Holomorphic Discrete Models of Semisimple Lie Groups and their Symplectic Constructions

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Let $G$ be a connected real semisimple Lie group which contains a compact Cartan subgroup such that it has non-empty discrete series. A holomorphic discrete model of $G$ is a unitary $G$-representation consisting of all its holomorphic discrete series with multiplicity one. We perform geometric quantization to a class of $G$-invariant pseudo-Kähler manifolds and construct a holomorphic discrete model. The construction of discrete series which are not holomorphic is also discussed.

Key Words: holomorphic discrete model; pseudo-Kähler

1. INTRODUCTION

Let $G$ be a connected real Lie group. A unitary representation of $G$ in which each irreducible representation occurs exactly once is called a model. This terminology is due to I. M. Gelfand and A. Zelevinski [6], who give several ingenious constructions of models for the classical compact groups. For compact semisimple Lie groups, the construction of a model is carried out in [4]. If $G$ is not compact, then looking for its model is overly ambitious since the unitary dual of $G$ is still unknown in general. However, suppose that $G$ is semisimple and has a compact Cartan subgroup, so that it has a non-empty discrete series [10]. Following the spirit of the above definition of a model, we define the holomorphic discrete model as a unitary $G$-representation consisting of all the holomorphic discrete series with multiplicity one. The major purpose of this article is to use symplectic techniques to construct such an object. Namely, we start with certain classes of $G$-invariant pseudo-Kähler manifolds and use the machinery of geometric quantization [14] to convert the pseudo-Kähler structures into holomorphic Hermitian line bundles. The square-integrable holomorphic
sections of these line bundles are used to construct the desired holomorphic discrete model. More generally, the $L^2$-cohomology of these line bundles provides discrete series which may not be holomorphic. We can of course define the discrete model in a similar manner. Unfortunately these $L^2$-cohomologies do not provide a discrete model due to multiplicity problems. Nevertheless, by applying symplectic reduction [15] to the pseudo-Kähler manifolds we show that these $L^2$-cohomologies obey the principle "quantization commutes with reduction" [7].

We now describe our projects in more detail. Throughout this paper, $G$ denotes a connected real semisimple Lie group which has a maximal compact Cartan subgroup. We also adopt the convention that the Lie algebra of a Lie group is always denoted by its lower case German letter; for instance, $\mathfrak{g}$ is the real Lie algebra of $G$.

Let $G^C$ be the adjoint group of the complex Lie algebra $\mathfrak{g}^C = \mathfrak{g} \otimes \mathbb{C}$. Then $G$ is a real form of $G^C$. Let $K$ be a maximal compact subgroup of $G$ satisfying $\text{rank } K = \text{rank } G$ and $T$ a Cartan subgroup of both $K$ and $G$. Thus $T \subset K \subset G$. The complex structure of $\mathfrak{g}^C$ sends $t$ to an abelian subalgebra $a$. We obtain a complex Cartan subalgebra $\mathfrak{h}^C = \mathfrak{t}^C + a^C$ and a Cartan subgroup $H = TA < G^C$. Consider the root system $\Delta \subset \mathfrak{h}^C$. Fix a system of positive roots $\Delta^+$, which determines a unipotent subgroup $N$ corresponding to the negative root spaces. Then $B = HN$ is a Borel subgroup of $G^C$.

The positive roots $\Delta^+$ are partitioned into compact roots $\Delta^+_c$ and non-compact roots $\Delta^+_n$. Namely, a root is said to be compact if its root space lies in $\mathfrak{k} \subset \mathfrak{g}$ and is said to be non-compact otherwise. Let $\Delta^+$ be the simple roots in $\Delta^+$ and consider the compact simple roots $\Delta^+_s = \Delta^+_c \cap \Delta^+$. Fix a subset $\sigma \subset \Delta^+_s$. We define $t_\sigma \subset \mathfrak{t}$ by

$$\mathfrak{t}_\sigma = \{x \in \mathfrak{t}; (x, x) = 0 \text{ for all } x \in \sigma \}. \tag{1.1}$$

Since $\mathfrak{g}^C$ is semisimple, its Killing form is non-degenerate. It identifies the subalgebras with their duals; for instance, we have $t^*_\sigma \subset \mathfrak{h}$. Whenever we pair two elements in the dual space to get a number, such as in (1.2) below, it is understood that this is always done via the Killing form.

Let $\sigma \subset \Delta^+$ be the positive roots generated by $\sigma$. Define the $\sigma$-regular elements

$$(t^*_\sigma)_{\text{reg}} = \{x \in t^*_\sigma; (x, x) \neq 0 \text{ for all } x \in \Delta^+ \setminus \sigma\}. \tag{1.2}$$

So $(t^*_\sigma)_{\text{reg}}$ is an open dense set in $t^*_\sigma$ whose connected components are cones. If $\sigma$ is the empty set $\emptyset$, we clearly have $(t^*_\emptyset)_{\text{reg}} = (t^*)_{\text{reg}}$. By the complex structure, the above constructions lead to $(a^*_\sigma)_{\text{reg}} \subset (a^*_\sigma)_{\text{reg}}$. We shall always identify $(t^*_\sigma)_{\text{reg}} \equiv (a^*_\sigma)_{\text{reg}}$. We define $\mathfrak{h}_\sigma = t_\sigma + a_\sigma \subset \mathfrak{h}$ and $H_\sigma = T_\sigma A_\sigma \subset H$ accordingly.
A subset $\sigma \subset A^*_e$ determines a parabolic subgroup $P \subset G^C$ via Langlands decomposition [13, p. 132]

$$B \subset P = M_\sigma A_\sigma N_\sigma; \quad A_\sigma \subset A, \quad N_\sigma \subset N.$$  \hspace{1cm} (1.3)

For example, $\rho = \emptyset$ corresponds to $P = B = HN$.

Let $\rho$ denote half the sum of all positive roots. We denote the discrete series of $G$ by Harish-Chandra’s notation $\Theta_{\lambda + \rho}$ for integral weights $\lambda$. Apply (1.3) and fix $\sigma, P$ for now. Let $Y_\sigma \subset G^C/P$ be the $G$-orbit containing the identity coset. It is a homogeneous $G$-space isomorphic to an elliptic orbit. Further, it is open in $G^C/P$ and thus is a complex manifold. Let $(P, P)$ be the commutator subgroup of $P$. We define $X_\sigma \subset G^C/(P, P)$ by the natural fibration

$$\pi: G^C/(P, P) \rightarrow G^C/P, \quad X_\sigma = \pi^{-1}(Y_\sigma).$$  \hspace{1cm} (1.4)

By studying homogeneous line bundles over $Y_\sigma$ corresponding to integral weights $\lambda \in h^*_e$, we obtain discrete series of $G$ [16]. However, each $\lambda$ gives at most one discrete series. Therefore, we shall work on line bundles over $X_\sigma$ instead to obtain the holomorphic discrete model.

Since $Y_\sigma$ is an open $G$-orbit in $G^C/P$, $X_\sigma$ is an open $G$-space in $G^C/(P, P)$. Thus $X_\sigma$ is a complex manifold. Since $H_\sigma$ normalizes $(P, P)$, it acts on $G^C/(P, P)$ on the right. Let $G^h$ be the centralizer of $T_\sigma$ in $G$ and let $G^h_\sigma \subset G^h$ be its commutator subgroup. We shall see (Proposition 3.1) that $X_\sigma = (G/G^h_\sigma) A_\sigma$. This implies that the right $H_\sigma$-action restricts to a right action on $X_\sigma$ and that $G$-invariant functions on $X_\sigma$ can be regarded as functions on $A_\sigma$.

A subscript of a Lie group shall always indicate invariance under the group action. For instance, $C^\infty_G(X_\sigma) = C^\infty(A_\sigma)$.

The exponential map is a diffeomorphism from $a_\sigma$ onto $A_\sigma$. Throughout this paper, we shall frequently make the identification

$$a_\sigma \cong A_\sigma, \quad F \cdot \exp \in C^\infty(a_\sigma) \leftrightarrow F \in C^\infty(A_\sigma).$$  \hspace{1cm} (1.5)

So if $F$ is a $G$-invariant function on $X_\sigma$, we can identify it with a function on $a_\sigma$. We shall say that $F$ is strictly convex if its Hessian matrix is positive definite everywhere and more generally that $F$ is non-degenerate if its Hessian matrix is non-degenerate everywhere. The image of the gradient function $\frac{1}{2} F: a_\sigma \rightarrow a^*_\sigma$ is denoted by $U_F$.

$$U_F = \frac{1}{2} F(a_\sigma) \subset a^*_\sigma.$$  \hspace{1cm} (1.6)

The inverse function theorem says that if $F$ is non-degenerate then $U_F$ is open. We shall see that $\frac{1}{2} F$ is essentially the moment map of pseudo-Kähler form. By definition, a pseudo-Kähler form is a symplectic form of
type \((1, 1)\). It is weaker than a Kähler form only on the positive definite condition.

The starting point of this paper is classifying all the \((G \times T_a)\)-invariant pseudo-Kähler structures on \(X_a\). Since \(G\) is semisimple, the \(G\)-actions on these pseudo-Kähler forms are necessarily Hamiltonian [8, Theorem 26.1].

Their \(G\)-moment maps are denoted by

\[
\Phi: X_a \to \mathfrak{a}^*.
\]

Since \(\Phi\) is \(G\)-equivariant it is determined by its restriction on \(A_a \subset X_a\).

Define the (possibly empty) chamber

\[
\mathcal{C} = \{ x \in \mathfrak{t}^*; (A_a^+, x) \geq 0, (A_a^+, x) < 0 \}. \tag{1.7}
\]

The complex structure also identifies \(\mathcal{C}\) as a subset of \(\mathfrak{a}^*\). Our theorems contain statements about \(\mathcal{C}\). If \(\mathcal{C} = \emptyset\) the results are still valid, and they simply say that the Kähler structures in question do not exist.

