KÄHLER STRUCTURES AND WEIGHTED ACTIONS ON THE COMPLEX TORUS

MENG-KIAT CHUAH

ABSTRACT

Let $T$ be the compact real torus, and $T_C$ its complexification. Fix an integral weight $\alpha$, and consider the $\alpha$-weighted $T$-action on $T_C$. If $\omega$ is a $T$-invariant Kähler form on $T_C$, it corresponds to a pre-quantum line bundle $L$ over $T_C$. Let $H^1_L$ be the square-integrable holomorphic sections of $L$. The weighted $T$-action lifts to a unitary $T$-representation on the Hilbert space $H^1_L$, and the multiplicity of its irreducible sub-representations is considered. It is shown that this is controlled by the image of the moment map, as well as the principle that ‘quantization commutes with reduction’.

1. Introduction

Let $T$ be the compact real $n$-torus, and $T_C$ its complexification. Then $T$ acts naturally on $T_C$, as subgroup of $T_C$. In [3], we study $T$-invariant Kähler structures on $T_C$, and the corresponding geometric quantization. The present paper follows a suggestion of V. Guillemin, and considers the more general $T$-actions with weights.

We write $T = \mathbb{R}^n/\mathbb{Z}^n$ and $T_C = \mathbb{C}^n/\mathbb{Z}^n$ as in [3], where

$$T_C = \{z = x + \sqrt{-1}y] : x \in \mathbb{R}^n, [y] \in \mathbb{R}^n/\mathbb{Z}^n = T\}.$$  \hspace{1cm} (1.1)

Let $\mathfrak{t}$ be the Lie algebra of $T$. The notation (1.1) automatically identifies $\mathfrak{t}, \mathfrak{t}^*, \mathbb{R}^n, \mathbb{R}^n$ with one another.

Consider now a weight $\alpha = (\alpha_1, \ldots, \alpha_n)$ in the integral lattice $\mathbb{Z}^n \subset \mathbb{R}^n = \mathfrak{t}^*$. We define the $\alpha$-weighted $T$-action on $T_C$ by

$$T \times T_C \rightarrow T_C, \quad ([t]) \times (x_j + \sqrt{-1}y_j) \mapsto (x_j + \sqrt{-1}y_j + \alpha_j t_j),$$  \hspace{1cm} (1.2)

where $t_j, x_j, y_j \in \mathbb{R}$ for all $j = 1, \ldots, n$. In particular, if $\alpha_j = 1$ for all $j$, then (1.2) is just the standard action of subgroup $T$ on $T_C$. We shall always deal with Kähler structures on $T_C$ that are invariant under this standard action, and we call them $T$-invariant. Let $D_{\alpha}$ be the diagonal matrix with entries $\alpha_1, \ldots, \alpha_n$ along the diagonal. We shall see that a $T$-invariant Kähler form is necessarily invariant under the weighted action (1.2), and has the expression $\omega = \sqrt{-1}D_\alpha \partial \bar{\partial} F$. In fact, the weighted action preserving $\omega$ is Hamiltonian, with moment map $\Phi: T_C \rightarrow \mathfrak{t}^*$ given by

$$\Phi(z) = \frac{1}{2} D_{\alpha} F'(z) = \frac{1}{2} \left( \alpha_j \frac{\partial F}{\partial z_j} (z) \right)$$

for all $z = x + \sqrt{-1}y] \in T_C$.

Since $\omega = \sqrt{-1}D_\alpha \partial \bar{\partial} F$, it has to be exact, and is in particular integral. We obtain a pre-quantum line bundle $L$ over $T_C$ [5, 6]. The Chern class of $L$ is the cohomology class $[\omega] = 0$, so $L$ is a trivial bundle. It is equipped with a connection $\nabla$ whose

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curvature is $\omega$, as well as an invariant Hermitian structure $(,)$. We say that a smooth section $s$ of $L$ is holomorphic if $\nabla v, s = 0$ for every anti-holomorphic vector field $v$. Let $H(L)$ denote the space of all holomorphic sections. The weighted action (1.2) leads to a $T$-representation on $H(L)$. Let $dV$ be the Haar measure on $T_C$. To obtain a unitary representation out of $H(L)$, let $H_\omega$ be the space of all holomorphic sections $s$ that satisfy

$$\int_{T_e} (s, s) dV < \infty.$$  \hspace{1cm} (1.3)

Then $H_\omega$ is a unitary $T$-representation space. Its infinitesimal $t$-representation is written as $\xi \cdot s \in H_\omega$ for $\xi \in t$, $s \in H_\omega$. The irreducible subrepresentations of $H_\omega$ are 1-dimensional, and each is a subspace of

$$(H_\omega)_\lambda = \{ s \in H_\omega : \xi \cdot s = (\lambda, \xi) s \text{ for all } \xi \in t\},$$

for some $\lambda \in \mathbb{Z}^* \subset t^*$. A basic question in geometric quantization is to compute the multiplicity of irreducible representations in $H_\omega$.

Let $\Omega$ be the image of the moment map. If the weight $\alpha$ in (1.2) contains no zero entry, then the multiplicity problem is solved by an easy generalization of [3], as follows.

**Theorem 1.1.** If $\alpha$ has no zero entry, then the unitary representation $H_\omega$ is multiplicity-free. It contains $(H_\omega)_\lambda$ if and only if $\lambda \in \Omega$ and $\lambda_j/\alpha_j \in \mathbb{Z}$ for all $j$.

We shall prove Theorem 1.1 in §2. Given a unitary representation of a Lie group, we call it a model if it contains every irreducible representation once. This terminology is due to I. M. Gelfand and A. Zelevinski [4]. From Theorem 1.1, Corollary 1.1 follows.

**Corollary 1.1.** $H_\omega$ is a model of $T$ if and only if the moment map is surjective and $\alpha_j = \pm 1$ for all $j$.

The main purpose of this paper is to consider the more complicated situation where the weight $\alpha$ in (1.2) contains zero entries. In this case, the multiplicity of $(H_\omega)_\lambda$ is no longer determined by the image of the moment map alone. To handle this problem, we introduce symplectic reduction, a process first explored by J. Marsden and A. Weinstein [7]. In the study of Hamiltonian group actions on symplectic manifolds, two of the central aspects are geometric quantization and symplectic reduction. A unifying theme between them is given by V. Guillemin and S. Sternberg [5] and is often called ‘quantization commutes with reduction’. A summary of recent developments of such concepts can be found in [8]. We shall see that it helps to solve our multiplicity problem.

