    \vdots \\
    u(Nk + N - 1) \end{bmatrix},$$

then the corresponding output, for each $k$, is given by

$$Y(k) = \begin{bmatrix} y(Nk) \\
    y(Nk + 1) \\
    \vdots \\
    y(Nk + N - 1) \end{bmatrix}.$$  

Conversely, suppose $(U(k))_{k=0}^N$ and $(Y(k))_{k=0}^N$ are respectively, the input and output of the system (1.2). If the input to the system (1.1) is

$$u(Nk + j) = U(j),$$

for $j = 0, \ldots, N - 1$, $k = 0, 1, \ldots$, where $U(k) = \begin{bmatrix} U_0(k) \\
    \vdots \\
    U_{N-1}(k) \end{bmatrix}$, then the corresponding output is

$$y(Nk + j) = Y(j),$$

for $j = 0, \ldots, N - 1$, $k = 0, 1, \ldots$, where $Y(k) = \begin{bmatrix} Y_0(k) \\
    \vdots \\
    Y_{N-1}(k) \end{bmatrix}$.

The transfer matrix of the system (1.2) is $G(z) = \bar{C}(z\bar{A} - \bar{B})^{-1}\bar{D}$, where $\bar{R}(z)$ is the set of proper rational functions in $z$ with real coefficients. We say that the system (1.1), or equivalently $(A_k, B_k, C_k, D_k)$, is an $N$-periodic realization of the transfer matrix $G(z)$ if it is claimed in [5] that if $G(z) \in \mathbb{R}(z)^{Nn_o \times Nn_o}$ with $G(z) = A(z)B(z)$, then there exists a causal $N$-periodic realization for $G(z)$.

The problem we study is the following: Given $G(z) \in \mathbb{R}(z)^{Nn_o \times Nn_o}$, what is the minimal number of states required for an $N$-periodic realization of $G(z)$ and how to obtain a minimal $N$-periodic realization?

II. Minimal Periodic Realizations

Suppose $G(z) \in \mathbb{R}(z)^{Nn_o \times Nn_o}$ is given with $G(z) = A(z)B(z)$ a minimal realization of $G(z)$ with $A(z) \in \mathbb{R}^{n \times n}$. Consider (1.3) with $A(z), B(z), C(z), D(z)$ given. It is easy to see that if $(A_k, B_k, C_k, D_k)$ is a solution of (1.3), then (1.1) is a minimal $N$-periodic realization of $G(z)$. On the other hand, if (1.3) has no solution, then there is no $N$-periodic realization of $G(z)$ of dimension $n$, the McMillan degree of $G(z)$. In other words, any $N$-periodic realization of $G(z)$ must be of dimension larger than $n$. In this section, we give a procedure to obtain a minimal $N$-periodic realization of the transfer matrix $G(z)$. We start by deriving a necessary and sufficient condition under which $G(z)$ has an $N$-periodic realization with its dimension equal to the McMillan degree of $G(z)$. The condition is then used to determine the minimal number of states that are required for an $N$-periodic realization of $G(z)$.

Now since $G(z) = A(z)B(z)$, $G(z)$ is $n_o \times n_i$-block lower triangular, $A(z)$ is also $n_o \times n_i$-block lower triangular. From (1.3), we have

$$B_{N-1} = \bar{B}_{N-1}, C_0 = \bar{C}_0, \text{ and } D_0 = \bar{D}_{0,0}.$$  

With (2.1), (1.3) reduces to

$$\begin{align*}
    A_{N-1}A_{N-2} \cdots A_1 A_0 &= \bar{A} \hspace{1cm} (2.2a) \\
    A_{N-1}A_{N-2} \cdots A_1 B_0 &= \bar{B}_0, \hspace{1cm} \text{for } i = 0, \ldots, N - 2 \hspace{1cm} (2.2b) \\
    C_i A_{i-1} \cdots A_1 A_0 &= \bar{C}_i, \hspace{1cm} \text{for } i = 1, \ldots, N - 1 \hspace{1cm} (2.2c) \\
    C_i A_{i-1} \cdots A_1 B_0 &= \bar{D}_{i,j}, \hspace{1cm} \text{for } i = 1, \ldots, N - 1, j = 0, \ldots, i - 1 \hspace{1cm} (2.2d)
\end{align*}$$

where $A_i \in \mathbb{R}^{n \times n}$, $B_j \in \mathbb{R}^{n \times n}$, and $C_i \in \mathbb{R}^{n \times n}$ are to be solved. Thus $G(z)$ has an $N$-periodic realization of dimension $n$ if and only if (2.2) has a solution. The following lemma is needed in the proof of the main result.