**Theorem 1.** Every \((G \times T_a)\)-invariant closed \((1, 1)\)-form on \(X_a\) is given by \(\omega = -1^2 \bar{\partial} \partial F\), where \(F \in C^\infty(A_a)\). It is pseudo-Kähler if and only if \(F\) is non-degenerate and \(U_F \subset (\mathfrak{a}_a^*)_{\text{reg}}\). It is Kähler if and only if \(F\) is strictly convex and \(U_F \subset (\mathfrak{a}_a^*)_{\text{reg}} \cap \mathcal{C}\). The moment map satisfies \(\Phi(a) = \frac{1}{2} F(a) \in \mathfrak{a}_a^* \simeq \mathfrak{t}_a^*\) for all \(a \in A_a\), so \(U_F = \Phi(A_a)\).

Fix a \((G \times T_a)\)-invariant Kähler form \(\omega\) on \(X_a\). By Theorem 1, \(\omega\) is exact. So there exists a pre-quantum line bundle \(L\) [14] whose Chern class is \([\omega] = 0\). Further, \(L\) is equipped with a connection \(V\) whose curvature is \(\omega\) as well as an invariant Hermitian structure \((, )\). A smooth section \(s\) on \(L\) is said to be holomorphic if \(V_s = 0\) for all anti-holomorphic vector fields \(v\). We shall show (Proposition 4.1) that \(X_a\) has \((G \times A_a)\)-invariant measure \(\mu_X\), which is unique up to scalar. A section \(s\) is said to be square-integrable if the integral

\[
\int_{X_a} (s, s) \mu_X \tag{1.8}
\]

converges. Consider the Hilbert space \(H_{\omega}\) of all square-integrable holomorphic sections. The \((G \times T_a)\)-action on \(X_a\) lifts to a unitary \((G \times T_a)\)-representation on \(H_{\omega}\). The next theorem describes the irreducible \(G\)-subrepresentations in \(H_{\omega}\).

Let \(\lambda \in \mathfrak{h}_a^*\) be an integral weight. We shall always write \(\chi = e^\lambda\) for its character. Namely, \(\chi: H_{\omega} \to \mathbb{C}^\times\) is the multiplicative homomorphism satisfying

\[
\chi(e^v) = \exp(\lambda, v), \quad v \in \mathfrak{h}_a. \tag{1.9}
\]
If $V$ is an $H_n$-module we say that $v \in V$ transforms by $\lambda$ if $h \cdot v = \varphi(h)v$ for all $h \in H_n$.

In the following theorem, $(H_n)_\lambda$ denotes the square-integrable holomorphic sections which transform by $\lambda$ under the right $T_n$-action. We also assume that $\lambda + \rho \in (t^*)_{reg}$, since this condition is needed for $\Theta_{\lambda + \rho}$ to exist.

**Theorem 2.** Let $\omega = \sqrt{-1} \partial \bar{\partial} F$ be a $(G \times T_n)$-invariant Kähler form on $X_n$. The holomorphic discrete series $\Theta_{\lambda + \rho}$ occurs in $H_n$ if and only if $\lambda \in U_F$. If so, it occurs with multiplicity one and is given by $(H_n)_\lambda$.

In Section 5, we apply Theorems 1 and 2 to construct a holomorphic discrete model for $G$. Namely, we show that for a suitable choice of Kähler form $\omega_n$ on $X_n$, $H_n$ contains every holomorphic discrete series $\Theta_{\lambda + \rho}$ in which $\lambda \in (a^*_n)_{reg} \cap \mathfrak{c}$ once. Consequently, if we vary $\sigma$ over all subsets of $A^*_n$, then $\bigoplus_n H_n$ becomes a holomorphic discrete model.

The square-integrable holomorphic sections $H_n$ can be generalized to the $L^2$-cohomology $H^*_n$, defined below. From $H^*_n$ we obtain the discrete series which may not be holomorphic. But unlike the construction $\bigoplus_n H_n$ above, these $H^*_n$ do not form the discrete model when we vary $\sigma$, due to the multiplicity problem. Nevertheless, we shall study $H^*_n$ in the context of symplectic reduction.

Let $\omega = \sqrt{-1} \partial \bar{\partial} F$ be a $(G \times T_n)$-invariant pseudo-Kähler form on $X_n$, where $F$ is strictly convex. Let $L$ be the pre-quantum line bundle as before. We shall construct the $L^2$-cohomology $H^*_n$ as follows. We denote the Dolbeault $(0, q)$-forms with coefficients in $L$ by $\Omega^0_q(X_n, L)$. Define a $G$-invariant Hermitian structure on it and then integrate over $\mu_X$ to obtain an $L^2$-structure $\langle x, y \rangle^L, x, y \in \Omega^0_q(X_n, L)$. We say that $x$ is square-integrable if $\langle x, x \rangle^L < \infty$. Let $\partial^* \bar{\partial}^*$ be the formal adjoint of $\partial \bar{\partial}$ relative to this $L^2$-structure. The differential forms which are annihilated by $\partial \bar{\partial}$ and $\partial^* \bar{\partial}^*$ are said to be harmonic. Let $H^*_n$ be the square-integrable harmonic $(0, q)$-forms. Let $\lambda \in \mathfrak{t}^*_n$ be an integral weight. The $(G \times T_n)$-action lifts to a $G \times T_n$-representation on $H^*_n$ and the right $T_n$-action defines $(H^*_n)_\lambda$ as before. Let

$$h(\lambda) = \sharp \{\lambda \in A^*_n; (\lambda + \rho, \alpha) < 0\} - \sharp \{\lambda \in A^*_n; (\lambda + \rho, \alpha) > 0\}.$$ \hspace{1cm} (1.10)

We now obtain the general discrete series from $H^*_n$.

**Theorem 3.** Let $\omega = \sqrt{-1} \partial \bar{\partial} F$ be a $(G \times T_n)$-invariant pseudo-Kähler form on $X_n$ with $F$ strictly convex. The discrete series $\Theta_{\lambda + \rho}$ occurs in $H^*_n$ if and only if $\lambda \in U_F$, $\lambda + \rho \in (t^*)_{reg}$, and $q = h(\lambda)$. If this is so, it occurs with multiplicity one and is given by $(H^*_n)_\lambda$. 
Let $\omega$ be a $(G \times T_\ast)$-invariant pseudo-Kähler form on $X_\ast$. The right $T_\ast$-action is Hamiltonian and we call its moment map
\[ \Phi_r: X_\ast \to t_\ast^* \]
the right moment map. Suppose that $\lambda \in t_\ast^*$ is in its image. We perform symplectic reduction [15] on it. This leads to the reduced space $R_\ast = \Phi_r^{-1}(\lambda)/T_\ast$ equipped with a symplectic form $\omega_\ast$, called the reduced form. The process
\[ (X_\ast, \omega, \lambda) \to (R_\ast, \omega_\ast) \] (1.11)
is known as symplectic reduction. We shall see (Proposition 7.2) that $R_\ast$ is a complex manifold and (Proposition 7.4) that $\omega_\ast$ is a $G$-invariant pseudo-Kähler form on $R_\ast$.

We want to study how $\omega$ and $\lambda$ determine the reduced space in (1.11). For $i = 1, 2$, let $\lambda_i \in t_\ast^*$ be in the image of the right moment maps of $\omega_i$. We introduce the notions of
\[ \lambda_1 \sim \lambda_2, \quad (\omega_1)_{\lambda_1} \sim (\omega_2)_{\lambda_2}, \quad (\omega_1)_{\lambda_1} = (\omega_2)_{\lambda_2} \] (1.12)
as follows. Regarding $\lambda_i$ as elements of $g^*$, we write $\lambda_1 \sim \lambda_2$ if they lie in the same coadjoint $G$-orbit. For the reduced forms, we write $(\omega_1)_{\lambda_1} \sim (\omega_2)_{\lambda_2}$ if there exists a $G$-equivariant symplectomorphism between them. In particular if this symplectomorphism can be made holomorphic and preserves the pseudo-Kähler structures we write $(\omega_1)_{\lambda_1} = (\omega_2)_{\lambda_2}$.

**Theorem 4.** The image of $\Phi_r$ lies inside $(t_\ast^*)_{\text{reg}}$. The connected components in $R_\ast$ are mutually isomorphic pseudo-Kähler manifolds, each of which is a copy of $Y_\ast$. They are Kähler if and only if $\lambda \in (t_\ast^*)_{\text{reg}} \cap E$. For $i = 1, 2$, suppose that $R_{\lambda_i}$ has the same number of connected components. Then $(\omega_1)_{\lambda_i} \sim (\omega_2)_{\lambda_i}$ if and only if $\lambda_1 \sim \lambda_2$, and $(\omega_1)_{\lambda_i} = (\omega_2)_{\lambda_i}$ if and only if $\lambda_1 = \lambda_2$.

By this theorem, the reduction process is independent of $\omega$ and depends uniquely on $\lambda$. Assume for simplicity that $R_\ast$ is connected. For example, this happens when $F$ is strictly convex or more generally when the gradient of $F$ is injective. By Theorem 4, (1.11) simplifies to
\[ (X_\ast, \lambda) \to (Y_\ast, \omega_\ast). \] (1.13)
Further, the process $\lambda \to \omega_\ast$ is injective.

The following theorem classifies the $G$-invariant pseudo-Kähler forms on $Y_\ast$ and shows that (1.13) is actually a bijective correspondence. The $G$-action on $Y_\ast$ preserving a pseudo-Kähler form is Hamiltonian and we let $\psi: Y_\ast \to g^*$ denote its moment map. Note that $Y_\ast = G/G^*$ (Proposition 3.1) and we write $e \in Y_\ast$ for the identity coset $eG^*$.
Theorem 5. The $G$-invariant pseudo-Kähler forms on $Y_\sigma$ are not exact and are classified by the points in $(t^*_\text{reg})\cap \mathcal{G}$, where $\mathcal{G}$ is the non-existant if $\mathcal{G} = \emptyset$. For example, take $G$ to be compact and $\sigma = \emptyset$, $P = B$. In this case $(t^*_\text{reg})\cap \mathcal{G}$ is just the interior of the Weyl chamber in $t^*$. We recover the classic result of Borel [2] that the $G$-invariant Kähler structures on $Y_\sigma = G/\mathcal{C}$ are classified by the interior points of the Weyl chamber.