We now perform symplectic reduction. Suppose that $\lambda$ is in the image $\Omega$ of the moment map $\Phi$. Then $T$ acts on $\Phi^{-1}(\lambda)$, and we call $B_\lambda = \Phi^{-1}(\lambda)/T$ the reduced space. Let

$$\iota: \Phi^{-1}(\lambda) \hookrightarrow T_C, \quad \pi: \Phi^{-1}(\lambda) \longrightarrow B_\lambda$$

respectively denote the natural inclusion and quotient. Then $B_\lambda$ is equipped with a symplectic structure $\omega_\lambda$, such that $\pi^*\omega_\lambda = \iota^*\omega$. This process is called symplectic reduction, and $\omega_\lambda$ is called the reduced symplectic form. We study certain properties of the reduced space $(B_\lambda, \omega_\lambda)$ in §3.
Let $k$ be the number of non-zero entries of the weight $\omega$, where $1 \leq k \leq n$. We may rearrange the indices and assume that $\omega = (\omega_1, \ldots, \omega_k, 0, \ldots, 0)$, where $\omega_1, \ldots, \omega_k$ are non-zero. Recall that $\omega = \sqrt{-1} \partial \bar{\partial} F$. Since $F$ can be regarded as a strictly convex function on $R^n$ (see [3]), the subset $X \subset R^n$ defined by

$$X = \{ x \in R^n : \frac{1}{2} \frac{\partial F}{\partial x_j} (x) = \frac{\lambda_j}{x_j} \text{ for } j = 1, \ldots, k \}$$

is a smooth submanifold of dimension $(n-k)$. Let $T^{n-k} \subset T$ be the real subtorus, spanned by the last $(n-k)$ coordinates. Then $X \times T^{n-k}$ imbeds into $T_c$, via

$$j : X \times T^{n-k} \longrightarrow R^n \times T = T_c.$$  

In §3, we prove the following theorem.

**Theorem 1.2.** The reduced space $(B_\lambda, \omega_\lambda)$ is symplectomorphic to the symplectic submanifold $(X \times T^{n-k}, j^* \omega)$.

By Theorem 1.2, we identify the reduced space $B_\lambda$ with the symplectic submanifold $X \times T^{n-k}$. Since $\omega_\lambda$ is exact, we again have the pre-quantum line bundle over $B_\lambda$, denoted $L_\lambda$. Here $L_\lambda$ is trivial, because its Chern class is the cohomology class $[\omega_\lambda] = 0$. It may be regarded as the restriction of $L$ to $B_\lambda$, due to Theorem 1.2. The space $B_\lambda \subset T_c$ is not complex, so there is no intrinsic polarization for an immediate definition of 'holomorphic' sections $H(L_\lambda)$. We shall define $H(L_\lambda)$ among the smooth sections of $L_\lambda$ in §4. This coincides with the usual holomorphic sections in cases where $B_\lambda$ happens to be complex.

The Haar measure of $T_c$ restricts to a measure on $B_\lambda = X \times T^{n-k}$, still denoted by $dV$. We again use the Hermitian structure on $L_\lambda$ to define an $L^2$-structure on $H(L_\lambda)$, and let $H_{\omega_\lambda}$ denote the square-integrable sections in $H(L_\lambda)$. In other words, $H_{\omega_\lambda}$ consists of all $s \in H(L_\lambda)$ in which

$$\int_{B_\lambda} (s, s) dV < \infty.$$

In §4, we prove that geometric quantization commutes with reduction, as stated in Theorem 1.3.

**Theorem 1.3.** $H_{\omega_\lambda}$ is a Hilbert space, and $(H_{\omega_\lambda}, \cdot) \cong H_{\omega_\lambda'}$.

Clearly, Theorem 1.1 is a special case of Theorem 1.3: if the weight $\omega$ has no zero entry, then for all $\lambda \in \Omega$, the reduced space $B_\lambda$ is just a point. Therefore $H_{\omega_\lambda} \subset C$, and Theorem 1.3 implies that $(H_{\omega_\lambda}, \cdot)$ occurs with multiplicity 1.

If the Hilbert spaces in Theorem 1.3 are infinite-dimensional, then of course the isomorphism in question is trivial. In §5, we justify the significance of this theorem by showing that their dimensions can be any of $0, 1, 2, \ldots, \infty$.

**2. Geometric quantization**

Let $\omega$ be a $T$-invariant Kähler form on the complex torus $T_c$. By [3],

$$\omega = d\beta = \sqrt{-1} \partial \bar{\partial} F,$$  

where $\beta$ and $F$ are $T$-invariant. We use the standard coordinates $z = x + \sqrt{-1} y$
introduced in equation (1.1). Here \( F \), being \( T \)-invariant, depends on the \( x \)-variables only. Since the weighted action defined in (1.2) acts along the \( [y] \)-variables, it preserves \( F \). Therefore, the weighted action also preserves \( \omega \).

Given \( \xi \in t = \mathbb{R}^n \), let \( \xi^x \) be the infinitesimal vector field on \( T_C \) induced by the weighted \( T \)-action. Hence

\[
\xi^x = \sum_j \partial_j \xi_j \frac{\partial}{\partial y_j}.
\]

In (2.1), \( \beta = \frac{1}{2} \sum_j \frac{\partial F}{\partial y_j} dy_j \). Hence the moment map \( \Phi: T_C \longrightarrow t^* \) of the weighted action is given by

\[
(\Phi(z), \xi) = (\beta, \xi^x) = \left( \frac{1}{2} \sum_j \frac{\partial F}{\partial x_j} (x), \sum_k \partial_k \xi_k \frac{\partial}{\partial y_k} \right)
= \sum_j \frac{1}{2} \partial_j \xi_j \frac{\partial F}{\partial x_j} (x),
\]

for all \( z \in T_C \) and \( \xi \in t \) (see [1, Theorem 4.2.10]). Therefore, the moment map is

\[
\Phi(z) = \frac{1}{2} \left( \partial_j \xi_j \frac{\partial F}{\partial x_j} (x) \right).
\]