Lemma 2.1 [7, page 25]: Consider matrix equations

$$\begin{align*}
    UX &= \bar{U} \hspace{1cm} (2.3a) \\
    XV &= \bar{V} \hspace{1cm} (2.3b)
\end{align*}$$

where $U \in \mathbb{R}^{m \times n}, \bar{U} \in \mathbb{R}^{n \times n}, V \in \mathbb{R}^{n \times m}$, and $\bar{V} \in \mathbb{R}^{n \times n}$. If
where $\mathcal{R}(A)$ is the range space of $A$.

The following theorem is a necessary and sufficient condition for $G(z)$ to have an $N$-periodic realization of dimension $n$, the McMillan degree of $G(z)$.

**Theorem 2.2:** Equation (2.2) has a solution $(A_1, \ldots, A_N, B_1, \ldots, B_N, C_1, \ldots, C_N)$ if and only if for $i = 0, \ldots, N - 2$

$$\rho(K_i) \leq n$$

(2.5)

where $\rho(K_i)$ denotes the rank of $K_i$ and

$$K_i = \begin{bmatrix} \bar{A} & \bar{B}_0 & \cdots & \bar{B}_i \\ \bar{C}_{i-1} & \bar{D}_{i-1,0} & \cdots & \bar{D}_{i-1,i} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{C}_i & \bar{D}_{i,0} & \cdots & \bar{D}_{i,i} \end{bmatrix}.$$

(2.6)

**Comments:** i) Thus, to check if $G(z)$ has $N$-periodic realization of dimension $n$, one only has to check the $N - 1$ rank conditions in (2.5). ii) It is easy to check that condition (2.5) does not depend on any particular LTI minimal realization chosen for $G(z)$. iii) Condition (2.5) is very strong when $A$ is nonsingular. The following corollary follows from the theorem and simple elementary operations [4, page 650].

**Corollary 2.3:** If $A$ is nonsingular, then (2.2) has a solution if and only if

$$\begin{align*}
\bar{D}_{ij} &= \bar{C}_i(A)^{-1} \bar{B}_j, & \text{for all } i = 0, \ldots, N - 1, j = 0, \ldots, i - 1.
\end{align*}$$

(2.7)

**Proof of Theorem 2.2:** $(\Rightarrow)$ Let $K_i$ be as defined in (2.6). Suppose $(A_1, \ldots, A_N, B_1, \ldots, B_N, C_1, \ldots, C_N)$ satisfy (2.2), then we must have for $i = 0, 1, \ldots, N - 2$

$$K_i = \begin{bmatrix} A_{i-1} & \cdots & A_{i+1} \\ C_{i-1} & \cdots & C_{i+1} \\ \vdots & \vdots & \vdots \\ A_i & \cdots & A_iB_0 & \cdots & A_iB_{i-1} & B_i \end{bmatrix}.$$

(2.8)

Since $A_i \in \mathbb{R}^{n \times n}$, it follows from (2.8) that $\rho(K_i) \leq n$, for $i = 0, 1, \ldots, N - 2$.

$(\Leftarrow)$ Suppose (2.5) holds. For each $i$, there exist $U_i \in \mathbb{R}^{(n + (N - i - 1)n) \times n}$ and $V_i \in \mathbb{R}^{n \times (n + (i + 1)n)}$ with $\mathcal{R}(U_i) = \mathcal{R}(V_i) = \rho(K_i)$ such that

$$K_i = U_iV_i.$$

(2.9)

Let us partition $U_i$ and $V_i$ as

$$U_i = \begin{bmatrix} U_{i,0} \\ U_{i,1} \\ \vdots \\ U_{i,N-i-1} \end{bmatrix}, \quad V_i = \begin{bmatrix} V_{i,0} & V_{i,1} & \cdots & V_{i,i+1} \end{bmatrix}.$$

(2.10)

where $U_{i,0}, V_{i,0} \in \mathbb{R}^{n \times n}$, $U_{i,j} \in \mathbb{R}^{n \times n}$, $j = 1, \ldots, N - i - 1$ and $V_{i,j} \in \mathbb{R}^{n \times n}$, $j = 1, \ldots, i + 1$. From (2.6), we have, for $i = 0, \ldots, N - 2$,

$$U_{i,0}V_{i,0} = \bar{A}_i,$$

$$U_{i,N-i-1} = \bar{B}_i,$$

$$U_{i,N-i-j} = \bar{D}_{i,j},$$

(2.11)