Let $\omega = \sqrt{-1} \partial \overline{\partial} F$ be a $(G \times T_\sigma)$-invariant pseudo-Kähler form on $X_\sigma$, where $F$ is strictly convex. We have quantized the left $G$-action and obtained a $G$-representation $H^q_\omega$ consisting of square-integrable harmonic forms. For integral weights $\lambda \in t^*_\text{int}$ we obtain the subrepresentation $(H^q_{\lambda})_\omega$ via the right $T_\sigma$-action. On the other hand, we can first perform symplectic reduction on the right $T_\sigma$-action and obtain $(R_x, \omega_x)$ and then quantize the $G$-action on $(R_x, \omega_x)$ to obtain a $G$-representation $H^q_{(\omega_x)}$. We compare the $G$-representations $(H^q_{\lambda})_\omega$ and $H^q_{(\omega_x)}$ and show that quantization commutes with reduction [7].

Theorem 6. Let $\omega = \sqrt{-1} \partial \overline{\partial} F$ be a $(G \times T_\sigma)$-invariant pseudo-Kähler form on $X_\sigma$, with $F$ strictly convex. Then $(H^q_{\omega})_\omega \cong H^q_{(\omega_x)}$.

In other words, quantizing the $G$-action followed by taking subrepresentation via the $T_\sigma$-action coincides with performing symplectic reduction via the $T_\sigma$-action followed by quantizing the $G$-action. Other results of this nature are summarized in [17].

We outline the structure of this paper as follows. In Section 2, we review some concepts of the Lie algebra and establish the common notations that will appear throughout the paper. In Section 3, we describe the spaces $X_\sigma$ and $Y_\sigma$ in terms of a torus and its centralizer. Also, we study the $(G \times T_\sigma)$-invariant pseudo-Kähler forms on $X_\sigma$ and their moment maps, leading to the proof of Theorem 1. In Section 4, we show that $X_\sigma$ has a $(G \times A_\sigma)$-invariant measure $\mu_x$. Using the Hermitian structure on the line bundle $L$ and the measure $\mu_x$, we construct an $L^2$-structure on the sections of $L$ and prove Theorem 2. In Section 5, we apply Theorems 1 and 2 to construct a holomorphic discrete model for $G$. In Section 6, we generalize Theorem 2 to Theorem 3: the Kähler structures, holomorphic discrete series, and square-integrable holomorphic sections $H^q_\omega$ are replaced by the pseudo-Kähler structures, discrete series, and $L^2$-cohomology $H^q_\omega$. In Section 7, we perform symplectic reduction to the right $T_\sigma$-action on $X_\sigma$ and prove Theorem 4. The reduced space is the flag domain $Y_\sigma$, and we study its
pseudo-Kähler structures in Section 8. This leads to the proof of Theorem 5. Finally, in Section 9 we quantize the $G$-action on the reduced space and prove Theorem 6.

2. LIE ALGEBRAS

In this section, we review some aspects of the Lie algebra which will be used later. We also establish some common notations.

Recall that $G$ is connected semisimple with compact Cartan subgroup and that it is a real form of $G^\mathbb{C}$. There exists a compact real form $U \subset G^\mathbb{C}$ such that $K = U \cap G$ is maximal compact in $G$. Let $\mathfrak{g}^\mathbb{C} = \mathfrak{h} + \sum g_{\pm i}$ be the root space decomposition, indexed over the positive roots $\alpha_i \in \Delta^+$. There exists $\xi_{\pm i} \in \mathfrak{g}_{\pm i}$ [11, p. 421] such that

$$u = t + R(\xi_{i} - \xi_{-i}, \sqrt{-1} (\xi_{i} + \xi_{-i})).$$

(2.1)

For convenience, write

$$e_i = \begin{cases} \
1 & \text{if } \alpha_i \in \Delta^+_c, \\
\sqrt{-1} & \text{if } \alpha_i \in \Delta^+_n. 
\end{cases}$$

(2.2)

Then

$$\xi_i = e_i (\xi_{i} - \xi_{-i}), \quad \gamma_i = e_i \sqrt{-1} (\xi_{i} + \xi_{-i}) \in \mathfrak{g}.$$  

(2.3)

From (2.3), it follows that for all $x \in t$,

$$[x, \xi_i] = \sqrt{-1} \alpha_i(x) \gamma_i, \quad [x, \gamma_i] = -\sqrt{-1} \alpha_i(x) \xi_i.$$  

(2.4)

The vectors in (2.1) can be normalized so that $[\xi_i - \xi_{-i}, \sqrt{-1} (\xi_{i} + \xi_{-i})] \in \mathfrak{t}$ is identified with the root $\alpha_i \in \mathfrak{t}^*$ by the Killing form. So by $e_i^2 = \pm 1$ in (2.2), the Killing form identifies

$$[\xi_i, \gamma_i] = \begin{cases} \
\alpha_i & \text{if } \alpha_i \in \Delta^+_c, \\
-\alpha_i & \text{if } \alpha_i \in \Delta^+_n. 
\end{cases}$$

(2.5)

Define $V$ and $V_i$ by

$$\mathfrak{g} = t + V, \quad V = \sum_{\alpha_i \in \Delta^+} V_i, \quad V_i = R(\xi_{i}, \gamma_{i}).$$

(2.6)

In fact, the Cartan decomposition $\mathfrak{g} = \mathfrak{t} + \mathfrak{q}$ is obtained by

$$\mathfrak{t} = t + \sum_{\alpha_i \in \Delta^+_c} V_i, \quad \mathfrak{q} = \sum_{\alpha_i \in \Delta^+_n} V_i.$$
The Lie bracket between $t$ and $V_i$ is given by (2.4). If we take the Lie bracket between $V_i$ and $V_j$, (2.3) gives

$$[x, y]_V = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

(2.7)

Since $g$ is semisimple, its Killing form is non-degenerate. It identifies $g$ with $g^*$, so that $t^*$ and $V_i^*$ can be regarded as subspaces of $g^*$. The subspaces $\{t, V_i\}$ are pairwise orthogonal under the Killing form, so

$$[t^*, V_i^*] = (V_i^*, t^*) = (V_i^*, V_j^*) = 0, \quad i \neq j.$$  

(2.8)

Consider the vectors

$$\zeta_i^*, \gamma_i^* \in V_i^* \subset V^* \subset g^*$$

(2.9)

dual to (2.3). The coadjoint representation $\text{ad}^*: g \to \text{End} g^*$ can be computed from the above identities. Namely, (2.4) says that for all $x \in t$,

$$\text{ad}^* \zeta_i^* = - \sqrt{-1} \pi(x) \gamma_i^*, \quad \text{ad}^* \gamma_i^* = - \sqrt{-1} \pi(x) \zeta_i^*.$$  

(2.10)

For $\sigma \subset \Delta_+$, we have defined the subalgebras $t_\sigma, g_\sigma^o, g_\sigma^q$ in Section 1. We now relate them to these $V_i$. Given $x \in \Delta^+$, we write $(x, t_\sigma) = 0$ if $x$ annihilates $t_\sigma$. Otherwise, if $(x, t_\sigma) \neq 0$ for some $x \in t_\sigma$, we simply write $(x, t_\sigma) = 0$. Let $t_\sigma^+ \subset t$ be the complement of $t_\sigma$ in $t$, under the Killing form. Then

$$g^o = t + \bigoplus_{(s_i, t_\sigma)} V_i, \quad g_\sigma^o = t_\sigma^+ + \bigoplus_{(s_i, t_\sigma)} V_i, \quad g = t + V$$

(2.11)

and

$$g^o = t + V_\sigma, \quad g^q = t_\sigma^+ + \bigoplus_{(s_i, t_\sigma) \neq 0} V_i.$$  

We also introduce the concept of the relative Lie algebra cohomology. Consider in general a Lie group $R$ with a closed subgroup $S$. Restricting the coadjoint representation to $S$, we get $\text{Ad}^*: S \to \text{Aut} r^*$. We extend this representation to the exterior algebras $\wedge^r r^*$, then differentiate to get $\text{ad}^*: s \to \text{End} (\bigwedge r^*)$. We define

$$\wedge^r (x, s)^* = \left\{ x \in \wedge^r r^*; \pi(v)x = \text{ad}^* x = 0 \text{ for all } v \in s \right\}.$$  

(2.12)
Here \( \iota_v \in \wedge^{q-1} \mathfrak{r}^* \) is the interior product. Then \( \{ \wedge^q (t, s)^* \}_q \) can be identified with the \( R \)-invariant differential forms on \( R/S \), and they become a chain complex under the de Rham operator \( d \). We write \( H^q (t, s) \) for the resulting cohomology.

3. PSEUDO-KÄHLER STRUCTURES

Fix a subset \( \sigma \) of the compact simple roots and it determines a parabolic subgroup \( P \) via (1.3). By (1.4), we define the domain \( X_\sigma \subset G^C (P, P) \) as a fibration \( \pi \) over the flag domain \( Y_\sigma \). The present section studies \( X_\sigma \) and its pseudo-Kähler structures and proves Theorem 1. Recall that \( H = TA \) is a Cartan subgroup of \( G^C \). Let \( T_\sigma \) be the subtorus whose Lie algebra is the kernel of \( \sigma \). It corresponds to a subgroup \( A_\sigma \subset A \), where \( H_\sigma = T_\sigma A_\sigma \) is complex. Let \( G_\sigma \) be the centralizer of \( T_\sigma \) in \( G \) and let \( G_\sigma^{ss} \) be its commutator subgroup. There is a natural action of \( G \times H_\sigma \) on \( X_\sigma \), a fact made clearer by the following description of \( X_\sigma \).