As discussed in §1, \( \omega \) corresponds to a pre-quantum line bundle \( L \) [5, 6], whose holomorphic sections are denoted by \( H(L) \). By [3], there exists a non-vanishing \( T \)-invariant holomorphic section \( s_0 \) satisfying

\[
(s_0, s_0) = e^{-F}.
\]  

(2.2)

For the rest of this section, we assume that in the weighted \( T \)-action (1.2), \( \alpha \) has no zero entry. The weighted action lifts to a \( T \)-representation on \( H(L) \). Each irreducible subrepresentation is one-dimensional, and is of the form \( C(e^{\xi^x} s_0) \) for some \( c \in \mathbb{Z}^n \). Since \( s_0 \) is \( T \)-invariant, the corresponding infinitesimal t-representation is given by

\[
\xi \cdot (e^{\xi^x} s_0) = (\xi \cdot e^{\xi^x}) s_0
= \frac{d}{dt} \bigg|_{t=0} \exp \left( \sum_j c_j (x_j + \sqrt{-1} y_j + \xi_j t) \right) s_0
= \sum_j c_j \partial_j \xi_j e^{\xi^x} s_0,
\]

(2.3)

for all \( \xi \in t \). Therefore, if we define

\[ H(L)_x = \{ s \in H(L) : \xi \cdot s = (\lambda, \xi) s \text{ for all } \xi \in t \}, \]

then (2.3) says that

\[
H(L)_x = C(e^{\xi^x} s_0) \iff \lambda_j = c_j \alpha_j; j = 1, \ldots, n.
\]

(2.4)

We consider the unitary \( T \)-representation \( H_\omega \) consisting of holomorphic sections which converge under the integral (1.3). Let

\[
(H_\omega)_x = H(L)_x \cap H_\omega.
\]

We want to consider the multiplicity of \((H_\omega)_x\) in \( H_\omega \) and prove Theorem 1.1.
Proof of Theorem 1.1. By (2.4), we only need to consider the cases with \( \lambda_j/\alpha_j \in \mathbb{Z} \) for all \( j \). For such cases, let \( c_j = \lambda_j/\alpha_j \in \mathbb{Z} \). Consider \( s = e^{\lambda_j}s_0 \in H(\mathcal{L})_c \), where \( s_0 \) is the holomorphic section in (2.2). Define \( G \in C^\infty(\mathbb{R}^n) \) by \( G(x) = F(x) - 2c \cdot x \). Then
\[
\int_{\mathcal{T}_c} (e^{\lambda_j}s_0, e^{\lambda_j}s_0) \, dV = \int_{\mathcal{T}_c} e^{2\lambda_j}e^{-\lambda_j} \, dV = \int_{\mathcal{T}_c} e^{-\lambda_j} \, dV. \tag{2.5}
\]
Since \( F \) and \( G \) have the same Hessian and \( F \) is strictly convex, so is \( G \). According to Proposition 3.3 of [3], the integral (2.5) converges if and only if \( G \) has a global minimum. This is equivalent to \( 2c \) being contained in the image of the gradient function \( F' \). Recall that \( D_z \) is the diagonal matrix with entries \( \alpha_1, \ldots, \alpha_n \), so that the moment map is \( \Phi = \frac{1}{2}D_z \cdot F' \). It follows that
\[
e^{\lambda_j}s_0 \in H_{\alpha_j} \iff 0 \in \text{Image}(G') 
\iff 2c \in \text{Image}(F')
\iff c \in \text{Image}(\frac{1}{2}F')
\iff D_z \cdot c \in \text{Image}(\frac{1}{2}D_z \cdot F')
\iff \lambda \in \text{Image}(\Phi).
\]
By (2.4), \( C(e^{\lambda_j}s_0) = H(\mathcal{L})_c \), which completes the proof of Theorem 1.1. \( \square \)

Since \( F \in C^\infty(\mathbb{R}^n) \) is a strictly convex function [3], the image of \( \frac{1}{2}F' \) is a convex set in \( \mathbb{R}^n \). Thus the image of the moment map \( \Phi = \frac{1}{2}D_z \cdot F' \) is convex, and it includes all \( \lambda \in \mathbb{Z}^n \) exactly when \( \Phi \) is surjective. Therefore, by Theorem 1.1, \((H_{\alpha_j}) = 0\) for all \( \lambda \in \mathbb{Z}^n \) if and only if \( \Phi \) is surjective and \( \alpha_j = \pm 1 \) for all \( j \). This proves Corollary 1.1.

3. Symplectic reduction

Let \( \omega \) be a \( T \)-invariant Kähler form on \( T_c \), preserved by the \( x \)-weighted \( T \)-action (1.2). From now on, we consider the more interesting case where \( x \) has zero entries, which is the main purpose of this paper. The square-integrable holomorphic sections \( H_{\alpha_j} \) now have a more complicated multiplicity problem. It turns out that symplectic reduction [7] can handle this problem. In this section, we describe the process of symplectic reduction, and prove Theorem 1.2.

The torus \( T \) has dimension \( n \). Let \( k \) be the number of non-zero entries of the weight \( \alpha \), where \( 1 \leq k \leq n \). We may arrange the indices so that the first \( k \) entries \( \alpha_1, \ldots, \alpha_k \) are non-zero. We identify \( \mathbb{R}^k \) with the subspace of \( \mathbb{R}^n \) spanned by the first \( k \) variables. Intuitively, we can think of it as being ‘horizontal’. In this way, the horizontal \( k \)-dimensional affine subspaces \( H^c \subset \mathbb{R}^n \) are defined by
\[
H^c = \mathbb{R}^k + v = \{(x_1, \ldots, x_k, 0, \ldots, 0) : x_j \in \mathbb{R}\} + v, \quad v \in \mathbb{R}^k. \tag{3.1}
\]
Similarly, we may regard \( \mathbb{R}^{n-k} \subset \mathbb{R}^n \) as the subspace spanned by the last \((n-k)\)-coordinates, and define the ‘vertical’ affine \((n-k)\)-subspaces \( V^c \subset \mathbb{R}^n \) by
\[
V^c = c + \mathbb{R}^{n-k} = c + \{(0, \ldots, 0, x_{k+1}, \ldots, x_n) : x_j \in \mathbb{R}\}, \quad c \in \mathbb{R}^n. \tag{3.2}
\]

Recall that \( \omega \) has potential function \( F \). Let \( \Omega \) be the image of the moment map, and let \( D_z \) be the diagonal matrix with entries \( \alpha_1, \ldots, \alpha_n \). Fix \( \lambda \in \Omega \), and consider
\[
X = (\frac{1}{2}D_z \cdot F')^{-1}(\lambda) = \{x \in \mathbb{R}^n : \frac{1}{2} \frac{\partial F}{\partial x_j}(x) = \frac{\lambda_j}{\alpha_j} \text{ for } j = 1, \ldots, k\}. \tag{3.3}
\]
The space \( X \) will play an important role in our study of symplectic reduction.