(2.12)

(2.13)

(2.14)

For $i = 0, 1, \ldots, N - 2$, let

$$\bar{U}_i = \begin{bmatrix} U_{i,0} \\ \vdots \\ U_{i,N-i-2} \end{bmatrix} \quad \text{and} \quad \bar{V}_i = \begin{bmatrix} V_{i,0} & \cdots & V_{i,i} \end{bmatrix}.$$

(2.15)

Note that $\bar{U}_i$ and $\bar{V}_i$ are obtained respectively by deleting the last $n_1$ rows from $U_i$ and the last $n_i$ columns from $V_i$. It follows from (2.11)-(2.15) that for $i = 1, \ldots, N - 2$

$$U_i\bar{V}_i = \bar{U}_{i-1}\bar{V}_{i-1}.$$

(2.16)

(2.17)

(2.18)

From (2.9) and that $\rho(U_i) = \rho(K_i)$, we know that

$$\mathcal{R}(U_i) = \mathcal{R}(K_i).$$

(2.19)

Let us partition $U_{i-1}$ as

$$U_{i-1} = \begin{bmatrix} U_{i-1,0} \\ U_{i-1,1} \end{bmatrix}.$$

(2.20)

From (2.20), (2.18) and that $\mathcal{R}(U_{i-1}) = \mathcal{R}(K_{i-1})$, we have

$$\mathcal{R}(U_{i-1}) = \mathcal{R}(W_i).$$

(2.21)

From (2.19), (2.21), and (2.17), it follows that

$$\mathcal{R}(U_{i-1}) \subseteq \mathcal{R}(U_i).$$

(2.22)

By similar arguments, it can be shown that

$$\mathcal{R}(V_{i-1}) \subseteq \mathcal{R}(V_i).$$

(2.23)

From (2.16), (2.22), (2.23), and Lemma 2.1, there exist $A_i$ such that, for $i = 1, \ldots, N - 2$

$$\begin{bmatrix} U_iA_i = U_{i-1} \\ A_iV_{i-1} = V_i \end{bmatrix}$$

(2.24)

or equivalently

$$U_{i,j}A_i = U_{i-1,j}, \quad \text{for } j = 0, \ldots, N - i - 1,$$

(2.25)

$$A_jV_{i-1,j} = V_{i,j}, \quad \text{for } j = 0, \ldots, i.$$
We now show that the $A_i$'s, $B_i$'s, and $C_i$'s defined in (2.24), (2.27)−(2.29) satisfy (2.2). From (2.29), (2.25), and (2.26), we get
\[
A_{N-1}A_{N-2} \cdots A_1A_0 = U_{N-1}U_{N-2} \cdots U_1U_0
\]
where $i$ can be any integer between 0 and $N - 2$. It follows from (2.31) and (2.30) that these $A_i |^{N-2}_0$ satisfy (2.2a).

The verification that the $(A_i |^{N-2}_0)$, $(B_i |^{N-2}_0)$, and $(C_i |^{N-2}_0)$ defined satisfy (2.2b), (2.2c), and (2.2d) is similar and straightforward and hence is omitted. □

If condition (2.5) is not satisfied, that is, $\rho(K_i) > n$ for some $i$, then any $N$-periodic realization of $G(z)$ must be of dimension larger than $n$. To find the minimum number of states that are required for an $N$-periodic realization, we note that the system
\[
\begin{align*}
\tilde{A} &= \begin{bmatrix} O_1 & O_2 & \cdots & O_n \\ O_1^T & A \end{bmatrix}, \\
\tilde{B} &= \begin{bmatrix} O_1^T \\ B \end{bmatrix}, \\
\tilde{C} &= \begin{bmatrix} O_1 & \tilde{C} \end{bmatrix}, \\
\tilde{D} &= \tilde{D}
\end{align*}
\]
(2.31)
where $O_1$ is a $\Delta n \times \Delta n$ zero matrix, $O_2, O_3, \ldots, O_n$ are zero matrices with compatible dimensions, has the same transfer matrix $G(z)$ as the system $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ and the matrix $\tilde{K}_i$, similarly defined through (2.6) with $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ replacing $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$, has the same rank as $K_i$ does. Note that (2.31) amounts to adding $\Delta n$ unobservable hidden modes at $z = 0$ to the original realization $(A, B, C, D)$ of $G(z)$.