**Proposition 3.1.** \( X_\sigma = (G/G_\sigma^{ss}) A_\sigma \) and \( Y_\sigma = G/G_\sigma \) and the fiber of \( X_\sigma \to Y_\sigma \) is \( H_\sigma \).

**Proof.** Recall that \( \sigma \subset A_\sigma^+ \). So (1.1) says that if \( \pi \in A_\sigma^+ \), then \( (\pi, t_\sigma) \not= 0 \). It follows that

\[
T \subset G_\sigma \subset K, \tag{3.1}
\]

and so \( G_\sigma \) is compact. Therefore, its commutator \( G_\sigma^{ss} \) is a compact semi-simple Lie group. Let \( P = M_\sigma A_\sigma N_\sigma \) be the Langlands decomposition (1.3). Let \( A_\sigma^+ \subset A \) be the subgroup whose Lie algebra is the orthocomplement of \( a_\sigma \) in \( a \) (under the Killing form). We have the Iwasawa decomposition \((G_\sigma^{ss})^C = G_\sigma^{ss} A_\sigma^+ (M_\sigma \cap N)\). Hence

\[
G_\sigma^{ss} A_\sigma^+ N = (G_\sigma^{ss})^C N_\sigma = \left( (G^C) (G^C)^C \right) N_\sigma = (M_\sigma A_\sigma, M_\sigma A_\sigma) N_\sigma = (P, P). \tag{3.2}
\]

By (3.2), \( X_\sigma = (G/G_\sigma^{ss} ((A/A_\sigma^+) = (G/G_\sigma^{ss}) A_\sigma \). The fiber of \( X_\sigma \) over \( Y_\sigma \) is the same as the fiber of \( \tilde{P} \) over \( (P, P) \), which is \( H_\sigma \). So \( Y_\sigma = G/G_\sigma \) and the proposition is proved. \( \blacksquare \)
With this proposition, the right action of $H_\alpha$ on $X_\alpha$ is clear: $A_\alpha$ acts by self-multiplication on $A_\alpha \subset (G/G_\alpha^*) A_\alpha$, while $T_\alpha$ acts on the right of $G/G_\alpha^*$ because it commutes with $G_\alpha^*$.

Let $\omega$ be a $(G \times T_\alpha)$-invariant closed $(1, 1)$-form on $X_\alpha$. To solve $\omega = \sqrt{-1} \partial \bar{\partial} F$ for $F$ we need the next two propositions. Here $H^\bullet_{\partial}(X_\alpha)$ denotes the $G$-invariant de Rham cohomology of $X_\alpha$ with real or complex coefficients.

**Proposition 3.2.** $H^1_{\partial}(X_\alpha) = H^2_{\partial}(X_\alpha) = 0$.

**Proof.** By the previous proposition, $X_\alpha = (G/G_\alpha^*) A_\alpha$. But $A_\alpha$ is contractible. So it suffices to consider $G$-invariant forms on the space $G/G_\alpha^*$, which can be identified with $\wedge^1 g_\alpha^* \otimes (2.12)$. Therefore, we can prove the proposition by showing that $H^1(g, g_\alpha^*) = H^2(g, g_\alpha^*) = 0$.

Note that $g$ is semisimple. So the Whitehead lemma [8] says that its Lie algebra cohomology satisfies

$$H^1(g) = H^2(g) = 0. \quad (3.3)$$

Consider a non-zero $\pi \in \wedge^1 (g, g_\alpha^*)^\ast$. We regard $\pi$ as an element of $\wedge^1 g^\ast$. Then $d\pi \neq 0$, because $H^1(g) = 0$ by (3.3). This proves that $H^1(g, g_\alpha^*) = 0$.

Next let $\omega \in \wedge^2 (g, g_\alpha^*)^\ast$ and suppose that $d\omega = 0$. Since $\omega \in \wedge^2 g^\ast$, (3.3) says that $\omega = d\beta$ for some $\beta \in \wedge^1 g^\ast$. To complete the proof, we need to show that $\beta \in \wedge^1 (g, g_\alpha^*)^\ast$. In other words, we check that for all $v \in g_\alpha^*$,

$$\langle \beta, v \rangle = \text{ad}^\ast_v \beta = 0. \quad (3.4)$$

By (3.1), $G^\ast$ is compact, so $G_\alpha^*$ is semisimple. Hence up to linear combination, $v \in g_\alpha^*$ can be written as $v = [x, y]$ for $x, y \in g_\alpha^*$. Then

$$\langle \beta, v \rangle = \langle \beta, [x, y] \rangle$$
$$= d\beta(x, y)$$
$$= \omega(x, y)$$
$$= (r(x) \omega)(y). \quad (3.5)$$

Since $\omega \in \wedge^2 (g, g_\alpha^*)^\ast$ and $x \in g_\alpha^*$, it follows that $r(x) \omega = 0$. Therefore, (3.5) vanishes.

For $x \in g_\alpha^*$ and $y \in g$, we apply the same argument as (3.5) and get

$$\langle \text{ad}^\ast_v \beta, y \rangle = \langle \beta, [x, y] \rangle = (r(x) \omega)(y) = 0.$$

Hence $\text{ad}^\ast_v \beta = 0$. This proves (3.4), and hence the proposition.

As a side remark, we note that in the statement of Proposition 3.2 the subscript $G$ is necessary for the vanishing of cohomology (unless of course

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$G$ is compact so that the $G$-invariant cohomology coincides with the usual cohomology): For non-compact $G$, the maximal compact subgroup $K$ has a center $Z$ of positive dimension. By (3.1), $G_\sigma = K$. Write the Cartan decomposition as $G = QK$, so that $G/G_\sigma = Q \times (K/G_\sigma)$. Since $Q$ is contractible,

$$H^q(X_\sigma) = H^q(G/G_\sigma) = H^q(K/G_\sigma) = H^q(\mathfrak{t}, g_\sigma^*)^\ast.$$ 

Since $\sigma \subset A^\ast$, we get $Z \subset \mathfrak{t}$, so (2.11) implies that $g_\sigma^* \cap Z = 0$. Hence for all $q \leq \dim \mathfrak{t}$, the non-zero elements of $\wedge^q Z^\ast \subset \wedge^q (\mathfrak{t}, g_\sigma^*)^\ast$ have non-trivial cohomology classes in $H^q(\mathfrak{t}, g_\sigma^*)^\ast$.

The next proposition deals with a Dolbeault cohomology taken over $(G \times T_\sigma)$-invariant forms, as indicated by its subscript.

**Proposition 3.3.** $H^{0,1}_{G/T_\sigma}(X_\sigma) = 0$.

**Proof.** The vector space $(g/g_\sigma^*)^\ast$, $\mathfrak{t}$ acquires the complex structure as a tangent space for $X_\sigma$. It contains $h_\sigma$ as a complex subspace, and so $(g/g_\sigma^* + \mathfrak{t}) = g/g_\sigma^*$ is a complex vector space. Therefore, it makes sense to define $\wedge^{0,1}(g/g_\sigma^*)^\ast$. A basis for this space is

$$\{ u_i = \zeta_i^* - \sqrt{-1} \gamma_i^* \}_{(\sigma_i, t_i) \neq 0} \subset \wedge^{0,1} (g/g_\sigma^*)^\ast, \tag{3.6}$$

where $\zeta_i^*, \gamma_i^* \in g_\sigma^*$ are the vectors in (2.9). By (2.10), for $x \in \mathfrak{t}$,

$$\text{ad}_x^* u_i = \sqrt{-1} x_i(x) u_i. \tag{3.7}$$

We apply Proposition 3.1 and express the $G$-invariant $(0,1)$-forms on $X_\sigma$ in a manner similar to (2.12), namely

$$\Omega^{0,1}_G(X_\sigma) = \left\{ \sum_i f_i w_i \in C^\infty(A_\sigma) \otimes \left( \bigwedge^{0,1} (g/g_\sigma^*)^\ast \otimes \bigwedge^{0,1} h_\sigma^* \right); \right. \ \ \ \text{ad}_x^* w_i = 0 \text{ for all } x \in g^\sigma \left. \right\}. \tag{3.8}$$

Let $z \in \Omega^{0,1}_G(X_\sigma)$. We want to express $z$ in terms of (3.8) but we omit the harmless linear combination $\sum_i f_i$ for convenience. So $z = f(u + v)$ where $f \in C^\infty(A_\sigma)$, $u \in \wedge^{0,1}(g/g_\sigma^*)^\ast$, and $v \in \wedge^{0,1} h_\sigma^*$. Here $u$ decomposes further to $u_i$ of (3.6), indexed over $(\sigma_i, t_i) \neq 0$. Let $L$ and $R$ be the left and right actions, and let $x_i = \exp \chi_i$ be the character of $\mathfrak{x}_i$. For all $t \in T_\sigma$,

$$R_t^* u_i = L_t^* R_t^* u_i \text{ \ by left invariance of } u_i \text{ \ by (3.7),}$$

$$= \text{Ad}_t^* u_i \text{ \ by left invariance of } u_i \text{ \ by (3.7).}$$
So each \( u_i \) transforms by \( x_i \) under the right \( T_x \)-action. On the other hand, since \( H_x \) is abelian, \( v \) is invariant under the right \( T_x \)-action. We conclude that if \( z = f(u + v) \) is \((G \times T_x)\)-invariant, then \( u = 0 \) and so \( z = fv \in C^\infty(A_x) \otimes \Lambda^{0,1} h_+^* \) with respect to (3.8). Indeed, since \( G_x \) commutes with \( H_x \) we get \( \text{ad}_x v = 0 \) for all \( x \in g^* \), so \( f \) satisfies the requirement for \( \Omega^0_{\mathbb{C}}(X_x) \) in (3.8).