If we let \( \bar{X} \) denote the closure of \( X \) in \( \mathbb{R}^n \), then the boundary of \( X \) is defined by \( \partial X = \bar{X} \setminus X \). The following proposition gives some properties of \( X \).

**Proposition 3.1.** The space \( X = (\frac{1}{2}D_z \cdot F^*)^{-1}(\lambda) \) is a closed, unbounded \((n-k)\)-dimensional submanifold of \( \mathbb{R}^n \), and \( \partial X = \emptyset \). For each \( v \in \mathbb{R}^n \), the horizontal affine \( k \)-space \( H^v \) intersects \( X \) at most once.

**Proof.** Since \( F \in C^\omega(\mathbb{R}^n) \) is strictly convex [3], \( \frac{1}{2}F^* \) maps \( \mathbb{R}^n \) diffeomorphically onto a domain \( U \subset \mathbb{R}^n \). Then \( D_z \) maps \( U \) onto the image \( \Omega \) of the moment map. Since \( D_z \) is a diagonal matrix whose last \((n-k)\) entries vanish, \( \lambda \in \Omega \) may be written as \( \lambda = (\lambda_1, \ldots, \lambda_k, 0, \ldots, 0) \).

Let \( v = (\lambda_1/x_1, \ldots, \lambda_k/x_k, 0, \ldots, 0) \), and let \( V \subset \mathbb{R}^n \) be the vertical affine \((n-k)\)-space defined in (3.2). Then \( D_z^{-1}(\lambda) \cap U = V \cap U \) is \((n-k)\)-dimensional. However, \( \frac{1}{2}F^* \) is a diffeomorphism between \( X \) and \( V \cap U \), so \( X \) is an \((n-k)\)-dimensional manifold.

Since \( V \cap U \) is closed in \( U \), we conclude from the diffeomorphism \( \frac{1}{2}F^* \) that \( X \) is closed in \( \mathbb{R}^n \). Since \( V \cap U \) is not compact, neither is \( X \). Hence \( X \), being non-compact and closed in \( \mathbb{R}^n \), is unbounded. Also, \( X \) equals its closure \( \bar{X} \) simply because \( X \) is closed, so the boundary \( \partial X \) is empty.

For \( v \in \mathbb{R}^n \), let \( H^v \) be the horizontal affine \( k \)-space defined in (3.1). It remains to show that \( X \) intersects each \( H^v \) at most once. Suppose that, for some \( v \in \mathbb{R}^n \), there exist distinct \( p, q \in X \cap H^v \). Let \( S = H^v \subset \mathbb{R}^n \) be the straight line joining \( p \) and \( q \). Let \( f \in C^\omega(S) \) be the restriction of \( F \) to \( S \). Since \( p, q \in X \), equation (3.3) says that

\[
\frac{\partial F}{\partial x_j}(p) = \frac{\partial F}{\partial x_j}(q) = \frac{1}{2} \frac{\lambda_j}{\lambda}
\]

for all \( j = 1, \ldots, k \). This means that \( f(t) \) has the same value at \( p \) and \( q \), where \( t \) is a linear variable on \( S \). This is a contradiction, because \( f \) should be strictly convex on \( S \). Hence, for all \( v \in \mathbb{R}^n \), \( X \cap H^v \) contains at most one point. This proves the proposition. \( \square \)

Since \( T \) is abelian, the moment map \( \Phi = \frac{1}{2}D_z \cdot F^* \) is \( T \)-invariant. By Proposition 3.1, \( \Phi^{-1}(\lambda) \) is a real \((2n-k)\)-submanifold of \( T_c \) given by

\[
\Phi^{-1}(\lambda) = X \times T = \{ x + \sqrt{-1} \text{Im}[y] : x \in X \}.
\]

Let \( i \) be the natural inclusion of \( \Phi^{-1}(\lambda) \) into \( T_c \). The torus \( T \) acts on \( \Phi^{-1}(\lambda) \), and we let \( B_\lambda = \Phi^{-1}(\lambda)/T \) be the quotient space. Let \( \pi \) be the quotient map from \( \Phi^{-1}(\lambda) \) onto \( B_\lambda \). There exists a symplectic form \( \omega_\lambda \) on \( B_\lambda \), satisfying \( \pi^*\omega_\lambda = i^*\omega \). The construction of the symplectic manifold \((B_\lambda, \omega_\lambda)\) is called symplectic reduction.

**Proof of Theorem 1.2.** When \( T \) acts on \( \Phi^{-1}(\lambda) \), the weight \( \alpha \) has \( k \) non-zero entries. We have arranged the indices so that \( T \) acts only along the first \( k \)-variables of \( [y] \). Therefore, equation (3.4) says that \( B_\lambda \) is diffeomorphic to the product manifold \( X \times T^{n-k} \), where \( T^{n-k} \) denotes the subtorus of \( T \) spanned by the last \((n-k)\) variables. To prove the theorem, it remains to check the assertion on symplectic forms. Consider the following diffeomorphism \( \sigma \) followed by two inclusions \( j \) and \( i \),

\[
B_\lambda \xrightarrow{\sigma} X \times T^{n-k} \xrightarrow{j} \Phi^{-1}(\lambda) \xrightarrow{i} T_c.
\]
Let $\pi$ be the quotient map from $\Phi^{\lambda}(\tilde{\lambda})$ to $B$. By the definition of $\omega_j$, $\tau^*\omega = \pi^*\omega_j$. Therefore, since $\pi \cdot \sigma$ is the identity function on $B$,

$$\sigma^* \cdot \tau^* \omega = \sigma^* \cdot \pi^* \omega_j = \omega_j.$$ 

This shows that the diffeomorphism $\sigma$ identifies $\omega_j$ with the pullback of $\omega$ to $X \times T^{n-k}$. Hence Theorem 1.2 holds.