Suppose $\Delta n$ is the smallest positive integer such that
\[
n + \Delta n \geq \rho(\tilde{K}_i) = \rho(K_i), \quad i = 0, \ldots, N - 2.
\]
Then any minimal $N$-periodic realization of $G(z)$ must be of dimension equal to $n + \Delta n$. Since it is impossible to obtain an $N$-periodic realization of $G(z)$ with dimension less than $n$, the McMillan degree of $G(z)$, we have that the dimension of the minimal $N$-periodic realization of $G(z)$ is
\[
n = \max \{ \max_{0 \leq i < N - 2} \rho(K_i), n \}.
\]
(2.32)

Based on Theorem 2.2 and the analysis above, we propose the following algorithm for obtaining a minimal $N$-periodic realization of a given transfer matrix $G(z) \in \mathbb{R}(z)^{N_m \times N_m}$.

**Algorithm 2.4:**

**Data:** $G(z) \in \mathbb{R}(z)^{N_m \times N_m}$, with $G(z)$ an $n \times n$ block lower triangular.

**Step 1:** Find a minimal realization $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ of $G(z)$, where $\tilde{A} \in \mathbb{R}^{n \times n}$.

**Step 2:** For $i = 0, \ldots, N - 2$, form the matrix $K_i$ as defined in (2.6).

**Step 3:** Compute $\tilde{n}$ as defined in (2.32).

**Step 4:** If $\tilde{n} \geq n$, then go to Step 5; else, form the corresponding $\tilde{K}_i$ for $i = 0, \ldots, N - 2$.

**Step 5:** Obtain $B_{N-1}, C_0$, and $(D_1 |^{N-2}_1)$ from (2.1).

**Step 6:** For $i = 0, \ldots, N - 2$, decompose $K_i$ into $U_i, V_i$ with $\rho(U_i) = \rho(V_i) = \rho(K_i)$ as in (2.9).

**Step 7:** Obtain $A_{N-1}, A_0, (B_0 |^{N-2}_0)$, and $(C_0 |^{N-2}_0)$ defined in (2.27)−(2.29).

**Step 8:** For $i = 1, \ldots, N - 2$, form $\tilde{U}_{i-1}$ and $\tilde{V}_{i-1}$ as in (2.15) and solve (2.24) for $A_i$.

We give an example to illustrate the proposed algorithm.

**Example:** Let
\[
G(z) = \frac{1}{z-1} \begin{bmatrix} z + 2 & 4 \\ 6z & 3z + 5 \\ 2z + 11 \\ z + 2 \end{bmatrix}
\]
We wish to obtain a 3-periodic minimal realization for $G(z)$. Compute a minimal LTI realization as
\[
\tilde{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \tilde{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\]
(2.33)

Based on (2.33), form the corresponding $K_0$, $K_1$, $K_2$. By Step 5, $D_0 = 1$, $D_1 = 3$, $D_2 = 1$, $B_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $C_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. By Step 6, decompose $K_0, K_1$ as
\[
K_0 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad K_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.
\]
(2.34)

By Step 7,
\[
A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\]
(2.35)

By Step 8, obtain $A_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$. The minimal 3-periodic realization is then given by $(A_k, B_k, C_k, D_k) |^{N-2}_0$.

**Remarks:** i) Formula (2.31) is not the only way to expand the dimension of $A$ such that Theorem 2.2 can still be used for realization, however, it is the most straightforward. In this fashion, the hidden modes introduced are both uncontrollable and unobservable. One can also add uncontrollable or unobservable hidden modes; ii) The result we obtained so far can also be used to determine whether a given periodic system contains redundant states. More precisely, suppose we are given an $N$-periodic system $(A_k, B_k, C_k, D_k)$, we can use (1.3) to obtain the corresponding $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ and the transfer matrix $G(z) = \tilde{C}(zI - \tilde{A})^{-1}\tilde{B} + \tilde{D}$. Following Algorithm 2.4, we can decide the minimal number of states required for the periodic system and obtain a new $N$-periodic realization which is minimal. Note that the new $N$-periodic realization has exactly the same input–output relation as the original one.

**III. CONCLUSION**

In this note, we develop a procedure to obtain a minimal $N$-periodic realization of a given transfer matrix $G(z) \in \mathbb{R}(z)^{N_m \times N_m}$. The result is useful in implementing periodic controllers designed by the so-called lifting technique; it can also be used to remove redundant states in a given periodic system.
Singular Perturbation of the Time-Optimal Soft-Constrained Cheap-Control Problem

M. U. Bikdash, A. H. Nayfeh, and E. M. Cliff

Abstract—We consider the solution of a time-optimal soft-constrained control problem with linear dynamics. The cost function has no penalty on the integral of the state. The solution is formulated in terms of the controllability Grammian and is obtained as the solution of a system of linear algebraic equations and a nonlinear scalar algebraic equation. As the state approaches the origin, or equivalently, as the control becomes cheap, the optimal final time becomes small. This introduces a highly degenerate hierarchy of amplitude scales. A new approach, solely based on traversing the controllability Grammian, is developed to obtain an asymptotic solution of the problem without resorting to boundary-layer theory.