The subcomplex \( C^\infty(A_x) \otimes \Lambda^{0,1} h_+^* \) of (3.8) can be identified with the \( T_x \)-invariant \((0, 1)\)-forms on \( H_x \). Since \( H_x \) is a Stein space, its \((0, 1)\)-Dolbeault cohomology vanishes. Consequently, if \( z \in C^\infty(A_x) \otimes \Lambda^{0,1} h_+^* \) is \( \partial \)-closed, then \( z \) has to be \( \partial \)-exact. This proves the proposition.

Let \( \omega \) be a \((G \times T_x)\)-invariant closed \((1, 1)\)-form on \( X_x \). We now apply Propositions 3.2 and 3.3 to obtain a potential function \( F \) for \( \omega \).

**Proposition 3.4.** Every \((G \times T_x)\)-invariant closed \((1, 1)\)-form on \( X_x \) can be written as \( \omega = -\sqrt{-1} \partial \bar{\partial} F \).

**Proof.** Since \( \omega \) is closed, Proposition 3.2 says that \( \omega = d\beta \) for some real 1-form \( \beta \). Since \( \beta \) is real, we write \( \beta = \alpha + \bar{\alpha} \), where \( \alpha \in \Omega^0_{\mathbb{C}}(X_x) \). Then, \( \omega \) being a \((1, 1)\)-form implies that \( \partial \bar{\partial} \alpha = 0 \). By Proposition 3.3, \( \alpha = \delta f \) for some \( f \in C^\infty_c(X_x) \). Define a real-valued function \( F = -\sqrt{-1} (f - \bar{f}) \). Then

\[
\omega = d\beta = \bar{\partial} f + \partial \bar{f} = \partial \bar{\partial} f = -\sqrt{-1} \partial \bar{\partial} F.
\]

This proves the proposition.

We remark that in the above proposition right \( T_x \)-invariance of \( \omega \) is both necessary and sufficient for the existence of the potential function \( F \). The necessity of right \( T_x \)-invariance is proved in [3].

Given \( \xi \in \mathfrak{g} \), we let \( \xi^* \) denote the infinitesimal vector field on \( X_x \) obtained from the left \( G \)-action. If \( J \) is the almost complex structure and \( J \xi \in \mathfrak{a} \) for \( \xi \in \mathfrak{t} \), we let \( (J \xi)^* = J (\xi^*) \). Let \( \omega \) be a \((G \times T_x)\)-invariant \((1, 1)\)-form on \( X_x \). By \( G \)-invariance, it suffices to study \( \omega_a \) for \( a \in A_x \subset X_x \). This is done in the next proposition.

The \( G \)-action preserving \((X_x, \omega)\) is Hamiltonian, with moment map

\[ \Phi: X_x \to \mathfrak{g}^*. \] (3.9)

Recall that \( \mathfrak{g} = t + V \) and \( V \) splits into \( V_i \) in (2.6).

**Proposition 3.5.** Let \( \omega \) be a \((G \times T_x)\)-invariant \((1, 1)\)-form on \( X_x \) and \( a \in A_x \). For \( i \neq j \), \( \omega(h_i^*, V_j^*)_a = \omega(V_i^*, V_j^*)_a = 0 \).
Proof. The complex structure of $X_\sigma$ sends $t_\sigma^i$ to $a_\sigma^i$. Since $\omega$ is of type (1, 1), it suffices to check that for all $a \in A_\sigma$,
\[ \omega(t_\sigma^i, V^j_\sigma)_a = \omega(V^i_\sigma, V^j_\sigma)_a = 0, \quad i \neq j. \quad (3.10) \]
By the Killing form, $V^*_\sigma \subset g^*$ satisfies (2.8). Let $\theta_i : g^* \to V^*_\sigma$ be the corresponding projection. We now prove the first part of (3.10). Let $x \in t_\sigma$ and $y \in V_i$. For all $t \in T_n$,
\[ \omega(x^\sigma, y^\sigma)_a = (\Phi(t a), [x, y]) = (\Phi(a), \text{Ad}_t[x, y]) \quad (3.11) \]
because $\Phi$ is $G$-equivariant. According to (2.4), $[x, y] \in V_i$ and $\text{Ad}_t$ acts on the two-dimensional $V_i$ by rotation. So $\text{Ad}_t[x, y] \in V_i$. Then (3.11) becomes
\[ \omega(x^\sigma, y^\sigma)_a = (\partial_t \Phi(a), [x, y]) \quad (3.12) \]
due to (2.8). On the other hand, since $H_\sigma$ is abelian,
\[ \omega(x^\sigma, y^\sigma)_a = \omega(x^\sigma, y^\sigma)_a = \{ R^*_t(\omega(x^\sigma, y^\sigma)) \}_a, \quad (3.13) \]
where $R$ denotes right action. Since $\omega, x^\sigma, y^\sigma$ are all right $T_n$-invariant, the function $\omega(x^\sigma, y^\sigma) \in C^\infty(X_\sigma)$ is right $T_n$-invariant too. So (3.13) becomes
\[ \omega(x^\sigma, y^\sigma)_a = \{ R^*_t(\omega(x^\sigma, y^\sigma)) \}_a \]
\[ = \omega(x^\sigma, y^\sigma)_a \quad \text{by right } T_n\text{-invariance} \]
\[ = (\Phi(a), [x, y]) \]
\[ = (\partial_t \Phi(a), [x, y]) \quad \text{by (2.8) and } [x, y] \in V_i. \quad (3.14) \]
By (3.12) and (3.14),
\[ (\partial_t \Phi(a), \text{Ad}_t[x, y]) = (\partial_t \Phi(a), [x, y]). \quad (3.15) \]
This equation is valid for all $t \in T_n$, $x \in t_\sigma$, and $y \in V_i$. Fix $x, y$ with $0 \neq [x, y] \in V_i$. As $t$ varies in $T_n$, $\text{Ad}_t[x, y]$ traces out a circle in the two-dimensional $V_i$. So for (3.15) to be valid, $\partial_t \Phi(a) \in V^*_\sigma$ has to be 0. Then (3.15) vanishes, and this implies the vanishing of (3.11), (3.12), (3.13), and (3.14). This proves the first part of (3.10).
Since $\partial_t \Phi(a) = 0$ for all $t$, by (2.8) we get $\Phi(a) \in t^*_\sigma$. Let $t^+_\sigma$ be the orthocomplement of $t_\sigma$ in $t$, via the Killing form. By (2.11), $T^+_\sigma \subset G_n^\sigma$. So the left action of $T^+_\sigma$ fixes $a$ and the statement $\Phi(a) \in t^*_\sigma$ can be sharpened to
\[ \Phi(a) \in t^*_\sigma. \quad (3.16) \]
We next show the second part of (3.10). Let \( x \in V_i \) and \( y \in V_j \), with \( i \neq j \). Then

\[
\omega(x^*, y^*)_{ij} = (\Phi(a), [x, y]) \in (t_a^*, V) \quad \text{by (2.7), (3.16)}
\]

\[
= 0 \quad \text{by (2.8)}.
\]

This proves the second part of (3.10). The proposition follows. \( \blacksquare \)

We next study the moment map (3.9). We shall describe it by the potential function \( F \). Since \( \Phi \) is \( G \)-equivariant and \( X_\# = (G/G_m) A_\# \), \( \Phi \) is determined entirely by its restriction to \( A_\# \subset X_\# \). By (3.16), we can write

\[
\Phi: A_\# \rightarrow t_a^*.
\]

On the other hand, the gradient of the potential function \( F \in \mathcal{C}^\infty(A_\#) \) is

\[
F': A_\# \rightarrow a^*_\# \cong t_a^*,
\]

where \( a^\# \cong t_a \) by the complex structure. The maps (3.17) and (3.18) are related by

**Proposition 3.6.** For \( a \in A_\# \), \( \Phi(a) = \frac{1}{\pi} F'(a) \).

**Proof.** Let \( i: H_\# \hookrightarrow X_\# \) be the natural imbedding. By (3.16), the moment maps of \( \omega \) and \( i^* \omega \) coincide when restricted to \( H_\# \). So we may consider \( \Phi \) as the moment map for the \( T_\# \)-invariant form \( i^* \omega \). To compute \( \Phi \), we introduce coordinates

\[
H_\# = T_\# \times A_\# = (R^r/Z^r) \times R^r
\]

\[
= \{ z_j = [x_i] + \sqrt{-1} y_i; i = 1, ..., r \}. \quad (3.19)
\]

By \( T_\# \)-invariance \( F(z) = F(y) \), and we get

\[
i^* \omega = \sqrt{-1} \partial \bar{\partial} F
\]

\[
= \sqrt{-1} \frac{\partial}{\partial z_i} \frac{\partial}{\partial \bar{z}_j} F \, dz_i \wedge d\bar{z}_j
\]

\[
= \frac{1}{2} \frac{\partial^2 F}{\partial y_i \partial y_j} \, dx_i \wedge dy_j. \quad (3.20)
\]

The \( T_\# \)-invariant 1-form \( \beta = -\frac{1}{2} (\partial F \partial y_i) dx_i \), satisfies \( i^* \omega = d\beta \). Write \( v = (v_i) \in t_a \cong R^r \). For all \( y \in A_\# \) and \( v \in t_a \),
\((\Phi(y), v) = -\langle \beta, v^2 \rangle_y \quad \text{by [1, Theorem 4.2.10]}\)
\[
\left( \frac{1}{2} \frac{\partial F}{\partial y_i} dx_i, v \frac{\partial}{\partial x_j} \right)_y = \frac{1}{2} \frac{\partial F}{\partial y_i} (y) v_i
\]
\[
= \left( \frac{1}{2} F'(y), v \right).
\]
This proves the proposition.

**Proof of Theorem 1.** Let \(\omega\) be a \((G \times T_a)\)-invariant closed \((1, 1)\)-form on \(X_a\). By Proposition 3.4 we get \(\omega = \sqrt{-1} \partial \bar{\partial} F\), and the moment map \(\Phi = \frac{1}{2} F'\) is given by Proposition 3.6. The Kähler and pseudo-Kähler conditions on \(\omega\) remain to be studied. By \(G\)-invariance, it suffices to consider \(\omega_a\) for \(a \in A_a\). In view of Proposition 3.5, we can study \(\omega_a\) on \(h_a^\#\) and each \(V_i^\#\) separately.