The realization of $B$ as $X \times T^{n-k}$ has the defect that the real submanifold $X \times T^{n-k} \subset T_c$ is generally not complex. This is because the tangent bundle of $X \times T^{n-k}$ may not be preserved by the almost complex structure of $B$. We remark that there is a complex realization of $B$, in terms of Reinhardt domain, with $\omega_k$ being identified with the 'linear' Kähler structure $\omega_k = \sqrt{-1}/2 \sum dz_j \wedge d\bar{z}_j$. However, this Kähler realization will not be used below, and we merely describe it in brief here. Recall that $U$ is the image of the gradient function $\frac{1}{2} F$. Consider the diffeomorphism

$$\tau: T_c \longrightarrow U \times T, \quad \tau(x + \sqrt{-1}[y]) = \frac{1}{2} F(x) + \sqrt{-1}[y].$$

By the definition (3.3) of $X$,

$$\tau(X \times T^{n-k}) = (V^c \cap U) \times T^{n-k} \subset V^c \times T^{n-k},$$

(3.5)

where $V^c$ is the vertical affine space (3.2) corresponding to $c = (\lambda_1/z_1, \ldots, \lambda_k/z_k, 0, \ldots, 0)$. Hence $\tau$ is a diffeomorphism from $X \times T^{n-k}$ to $R = (V^c \cap U) \times T^{n-k}$. In particular, $R$ is a Reinhardt domain in a complex torus $V^c \times T^{n-k}$. Consider the standard Kähler structure on $T_c$, $\omega_c = (\sqrt{-1}/2) \sum dz_j \wedge d\bar{z}_j$. It satisfies

$$\tau^* \omega_c = \tau^* \frac{\sqrt{-1}}{2} \sum dz_j \wedge d\bar{z}_j$$

$$= \tau^* \sum_j dx_j \wedge dy_j$$

$$= \sum_j d(\tau^* x_j) \wedge dy_j$$

$$= \frac{1}{2} \sum_j \hat{c}^2 F \hat{c} x_j \wedge dy_j$$

$$= \omega.$$

Hence, by (3.5), $\tau$ identifies $\omega_j$ with the pullback of $\omega_j$ to $R$. In other words, $\tau$ is a symplectomorphism between the pullback of $\omega$ to $X \times T^{n-k}$ and the pullback of $\omega_j$ to $R$. Unfortunately, the Kähler realization $\omega_j$ does not reflect the geometry of the original Kähler form $\omega$, and its quantization is not so interesting. Hence we will always stick to the other realization, $(X \times T^{n-k}, \omega)$, from now on, we think of $B$ as the symplectic submanifold $X \times T^{n-k} \subset T_c$, and regard the reduced symplectic form $\omega_j$ as the pullback of $\omega$ to $B$.

4. Quantization commutes with reduction

Recall that $T$ acts on $T_c$ by mapping (1.2), preserving $\omega = \sqrt{-1} \hat{c} \hat{c} F$. We have assumed the weight of this action to be $\mathbf{x} = (x_1, \ldots, x_k, 0, \ldots, 0)$, where $x_1, \ldots, x_k$ are non-zero. In this way, every integral point in the image of the moment map is of the form $\lambda = (\lambda_1, \ldots, \lambda_k, 0, \ldots, 0) \in \Omega$. In the previous section, we performed symplectic reduction to $\lambda$, and obtained the reduced symplectic manifold $(B_{\lambda}, \omega_{\lambda})$. 

In this section, we apply geometric quantization to $B_j$, and prove Theorem 1.3. Using Theorem 1.2, we shall always identify $B_j$ with the real submanifold $X \times T^{n-k} \subset T_c$. In this way, the reduced symplectic form $\omega_j$ is just the pullback of $\omega$ to $B_j$. Since $\omega$ is exact, so is $\omega_j$. We let $\mathbf{L}_j$ be the pre-quantum line bundle over $B_j$, with Chern class $[\omega_j] = 0$. Hence $\mathbf{L}_j$ is a trivial bundle, and in fact is the restriction of $\mathbf{L}$ to $B_j$.

As remarked at the end of the previous section, the submanifold $B_j \subset T_c$ is not complex. Therefore, there is no intrinsic polarization on the space of smooth sections $C^{\omega}(\mathbf{L}_j)$. To overcome this problem, consider

$$\mathcal{C} = \{ x + \sqrt{-1}\xi | \xi \in T_c : H^\frac{k}{n} \text{ intersects } X \},$$

where $H^k$ is the horizontal $k$-space introduced in (3.1). Clearly, $\mathcal{C} \subset T_c$ is open. In fact, since $T$ acts along the first $k$ variables of $[y]$, $\mathcal{C}$ is the smallest complex submanifold of $T_c$ which contains $B_j$, and is preserved by the weighted $T$-action. Therefore, we can define $H(\mathcal{C}, \mathbf{L}_j)$ to be the holomorphic sections over $\mathcal{C}$ which transform by the weight $\lambda$ under the $T$-action. Consequently, among the smooth sections $C^{\omega}(\mathbf{L}_j)$ over $B_j$, we can define

$$H(\mathbf{L}_j) = \{ s \in C^{\omega}(\mathbf{L}_j) : s \text{ extendable to } H(\mathcal{C}, \mathbf{L}_j) \}.$$  

(4.2)

In other words, $H(\mathbf{L}_j)$ consists of all smooth sections of $\mathbf{L}_j$ obtained from the restriction of $H(\mathcal{C}, \mathbf{L}_j)$ to $B_j$.

We restrict the Haar measure $dV$ to $B_j$, and use the Hermitian structure of $\mathbf{L}_j$ to define an $L^2$-structure on $H(\mathbf{L}_j)$. Let $H^{(\omega_j)}$ be the corresponding square-integrable sections:

$$H^{(\omega_j)} = \left\{ s \in H(\mathbf{L}_j) : \int_{B_j} (s, s) dV < \infty \right\}.$$  

(4.3)

In this way, $H^{(\omega_j)}$ is a complex inner product space.

**Proposition 4.1.** $H^{(\omega_j)}$ is a Hilbert space.

**Proof.** The only thing to check is completeness. We do this by constructing an inner product space isomorphism between $H^{(\omega_j)}$ and a Hilbert space.