I. INTRODUCTION

We are concerned with the class of time-optimal soft-constrained control problems that minimize the cost function

\[ J = t_f + \frac{1}{2} \int_0^{t_f} \left( x^T(t)Qx(t) + u^T(t)Ru(t) \right) dt \]  \hspace{1cm} (1)

subject to

\[ \dot{x}(t) = Ax(t) + Bu(t) \]  \hspace{1cm} (2)

\[ x(0) \text{ specified}, \quad \text{and} \quad x(t_f) = 0. \]  \hspace{1cm} (3)

The conventional “cheap”-control problem \[ J = t_f \] is obtained by letting \( Q = 0 \) with the final time \( t_f \) being held fixed, in which case the control action becomes increasingly cheap. In the case where \( t_f \to \infty \), the cheap-control linear quadratic regulator (LQR) problem yields a high-gain linear state feedback control policy with many desirable properties, such as disturbance rejection, insensitivity to parameter errors and distortions, improved tracking, and minimum steady-state error [2]. The case of finite final time is considered in [3] and [4].

A typical solution of the cheap-control problem proceeds by transforming the problem into a singularly perturbed system [2]. The analysis of such a system is well understood if it is written in the so-called standard form [6], [7]. The method of matched asymptotic expansions can then be used to determine an approximate solution consisting of an outer expansion and a boundary-layer correction (inner expansion) [6], [8], [9]. Impulsive controls are needed to get onto and away from the singular arcs [10], [11]. If, an addition, the control is constrained by inequalities, then both bang-bang and singular arcs exist [3].

We consider the time-optimal soft-constrained control problem. Moreover, we take \( Q = 0 \). This case is not included in the classification introduced in [4]. The solution of this case is derived in [12] in terms of the controllability Grammian matrix [13]. It consists of the solution of a system of linear algebraic equations parametrized on the final time, coupled with a nonlinear scalar algebraic equation that yields the optimal final time. When the control becomes increasingly cheap, a very fast maneuver is expected, and the optimal final time becomes small even for large initial conditions. In this note, we obtain an asymptotically valid solution for the soft-constrained time-optimal cheap-control problem without resorting to the method of matched asymptotic expansions.

II. PROBLEM FORMULATION

The open-loop, soft-constrained, time-optimal control problem is to find a measurable function \( u(t) \) that drives the state \( x(0) \) into the target point \( x(t_f) = 0 \) while minimizing (1) with \( Q = 0 \). The solution was described in [12] in terms of the controllability Grammian matrix

\[ W(0, t_f) = \int_0^{t_f} v(\tau)v^T(\tau) d\tau, \quad \text{where} \quad v(t) = e^{-At}b. \]  \hspace{1cm} (4)

The costate is given by \( \lambda(t) = e^{-At}\mu \), where \( \mu = \lambda(0) \) is the initial costate and

\[ W(0, t_f)\mu = \xi \]  \hspace{1cm} (5)

where \( \xi = e^x(0) \). The control function can be written as \( u(t) = -e^{-At}(0)v(t) \). For \( t = 0 \), we have \( u(0) = -e^x(0)\mu(t_f) \), where \( \lambda(t_f) = [W(0, t_f)]^{-1}b \). Furthermore, the cost function is given by

\[ J = t_f + \frac{1}{2} e^{-x(0)}[W(0, t_f)]^{-1}e(0). \]  \hspace{1cm} (6)

The optimal final time has to satisfy the transversality condition

\[ 2 = [z^T(t_f)e(0)]^2, \quad \text{where} \quad z(t_f) = [W(0, t_f)]^{-1}e(t_f). \]  \hspace{1cm} (7)

We note that the mapping \( t_f \to W(0, t_f) \) and \( t_f \to e(t_f) \) depend only on the problem data (specifically \( A \) and \( b \)). Then, with fixed initial data \( x(0) \), (7) is a scalar equation in \( t_f \). The roots of (7) specify all the candidates for the minimum \( t_f \). If many roots \( t_f, t_{f2}, \ldots \), exist, the corresponding costs can be computed via (6). Direct comparison of the costs can then be used to find the

REFERENCES