We first consider \(\omega_a\) on \(h_a^\#\). Let \(\tau\) be the imbedding of Proposition 3.6. Then \(\omega_a\) is Kähler or pseudo-Kähler exactly when \(\tau^* \omega\) is Kähler or pseudo-Kähler. By the Hessian matrix \((\partial^2 F/\partial y_i \partial y_j)_y\) from (3.20), \(\tau^* \omega\) is Kähler or pseudo-Kähler exactly when \(F\) is strictly convex or non-degenerate.

We next restrict \(\omega_a\) to \(V_i^\#\). Here \(i\) is indexed according to \(\pi_i \in \mathbb{A}^+\). If \((\pi_i, t_a) = 0\), then (2.11) says that \(V_i \subset g_a^\#\), so \((V_i^\#)_{\omega_a} = 0\) because \(G_a^\#\) fixes \(a\). Therefore, it suffices to consider \((\pi_i, t_a) \neq 0\). By (2.5) and Proposition 3.6, the vectors \(\zeta_i, \gamma_i \in V_i\) satisfy
\[
\omega(\zeta_i^\#, \gamma_i^\#)_a = (\Phi(a), [\zeta_i, \gamma_i]) = \pm \left( \frac{1}{2} F'(a), \pi_i \right). \tag{3.21}
\]
Recall the definition of \((1^a_{\#})_{\text{reg}} \equiv (a^*_{\#})_{\text{reg}}\) from (1.2). The last expression of (3.21) is non-zero if and only if \(\frac{1}{2} F'(a)\) is not perpendicular to \(\pi_i\), or equivalently \(\frac{1}{2} F'(a) \in (a^*_{\#})_{\text{reg}}\). We determine the sign \(\pm\) by (2.5). If \(\pi_i \in \mathbb{A}^+_a\), then (3.21) is positive if and only if \(\left( \frac{1}{2} F'(a), \pi_i \right) > 0\). If \(\pi_i \in \mathbb{A}^+_a\), then (3.21) is positive if and only if \(\left( \frac{1}{2} F'(a), \pi_i \right) < 0\). So (3.21) is positive for all \(\pi_i \in \mathbb{A}^+\) exactly when \(\frac{1}{2} F'(a) \in (a^*_{\#})_{\text{reg}} \cap \mathbb{C}\). This proves Theorem 1.

### 4. GEOMETRIC QUANTIZATION

Let \(\omega\) be a \((G \times T_a)\)-invariant Kähler form on \(X_a\). The purpose of this section is to apply the standard scheme of geometric quantization [14] to \((X_a, \omega)\), and prove Theorem 2.

We need a nice invariant measure on \(X_a\) to perform integration later. This is given by the next proposition.
Proposition 4.1. There exists a \((G \times A)\)-invariant measure on \(X_a\) which is unique up to scalar.

Proof. Recall that \(X_a = (G/G_m^a) A_a\). Since \(G\) and \(G_m^a\) are semisimple, they are in particular unimodular, so \(G/G_m^a\) has a \(G\)-invariant volume form \([12, p. 89]\). Taking its product with the Haar measure on \(A_a\) creates a \((G \times A)_\cdot\)-invariant measure on \(X_a\). Further, since \(G \times A\) acts transitively on \(X_a\), such a measure has to be unique up to scalar. 

By Theorem 1, \(\omega\) has a strictly convex potential function \(F\). As explained in the Introduction, \(\omega\) corresponds to a pre-quantum line bundle \(L\) with compatible connection \(\nabla\) and Hermitian structure \((\cdot , \cdot )\). The \((G \times T)_\cdot\)-action on \(X_a\) lifts to a \((G \times T)_\cdot\)-representation on the holomorphic sections on \(L\).

Proposition 4.2. There exists a \((G \times T)_\cdot\)-invariant non-vanishing holomorphic section \(s_0\) on \(L\) satisfying \((s_0, s_0)_a = e^{-F(a)}\) for all \(a \in A_a \subset X_a\).

Proof. Since the Chern class of \(L\) is \([\omega]\) = 0, we can pick a \((G \times T)_\cdot\)-invariant non-vanishing smooth section \(s\). Let \(\gamma = \sqrt{-1} \frac{dz}{2}\), so that by the definition of the curvature form \(d\gamma = \omega\). Since \(s\) is \(G\)-invariant, so are \(\nabla s\) and \(s\). Let \(\gamma = -\sqrt{-1} dF\), so that \(d\gamma = \sqrt{-1} \gamma = \omega\). Hence \(\gamma\) and \(\gamma\) are complex \(G\)-invariant 1-forms satisfying \(d(\gamma - \gamma) = 0\). By Proposition 3.2, there exists a \(G\)-invariant complex-valued function \(f \in C^\omega(A_a)\) such that

\[
\gamma - \gamma = df.
\]

Let \(s_0 = e^{-\sqrt{-1} F/2}\). Then

\[
\nabla s_0 = \nabla (e^{-\sqrt{-1} F/2}) = e^{-\sqrt{-1} F/2} ds + e^{-\sqrt{-1} F/2} \nabla s = -\sqrt{-1} ds \cdot s_0 - \sqrt{-1} \gamma s_0 = -\sqrt{-1} \gamma s_0.
\]

(4.1)

It follows from \(s_0 = e^{-\sqrt{-1} F/2}\) that \(s_0\) is non-vanishing and \((G \times T)_\cdot\)-invariant. Since \(\gamma\) is a \((1, 0)\)-form, it follows from \(\nabla s_0 = -\sqrt{-1} \gamma s_0\) that \(\nabla s_0\) annihilates anti-holomorphic vector fields. So \(s_0\) is holomorphic.

Since \(s_0\) is \(G\)-invariant, \((s_0, s_0)\) is determined by its value on \(A_a \subset X_a\). This is found by differentiating along the \(A_a\) direction. We write \(z = [x] + \sqrt{-1} y\) as in (3.19). Since \(\gamma = -\sqrt{-1} dF\), (4.1) gives

\[
\nabla s_0 = \frac{-\sqrt{-1}}{2} \frac{dF}{\partial y} dz s_0.
\]

(4.2)
We evaluate \((s_0, s_0)\) against \(\partial_{\bar{y}}\) and get
\[
\frac{\partial}{\partial y} (s_0, s_0) = (\sqrt{-1} V_{\bar{y}0} s_0, s_0) + (s_0, \sqrt{-1} V_{\bar{y}0} s_0)
\]
since \(s\) is holomorphic
\[
= (-\frac{1}{2} \frac{\partial F}{\partial y} s_0, s_0) + (s_0, -\frac{1}{2} \frac{\partial F}{\partial y} s_0) \quad \text{by (4.2)}
\]
\[
= -\frac{\partial F}{\partial y} (s_0, s_0).
\]
Replacing \(s_0\) by a constant multiple if necessary, it follows that \((s_0, s_0) = e^{-F}.\)
Hence we have the proposition. \(\square\)

Let \(\mu_Y\) be the \((G \times A_\times)\)-invariant volume form on \(X\) given by Proposition 4.1. We define an \(L^2\)-structure on the holomorphic sections \(s\) on \(L\) via (1.8). The holomorphic sections which converge under this \(L^2\)-structure are said to be square-integrable and are denoted by \(H_\times\). Since the \(L^2\)-structure is \((G \times T_\times)\)-invariant, \(H_\times\) is a unitary \((G \times T_\times)\)-representation. For an integral weight \(\lambda \in \mathfrak{t}_+^\times\), let \(H(L)\), be the holomorphic sections which transform by \(\lambda\) under the right \(T_\times\)-action. We want to evaluate the conditions on \(\lambda\) in which \(H(L)\), and prove Theorem 2.

We think of the integral weight \(\lambda\) as an element of \(h_\times^* \otimes \mathfrak{t}_+^\times\), or \(a_\times^*\). Thus \(\lambda \in h_\times^*\) defines a homogeneous line bundle over \(Y_\times\).

Let \(L_\lambda \rightarrow Y_\times\).

Let \(C^\infty (Y_\times, L_\lambda)\) be the smooth sections on \(L_\lambda\). There is a natural assignment
\[
s \in C^\infty (Y_\times, L_\lambda) \mapsto f_s \in C^\infty (X_\times).
\]
(4.3)

Here \(f_s\) transforms by \(\lambda \in h_\times^*\) under the right \(H_\times\)-action. Recall from (1.9) that \(\chi = e^{t}\) is a multiplicative homomorphism from \(H_\times\) to \(C^\times\). Its restriction to \(A_\times\) is
\[
\chi : A_\times \rightarrow \mathbb{R}^+.
\]
(4.4)
The right action \(R\) of \(a \in A_\times\) gives
\[
R_a^\times (f_s, \tilde{f}_t) = (\bar{\chi} (a) f_s)(\bar{\chi} (a) \tilde{f}_t) = \chi (a)^2 f_s, \tilde{f}_t,
\]
(4.5)
for all \(s, t \in C^\infty (Y_\times, L_\lambda)\).

Define a \(G\)-invariant function
\[
\chi_A \in C^\infty (X_\times)
\]
for all \(s, t \in C^\infty (Y_\times, L_\lambda)\).
simply by extending (4.4) to \(X_a\) by \(G\)-invariance. The function \(\chi_A\) is non-vanishing, so it makes sense to consider its negative power \(\chi_A^{-2} \in \mathcal{O}^\infty(X_a)\).

From \(\pi\) of (1.4), we get \(\pi^*: \mathcal{O}^\infty(Y_a) \to \mathcal{O}^\infty(X_a)\).