From Proposition 3.1, $X \subset \mathbb{R}^n$ is an $(n-k)$-submanifold which intersects every horizontal affine $k$-space $H^k$ (3.1) at most once. Define

$$W = \{ v \in \mathbb{R}^{n-k} \subset \mathbb{R}^n : H^k \text{ intersects } X \}.$$  

(4.4)

Then we get a diffeomorphism

$$\psi : X \longrightarrow W, \quad x \longmapsto (0, \ldots, 0, x_{k+1}, \ldots, x_n).$$  

(4.5)

Let $dV$ be the restriction of the Lebesgue measure of $\mathbb{R}^{n-k}$ to its open set $W = \psi(X)$, and $dV_j$ the restriction of the Lebesgue measure of $\mathbb{R}^n$ to $X$. Also, let $c_j = \lambda_j / x_j \in \mathbb{Z}$ for $j = 1, \ldots, k$, and let $c \cdot x$ denote $\sum c_j x_j$. Recall that $F$ is the potential function of $\omega$. Since $\psi$ is a diffeomorphism, it has a Jacobian $J_\psi \in C^\omega(W)$ between the volume forms $e^{2\pi z - F(x)} dV_j$ on $X$ and $dV$ on $W.$ In other words,

$$e^{2\pi z - F(x)} dV_j = \psi^*(J_\psi dV).$$  

(4.6)

Let $W_c \subset T_c^{n-k}$ be the Reinhardt domain

$$W_c = \{ x + \sqrt{-1}\xi | \xi \in T_c^{n-k}, x \in W \}.$$  

(4.7)
The function $J_\mu$ extends naturally to $W_C$ by $T^{n+2}$-invariance. Let $B(W_C, J_\mu)$ be the Bergman space of $J_\mu$-weighted $L^2$-holomorphic functions on the Reinhardt domain $W_C$. In other words, it is the Hilbert space defined by

$$B(W_C, J_\mu) = \left\{ h \in C^\infty(W_C) : h \text{ holomorphic, } \int_{W_C} h\bar{h}J_\mu dV < \infty \right\}.$$ 

We now check that the inner product space $H_{(\omega_j)}$ is isomorphic to the weighted Bergman space $B(W_C, J_\mu)$, which implies that $H_{(\omega_j)}$ is a Hilbert space.

From the definition (4.2) of $H(L_j)$, we have the natural restriction map

$$\kappa : H(\mathcal{E}, L_j) \longrightarrow H(L_j).$$

Let $s_0$ be the $T$-invariant holomorphic section of (2.2), restricted to $\mathcal{E}$. Pick $s \in H_{(\omega_j)}$. By the definition of $H(L_j)$, $s$ is of the form $s = \kappa(he^{-z}s_0)$, where $h$ is a holomorphic function on $\mathcal{E}$ and depends only on the variables $z_{k+1}, \ldots, z_n$ because the section $he^{-z}s_0 \in H(\mathcal{E}, L_j)$ needs to transform by $\lambda$. Define

$$L : H_{(\omega_j)} \longrightarrow B(W_C, J_\mu), \quad \kappa(he^{-z}s_0) \longmapsto h|_{W_C}. \quad (4.8)$$

We claim that $L$ is an inner product space isomorphism.

Since $h$ is independent of the first $k$ variables, the function $hh$ satisfies

$$(hh)(q) = (h\bar{h})(\psi(q)) \quad (4.9)$$

for all $q \in X$. We let $\| \cdot \|_1$ and $\| \cdot \|_2$ denote the norms of $H_{(\omega_j)}$ and $B(W_C, J_\mu)$ respectively. For $s = \kappa(he^{-z}s_0) \in H_{(\omega_j)}$,

$$\|s\|_1^2 = \int_X hh\bar{e} e^{-P(\psi)} dV \quad \text{by (2.2)}$$

$$= \int_X hh\bar{h} \psi^*(J_\mu) dV \quad \text{by (4.6)}$$

$$= \int_X \psi^*(hhJ_\mu) dV \quad \text{by (4.9)}$$

$$= \int_X hhJ_\mu dV \quad \text{by (4.5)}$$

$$= \int_{W_C} hhJ_\mu dV \quad \text{by (4.8)}$$

$$= \|L(s)\|_2^2. \quad (4.10)$$

Let $T^k$ be the complex subtorus spanned by the first $k$ variables. It follows from the definitions (4.1), (4.4) and (4.7) that $\mathcal{E} = T^k_C \times W_C$. Hence the operation $h \longmapsto h|_{W_C}$ in (4.8) is bijective, because a holomorphic function $h$ on $\mathcal{E}$ independent of $z_1, \ldots, z_k$ is equivalent to a holomorphic function on $W_C$. Therefore, $L$ is a bijection. Then (4.10) says that $L$ is an isomorphism of inner product spaces from $H_{(\omega_j)}$ to $B(W_C, J_\mu)$. Therefore $H_{(\omega_j)}$ is a Hilbert space.}

Recall that $(H_{\omega_j})_J = H(L_j) \cap H_{\omega_j}$. Let $c_j = \lambda_j/\pi_j$ for $j = 1, \ldots, k$. From an argument similar to the one leading to (2.4), we know that, if any $c_j$ is not an integer, then $(H_{\omega_j})_J$
vanishes. Assuming \(c_j \in \mathbb{Z}\) from now on, our goal is to construct a natural Hilbert space isomorphism \((H_{\omega j})_k \cong H_{(\omega j)}\), and prove Theorem 1.3. In order to compare these two Hilbert spaces, the next two propositions provide integrability conditions.

Let \(s_\eta\) be the holomorphic section of equation (2.2). Given \(b = (b_{k+1}, \ldots, b_n) \in \mathbb{Z}^{n-k}\), we define a one-dimensional subspace

\[
S_b = \left\{ a \exp \left( \sum_{i=1}^{k} c_i z_i \right) \exp \left( \sum_{j=k+1}^{n} b_j z_j \right) s_\eta : a \in \mathbb{C} \right\} \subseteq H(L).
\] (4.11)

Proposition 4.2 follows, from [3].

PROPOSITION 4.2 [3]. Let \(0 \neq s \in S_b\). Then \(\int_{\nabla} (s, s) dV\) converges if and only if \((c_1, \ldots, c_k, b_{k+1}, \ldots, b_n)\) is in the image of \(\frac{1}{n} F\).