**Proposition 4.3.** A \(G\)-invariant Hermitian structure \((\cdot, \cdot)^{t_\|}_{\mathcal{O}}\) on \(\mathcal{O}^\infty(Y_a, L_\|)\) is given by \(\pi^*(s, t)t_\| = f_s f_t \chi_A^{-2}\). It is unique up to scalars.

**Proof.** We claim that \(f_s f_t \chi_A^{-2}\) is right \(H^r\)-invariant. By (4.5), it is right \(A^r\)-invariant. So it only remains to check the right \(T^r\)-action.

Since \(\chi_A^{-2}\) is defined to be \(G\)-invariant, it is necessarily right \(T^r\)-invariant. So it suffices to consider \(f_s f_t\). Note that \(\pi^*: T^r \to S^1 \subset \mathbb{C}^\times\). For all \(\theta \in T^r\),

\[
R^r_\theta(f_s f_t) = (\chi(\theta) f_s)(\chi(\theta) f_t) = \chi(\theta) \frac{f_s f_t}{\chi(\theta)} = f_s f_t.
\]

This shows that \(f_s f_t\) is right \(T^r\)-invariant and so is \(f_s f_t \chi_A^{-2}\). We conclude that \(f_s f_t \chi_A^{-2}\) is right \(H^r\)-invariant, as claimed.

Recall that in (1.6) we define \(U_F \subset a^r\) to be the image of \(\frac{1}{2} F\). The following proposition will be helpful in proving Theorem 2.

**Proposition 4.4.** The integral \(\int_{A^r} \chi(a)^2 e^{-F(a)} da\) converges if and only if \(a^r \in \mathbb{U}_F\).

**Proof.** We change the variables by the diffeomorphism

\[
e^c a = a^r, \quad e^{c} = a.
\]

By (1.9), \(\chi(a)^2 = \exp(2\lambda, v)\) whenever \(e^c = a\). Let \(dV\) be the Lebesgue measure of \(a^r\). We identify \(F(a)\) with \(F(v)\) via (1.5). Then

\[
\int_{A^r} \chi(a)^2 e^{-F(a)} da = \int_{a^r} \exp(2\lambda, v) - F(v)) dV.
\]

Since \(F\) is strictly convex, [5, Appendix] says that the RHS of (4.6) converges if and only if \(a^r \in U_F\). Hence we have the proposition.

**Proof of Theorem 2.** Consider the natural fibrations

\[
\pi_1: X_a \to G/G_{\text{ss}}, \quad \pi_2: G/G_{\text{ss}} \to G/G^r = Y_a.
\]

Thus we have that (1.4) is \(\pi = \pi_2 \cdot \pi_1\).
Let \( s \in H(L) \). We want to determine if \( s \) is square-integrable with respect to (1.8). By Proposition 4.2, we write \( s = f_{s_0} \), where \( f \in H(X_\alpha) \). The \((G \times A_\alpha)\)-invariant measure of Proposition 4.1 is of the form \( \mu_X = dg \, da \), where \( dg \) is the \( G \)-invariant measure on \( G/G^*_\alpha \) and \( da \) is the Haar measure on \( A_\alpha \). We have

\[
\int_{X_\alpha} (s, s) \, \mu_X = \int_{g \in (G/G^*_\alpha) \cdot A_\alpha} (f_{s_0}, f_{s_0})_{g} \, dg \, da
\]

\[
= \int_{g \in (G/G^*_\alpha) \cdot A_\alpha} (ff)_{g} \, e^{-F(\alpha)} \, dg \, da. \tag{4.8}
\]

Since \( f \) is holomorphic and transforms by \( \lambda \in t^*_\alpha \) under the right \( T_\alpha \)-action, it necessarily transforms by the complexified \( \lambda \in h^*_\alpha \) under the right \( H_\alpha \)-action.
So \( f \) is in the image of (4.3), i.e., \( f = f_u \) for some holomorphic section \( u \) on \( L_\alpha \).

By Proposition 4.3, Eq. (4.8) becomes

\[
\int_{X_\alpha} (s, s) \, \mu_X = \int_{g \in (G/G^*_\alpha) \cdot A_\alpha} \pi^*(u, u)^{L_\alpha} \, \chi(\alpha)^2 \, e^{-F(\alpha)} \, dg \, da
\]

\[
= \int_{G/G^*_\alpha} \pi^*(u, u)^{L_\alpha} \, \int_{A_\alpha} \chi(\alpha)^2 \, e^{-F(\alpha)} \, da, \tag{4.9}
\]

where \( \pi^* \) is the fibration in (4.7). The flag domain \( Y_\alpha \) has a \( G \)-invariant measure \( \mu_Y \) [9]. The fiber of \( \pi^* \) is \( T_\alpha \), which is compact. So by [12, Proposition 1.13, p. 95],

\[
\int_{G/G^*_\alpha} \pi^*(u, u)^{L_\alpha} \, dg = \int_{\gamma \in Y_\alpha} (u, u)^{L_\gamma} \, \mu_Y. \tag{4.10}
\]

By (4.8), (4.9), and (4.10),

\[
\int_{X_\alpha} (s, s) \, \mu_X = \int_{\gamma \in Y_\gamma} (u, u)^{L_\gamma} \, \mu_Y \int_{A_\alpha} \chi(\alpha)^2 \, e^{-F(\alpha)} \, da. \tag{4.11}
\]

We may assume that \( \lambda + \rho \in (1^*)_{\text{reg}} \), for otherwise there is no discrete series in \( \Theta_{\lambda + \rho} \). If \( \lambda \not\in U_\rho \), then Proposition 4.4 says that \( \int_{A_\alpha} \chi(\alpha)^2 \, e^{-F(\alpha)} \, da \) diverges and so Eq. (4.11) diverges.

Conversely, suppose that \( \lambda \in U_\rho \). By Proposition 4.4, \( \int_{A_\alpha} \chi(\alpha)^2 \, e^{-F(\alpha)} \, da \) converges. Also, Theorem 1 says that \( \lambda \in U_\rho \subset (\mathfrak{a}_2^*)_{\text{reg}} \cap \mathfrak{g} \). By [9], the holomorphic sections \( u \) on \( L_\alpha \) converge in (4.11) and form the discrete series \( \Theta_{\lambda + \rho} \). Theorem 2 follows.
A unitary \( G \)-representation is said to be a holomorphic discrete model if it contains every holomorphic discrete series exactly once. We now apply Theorems 1 and 2 to construct such a representation.

Let \( W \) be a real vector space of dimension \( r \) and let \( \lambda_1, ..., \lambda_r \) be a basis of its dual \( W^* \). The following proposition defines a function \( F \in C^\infty(W) \) and studies its gradient \( F' \). Let \( W \) be a real vector space of dimension \( r \) and let \( \lambda_1, ..., \lambda_r \) be a basis of its dual \( W^* \).

**Proposition 5.1.** Let \( F(y) = \sum_i \exp(\lambda_k, y) \), for \( y \in W \). Then \( F(0) = \sum_i \lambda_k \), and the image of \( F' \) is all positive linear combinations of \( \lambda_1, ..., \lambda_r \). Also, \( F \) is strictly convex.

**Proof.** The basis \( \{\lambda_k\} \) identifies \( W \) with \( \mathbb{R}^r \) and introduces coordinates \( y_1, ..., y_r \) on \( W \). Under such coordinates, \( F \) becomes \( F(y) = \sum_i \exp(y_i) \). Its gradient is evaluated at \( y_1, ..., y_r \) at \( (\exp(y_i))_i \) which is identified with the positive linear combinations of \( \lambda_1, ..., \lambda_r \). In particular, \( F(0) = (\exp(0))_i \), which is identified with the element \( \sum_i \lambda_k \in W^* \).

The Hessian matrix of \( F \) is \( \left( (\partial^2 F/\partial y_i \partial y_j) \right)_{ij} \), and this is a diagonal matrix with entries \( \exp(y_i) \). It is a positive definite matrix, so \( F \) is strictly convex. Hence we have the proposition.

Let \( \sigma = A_c^* \). Recall that \( (t^*_{\sigma})_{\text{reg}} \cong (a^*)_{\text{reg}} \) and that \( \emptyset \) are defined in (1.2) and (1.7). We assume that \( \emptyset \neq \emptyset \) in this section. Let \( \lambda_1, ..., \lambda_r \in a^* \) be the fundamental weights for the cone \( (a^*)_{\text{reg}} \cap \emptyset \), namely

\[
(a^*)_{\text{reg}} \cap \emptyset = \left\{ \sum_i c_i \lambda_i; c_i > 0 \right\}.
\]

We define \( F_\sigma \in C^\infty(a_\sigma) \) by \( F_\sigma(y) = \sum_i \exp(\lambda_i, y) \). By Proposition 5.1, the image of \( F_\sigma \) is \( (a^*)_{\text{reg}} \cap \emptyset \) and \( F_\sigma \) is strictly convex. By (1.5), we identify it with \( F_\sigma \in C^\infty(A_\sigma) \). By Theorem 1, \( \omega_\sigma = -\partial \bar{\partial} F_\sigma \) is Kähler. By Theorem 2, the representation \( H_{\omega_\sigma} \) contains every holomorphic discrete series \( \Theta_{\lambda+\rho} \) in which \( \lambda \in U_F = (a^*)_{\text{reg}} \cap \emptyset \).

The holomorphic discrete series \( \Theta_{\lambda+\rho} \) are parametrized by integral weights \( \lambda \) in which \( \lambda + \rho \in (t^*)_{\text{reg}} \) and \( \lambda \) lies in the region

\[
\{ x \in a^*; (A_c^+, x) \geq 0, (A_c^+, x + p) < 0 \}.
\]
This region is contained in \( \mathcal{C} \). Observe that \( \{ (a_{\sigma}^*)_{\mu} : \mathcal{C} \subset A_{\sigma}^* \} \) is a partition of \( \mathcal{C} \). Therefore, as \( \sigma \) varies over all subsets of \( A_{\sigma}^* \),

\[
\bigoplus_{\sigma} H_{a_{\sigma}^*}
\]

is a holomorphic discrete model of \( G \).