Recall that \(\Omega\) is the image of the moment map. Following Proposition 4.2, it is clear that

\[
(H_{\omega j})_k \neq 0 \Leftrightarrow \lambda \in \Omega, \frac{\lambda}{\partial x_j} \in \mathbb{Z} \quad \text{for} \quad j = 1, \ldots, k.
\] (4.12)

Assume that \(\lambda \in \Omega\), so that we have the reduced space \(B_j\). Let \(s \in S_b\). In Proposition 4.2, we have given a necessary and sufficient condition for \((s, s) dV\) to be integrable over \(T_c\). We now restrict it to \(B_j \subset T_c\), but for simplicity, we still denote it by \((s, s) dV\).

The next proposition considers its integrability over \(B_j\).

PROPOSITION 4.3. Let \(0 \neq s \in S_b\). Then \(\int_{B_j} (s, s) dV\) converges if and only if \((c_1, \ldots, c_k, b_{k+1}, \ldots, b_n)\) is in the image of \(\frac{1}{n} F\).

Proof. Let \(s \in S_b\). Suppose that \((c_1, \ldots, c_k, b_{k+1}, \ldots, b_n)\) is in the image of \(\frac{1}{n} F\). By Proposition 4.2, \(\int_{T_c} (s, s) dV\) converges. Therefore, when restricted to \(B_j \subset T_c\), \(\int_{B_j} (s, s) dV\) also converges.

Therefore, it only remains to prove the converse. Suppose that \(\int_{B_j} (s, s) dV\) converges for all \(s \in S_b\). Define \(G \in C^\infty(\mathbb{R}^n)\) by

\[
G(x) = F(x) - 2 \sum_{i=1}^{k} c_i x_i - 2 \sum_{i=k+1}^{n} b_j x_j.
\] (4.13)

Since \(F\) and \(G\) have the same Hessian, \(G\) is strictly convex. Since \((s_\eta, s_0) = e^{-\rho}\), equations (4.11) and (4.13) imply that up to a positive constant, \(e^{-\rho} = (s, s)\). Hence

\[
\int_X e^{-\rho} dV_j = \int_{B_j} (s, s) dV < \infty,
\] (4.14)

where \(dV_j\) is the restriction of the Lebesgue measure to \(X \subset \mathbb{R}^n\). By Proposition 3.1, \(X\) is unbounded and has no boundary. Thus (4.14) implies that \(e^{-\rho}\) approaches 0 along every direction of \(X\), in the sense that, for any \(\epsilon > 0\), there exists a compact subset of \(X\) such that \(e^{-G(x)} < \epsilon\) for \(x\) outside this compact set. This means that \(e^{-\rho}\) acquires a maximum point in \(X\). Equivalently, \(G\) has a minimum point \(p\) in \(X\):

\[
G(p) \leq G(x), \quad x \in X.
\] (4.15)

Recall the notions of horizontal and vertical affine spaces, defined in equations (3.1) and (3.2). Let \(G|_{V^p}\) denote the restricted function on the vertical space \(V^p\). We
want to show that \( p \) is the global minimum of \( G_{|_{V^p}} \). However, since \( G_{|_{V^p}} \) is strictly convex, it suffices to show that \( p \) is a local minimum of \( G_{|_{V^p}} \). By Proposition 3.1, \( X \) intersects \( H^p \) exactly once, at \( p \). Hence, for each \( v \in V^p \) that is sufficiently near \( p \),

\[
X \cap H^v = \{ q_v \}
\]  

(4.16)

for some \( q_v \). By the definition of \( X \) (3.3) and the definition of \( G \) in equation (4.13),

\[
\frac{\partial G}{\partial x_j}(q_v) = 0, \quad j = 1, \ldots, k.
\]  

(4.17)

Since the restriction of \( G \) to \( H^v \) is strictly convex, equation (4.17) says that \( q_v \) is the global minimum of the restriction of \( G \) to \( H^v \). In particular, since \( v \in H^v \), it gives

\[
G(q_v) \leq G(v).
\]  

(4.18)

Since \( q_v \in X \), equations (4.15) and (4.18) imply that \( G(p) \leq G(v) \) whenever \( v \in V^p \) is sufficiently near \( p \). This proves that \( p \) is a local minimum of \( G_{|_{V^p}} \). However, \( G_{|_{V^p}} \) is strictly convex, so \( p \) is a global minimum of it. Therefore,

\[
\frac{\partial G}{\partial x_j}(p) = 0, \quad j = k + 1, \ldots, n.
\]  

(4.19)

Set \( v = p \) in (4.16), so that \( q_v = p \). Then (4.17) becomes

\[
\frac{\partial G}{\partial x_j}(p) = 0, \quad j = 1, \ldots, k.
\]  

(4.20)

Using equations (4.19) and (4.20), we conclude that \( p \) is the global minimum of \( G \), and so 0 is in the image of \( G' \). Then (4.13) implies that \((c_1, \ldots, c_k, b_{k+1}, \ldots, b_n)\) is in the image of \( \frac{1}{\lambda}F' \). Hence the proposition holds.

\[\square\]

If \( s \in H(\mathbb{L}) \), we let \( \rho(s) \) be its restriction to \( B \). By the definition (4.2) of \( H(\mathbb{L}) \), \( \rho(s) \in H(\mathbb{L}_o) \). Therefore, we have the restriction map

\[
\rho: H(\mathbb{L}) \longrightarrow H(\mathbb{L}_o).
\]

We apply \( \rho \) to the one-dimensional spaces \( S_b \) of (4.11). Recall that \( c_j = \lambda_j/x_j \in \mathbb{Z} \), for \( j = 1, \ldots, k \). Let

\[
I = \{(b_{k+1}, \ldots, b_n) \in \mathbb{Z}^{n-k} : (c_1, \ldots, c_k, b_{k+1}, \ldots, b_n) \in \text{Image} \left( \frac{1}{\lambda}F' \right) \}.
\]  

(4.21)

By Propositions 4.2 and 4.3,

\[
S_b \subset (H_{\omega}), \Longleftrightarrow \rho(S_b) \subset H_{\omega_o}, \Longleftrightarrow b \in I.
\]  

(4.22)

It follows from definition (4.11) that, if \( 0 \neq s \in S_b \), then \( s \) is non-vanishing, so in particular its restriction \( \rho(s) \) is non-zero. Thus \( \rho \) is injective on each \( S_b \). Therefore, since each \( S_b \) is one-dimensional, we obtain a constant \( m_b > 0 \) for each \( b \in I \) by

\[
\| s \| = m_b \| \rho(s) \|, \quad s \in S_b.
\]  

(4.23)

Here \( \| \cdot \| \) denotes the norms of both \( (H_{\omega}), \) and \( H_{\omega_o} \). For \( b \in I \), define \( \tilde{\rho} \) on \( S_b \) by

\[
\tilde{\rho}(s) = m_b \rho(s), \quad s \in S_b.
\]  

(4.24)

Proof of Theorem 1.3. If \( \lambda \) is not in the image \( \Omega \) of the moment map, then there is no reduced space \( B_\lambda \) or \( H_{\omega_o} \). Also, \( (H_{\omega}), \lambda = 0 \) by (4.12), and there is nothing to prove. Therefore, we may assume that \( \lambda \in \Omega \).
Proposition 4.1 says that \( H_{\omega_0} \) is a Hilbert space. To complete the proof, we show that \( \hat{\rho} \) gives the desired Hilbert space isomorphism. It follows from (4.22), (4.23) and (4.24) that
\[
\| \hat{\rho}(s) \| = \| s \| < \infty \quad \text{for} \quad s \in S_b, \; b \in I. \tag{4.25}
\]

Consider the standard subgroup action of \( T^{n-k} \) on \( T \), which lifts to a \( T^{n-k} \)-representation on \( H(L) \). This restricts to a \( T^{n-k} \)-representation on \( H(L)_0 \), because the standard \( T^{n-k} \)-action commutes with the weighted \( T \)-action defined in (1.2). Since \( T^{n-k} \) preserves the \( L^2 \)-structure (1.3) on \( H(L)_0 \), we get a unitary representation
\[
\pi_1: T^{n-k} \longrightarrow \text{Aut}(H_{\omega_0}).
\]

Since \( T^{n-k} \) also acts on \( B_0 \) by acting on its toral component, we similarly get a \( T^{n-k} \)-representation on \( H(L)_0 \). It preserves the \( L^2 \)-structure (4.3) on \( H(L)_0 \), so we get a unitary representation
\[
\pi_2: T^{n-k} \longrightarrow \text{Aut} H_{\omega_0}.
\]

By the definition (4.11) and property (4.22) of \( S_0 \), the irreducible subrepresentations of \( \pi_1, \pi_2 \) are given by \( \{ S_0 \}_{b \in I} \) and \( \{ \rho(S_0) \}_{b \in I} \) respectively. Apply the Peter–Weyl theorem \([2, \text{Chapter III}]\) to these subrepresentations. It says that \( \{ S_0 \}_{b \in I} \) and \( \{ \rho(S_0) \}_{b \in I} \) are collections of mutually orthogonal subspaces in \( (H_{\omega_0}),_k \) and \( H_{\omega_0} \) respectively, and their linear spans are dense in these Hilbert spaces. Let
\[
S \subset (H_{\omega_0}), \quad R \subset H_{\omega_0}
\]
be the dense subsets given by their linear spans. Since both \( \{ S_0 \}_{b \in I} \) and \( \{ \rho(S_0) \}_{b \in I} \) are collections of mutually orthogonal subspaces, definition (4.25) says that \( \hat{\rho} \) is an isometry from \( S \) to \( R \).

If \( I \) is finite, then the Hilbert spaces are finite-dimensional, and \( S = (H_{\omega_0}),, \; R = H_{\omega_0} \). Thus \( \hat{\rho} \) is the required isomorphism. Suppose that \( I \) is infinite. Since \( \hat{\rho} \) is an isometry between the dense subsets \( S \) and \( R \), it extends continuously to a Hilbert space isomorphism \( \hat{\rho}: (H_{\omega_0}), \longrightarrow H_{\omega_0} \). This proves Theorem 1.3. \( \square \)

5. Open cones

In this section, we give some simple examples to show that the Hilbert spaces of Theorem 1.3 can have any dimension. It suffices to consider a torus of dimension 2.

From the previous section, we see that the dimension of \( (H_{\omega_0}), \) is the cardinality of the index set \( I \) of (4.21). Consider \( \omega = \sqrt{-1} \epsilon \tilde{c} F \), invariant under the \( T \)-action (1.2) with weight \( \alpha = (1, 0) \). Let \( \lambda = (0, 0) \in \mathbb{R}^2 \). Then the set \( I \) of (4.21) becomes
\[
I = \{ b \in \mathbb{Z} : (0, b) \in \text{Image} \frac{1}{2} F' \}. \tag{5.1}
\]

We now show that \( |I| = \text{dim}(H_{\omega_0}), \) can be any of 0, 1, 2, ..., \( \infty \).

Let \( v_1, v_2 \in \mathbb{R}^2 \) be a basis. Define \( G \in C^\infty(\mathbb{R}^2) \) by
\[
G(x) = \exp(v_1 \cdot x) + \exp(v_2 \cdot x),
\]
where \( v_1 \cdot x \) is the usual dot product. Then \( G \) is strictly convex. The image of \( \frac{1}{2} G' \) is the open cone consisting of all positive linear combinations of \( v_1, v_2 \). By adjusting \( v_1 \) and \( v_2 \), we get all the open cones of \( \mathbb{R}^2 \) that emit from the origin.

In fact, if \( G \) is any strictly convex function and \( w \in \mathbb{R}^2 \), then \( F(x) = G(x) + w \cdot x \) is also strictly convex. Further, the images of \( \frac{1}{2} F' \) and \( \frac{1}{2} G' \) differ by an affine translation of \( \frac{1}{2} w \). From this observation, we consider the strictly convex function
\[
F(x) = \exp(v_1 \cdot x) + \exp(v_2 \cdot x) + w \cdot x.
\]
By choosing different parameters $v_1, v_2, w \in \mathbb{R}^2$ for $F$, the image of $\frac{1}{2}F$ can be any given open cone $C \subset \mathbb{R}^2$. For every $s = 0, 1, 2, \ldots, \infty$, we can always find an open cone $C$ whose intersection with the $y$-axis contains $s$ integral points, so that $I$ in (5.1) has $s$ elements.

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References

Department of Applied Mathematics
National Chiao Tung University
1001 Ta Hsueh Road
Hsinchu
Taiwan

chuah@math.nctu.edu.tw