6. \( L^2 \)-COHOMOLOGY

The space of holomorphic sections on the pre-quantum line bundle has allowed us to construct the holomorphic discrete series of \( G \). In this section, we consider the \( L^2 \)-cohomology in order to obtain other discrete series and prove Theorem 3. We fix \( \sigma, P \) via (1.3) and define \( X_{\sigma}, Y_{\sigma} \) as before.

Since \( g^C \) is semisimple, its Killing form is non-degenerate. Recall that we let \( u \) be a compact real form of \( g^C \) in (2.1). So the Killing form is negative definite on \( u \) and positive definite on \( -1 u \). By making suitable sign changes on the Killing form, we obtain a positive definite inner product \((-,-)\) on \( g^C \). The subspaces \( V_i \) of (2.6) satisfy

\[
(h_i, h_j^+)= (h_i, V_j)= (h_i^+, V_j)= (V_i, V_j) = 0 \quad (6.1)
\]

for \( i \neq j \). The parabolic subalgebra \( p \) and its commutator \([p, p]\) are built by piecing together the various subspaces of \( g^C \) which appear in (6.1). Therefore, the orthogonality conditions in Eq. (6.1) imply that it descends to inner products on \( g^C/[p, p] \) and \( g^C/p \). Taking their duals, we get inner products on \((g^C/[p, p])^* \) and \((g^C/p)^* \). These dual spaces can be identified with the cotangent spaces of \( X_{\sigma} \) and \( Y_{\sigma} \) respectively at the identity cosets \( e \). We now have the inner products

\[
(,): T^*_e X_{\sigma} \times T^*_e X_{\sigma} \rightarrow \mathbb{C} \quad (6.2)
\]

and

\[
(,): T^*_e Y_{\sigma} \times T^*_e Y_{\sigma} \rightarrow \mathbb{C}. \quad (6.3)
\]

The Killing form is adjoint invariant. This means that if we translate (6.2) to all of \( X_{\sigma} \) by the \((G \times A_{\sigma})\)-action, we get a \((G \times A_{\sigma})\)-invariant Hermitian structure on the cotangent bundle

\[
(,)^X: T^* X_{\sigma} \times T^* X_{\sigma} \rightarrow \mathbb{C}. \quad (6.4)
\]
Similarly, (6.3) leads to a $G$-invariant Hermitian structure on the cotangent bundle

$$ (\cdot, \cdot)^Y: T^* X_\alpha \times T^* X_\alpha \to \mathbb{C}. $$

(6.5)

Recall that $\pi$ denotes the fibration of $X_\alpha$ over $Y_\alpha$. We get an injection between the Dolbeault $(0, q)$-forms,

$$ \pi^*: \Omega^0(\Omega^q Y_\alpha) \to \Omega^0(\Omega^q X_\alpha). $$

(6.6)

These Dolbeault differential forms inherit Hermitian structures from (6.4) and (6.5). We still use the same notation and write

$$ (\cdot, \cdot)^X: \Omega^0(\Omega^q X_\alpha) \times \Omega^0(\Omega^q X_\alpha) \to \mathbb{C}(X_\alpha) $$

(6.7)

as well as

$$ (\cdot, \cdot)^Y: \Omega^0(\Omega^q Y_\alpha) \times \Omega^0(\Omega^q Y_\alpha) \to \mathbb{C}(Y_\alpha). $$

(6.8)

They are related by the injection (6.6).

**Proposition 6.1.** For all $\alpha, \beta \in \Omega^0(\Omega^q Y_\alpha)$, $(\pi^* \alpha, \pi^* \beta)^X = \pi^*(\alpha, \beta)^Y$.

**Proof.** Consider the natural map $\pi: \mathfrak{g}[p, p] \to \mathfrak{g}/p$. The inner products (6.2) and (6.3) induced by the Killing form satisfy

$$ (u, v) = (\pi^* u, \pi^* v) $$

(6.9)

for all $u, v \in (\mathfrak{g}/p)^*$. Since $\Omega^q(\Omega^q Y_\alpha)$ is obtained from the linear combinations and exterior powers of $\Omega^q(\Omega^q Y_\alpha)$, it suffices to prove the proposition for $q = 1$. So let $\alpha, \beta \in \Omega^0(\Omega^1 Y_\alpha)$. Then (6.9) says that at $e$,

$$ (\alpha, \beta)^Y = (\alpha_+, \beta_+) = (\pi^* \alpha_+, \pi^* \beta_+) = (\pi^* \alpha, \pi^* \beta)^X. $$

By $G$-equivariance of $\pi$ this suffices, since checking $(\alpha, \beta)^Y \in \mathbb{C}(\Omega^q Y_\alpha)$ at another point $L_\lambda(e) \in Y_\alpha$ is the same as checking $(L_\lambda^* \alpha, L_\lambda^* \beta)^Y \in \mathbb{C}(\Omega^q Y_\alpha)$ at $e$. The proposition follows.

Let $\omega = \sqrt{-1} \partial \bar{\partial} F$ be a $(G \times T_\alpha)$-invariant pseudo-Kähler form on $X_\alpha$ with $F$ strictly convex. As before, it leads to a pre-quantum line bundle $L$ over $X_\alpha$. Fix an integral weight $\lambda \in \mathfrak{t}_\alpha^*$. Let $\Omega^\lambda(\Omega^q X_\alpha)$ be the $(0, q)$-forms on $X_\alpha$ which transform by $\lambda$ under the right $T_\alpha$-action. We similarly define $\Omega^\lambda(\Omega^q X_\alpha, L)$ for coefficients in $L$. This space acquires a Hermitian structure $(\cdot, \cdot)^L$ by taking the product of (6.7) and the Hermitian structure of $L$. The
holomorphic section $s_0$ of Proposition 4.2 leads to a $(G \times T_\alpha)$-equivariant trivialization

$$\Omega^0_q(X_\alpha) \to \Omega^0_q(X_\alpha, L); \ x \mapsto x \otimes s_0. \quad (6.10)$$

Extend $\lambda \in \mathfrak{t}_\alpha^*$ to $\mathfrak{h}_\alpha^*$ by complex linearity, and let $L_\alpha$ be the homogeneous line bundle over $Y_\alpha$ corresponding to $\lambda$. We let $\Omega^0_q(Y_\alpha, L_\alpha)$ be the $(0, q)$-forms on $Y_\alpha$ with coefficients in $L_\alpha$. Taking the product of Hermitian structures of Proposition 4.3 and (6.8), we obtain a Hermitian structure on

$$\Omega^0_q(Y_\alpha, L_\alpha) \quad (6.11)$$

In fact, this is the Hermitian structure discussed in [16].

Recall from Section 4 that $\mu_x$ and $\mu_y$ are respectively the $(G \times A_\alpha)$-invariant measure on $X_\alpha$ and the $G$-invariant measure on $Y_\alpha$. By integrating $(\cdot, \cdot)^L_{\mu_x}$ we obtain an $L^2$-structure $\langle \cdot, \cdot \rangle^L$ on $\Omega^0_q(X_\alpha, L)$. Similarly, by integrating $(\cdot, \cdot)^L_{\mu_y}$, we obtain an $L^2$-structure $\langle \cdot, \cdot \rangle^L_{\mu_y}$ on $\Omega^0_q(Y_\alpha, L_\alpha)$.

By (4.3), a smooth section $s$ on $L_\alpha$ can be naturally identified with $f_s \in \Omega^0_q(X_\alpha)$. We use (6.6) and (6.10) to define

$$\pi^*_x : \Omega^0_q(Y_\alpha, L_\alpha) \to \Omega^0_q(X_\alpha, L); \quad \pi^*_x(s \otimes x) = f_s \pi^* x \otimes x_0. \quad (6.12)$$

For the rest of this section, we use $\pi^*_x$ to compare the $L^2$-structures $\langle \cdot, \cdot \rangle^L$ and $\langle \cdot, \cdot \rangle^L_{\mu_y}$.

**Proposition 6.2.** Suppose that $\phi, \psi \in C^\infty_c(X_\alpha) = C^\infty(A_\alpha)$ and that $x, y \in \Omega^0_q(Y_\alpha, L_\alpha)$ are square-integrable. Then

$$\langle \phi \pi^*_x x, \phi \pi^*_y y \rangle^L = \int_{A_\alpha} \phi \pi^*_x x \overline{\phi \pi^*_y y} \chi^2 e^{-F} ds \quad (6.13)$$

**Proof.** Write $x = x \otimes x_0, y = y \otimes y_0 \in \Omega^0_q(Y_\alpha, L_\alpha)$. Then

$$\langle \phi \pi^*_x(x \otimes x_0), \phi \pi^*_y(y \otimes y_0) \rangle^L$$

$$= \int_{x_\alpha} \phi \pi^*_x x, \overline{\phi \pi^*_y y} \chi^2 e^{-F} ds \quad \text{by (6.12)}$$

$$= \int_{x_\alpha} \phi \pi^*_x x, \overline{\phi \pi^*_y y} \chi^2 e^{-F} \mu_x \quad \text{by Propositions 4.2 and 6.1}$$

$$= \int_{x_\alpha} \phi \pi^*_x x, \overline{\phi \pi^*_y y} \chi^2 e^{-F} \mu_x \quad \text{by Proposition 4.3}$$

$$= \int_{x_\alpha} \overline{\phi \pi^*_x x}, \pi^*_y y \chi^2 e^{-F} \mu_x \quad \text{by (6.11)}. \quad (6.13)$$
Recall the fibrations $\pi_1, \pi_2$ of (4.7) satisfying $\pi = \pi_2 \cdot \pi_1$. Write $\mu_x = dg \, da$.

The last expression of (6.13) becomes

$$
\int_{G/G^*} \pi_2^*(x, y)^{L_2} \, dg \int_{A_2} \phi \varphi \, e^{-F} \, da.
$$

(6.14)

Each fiber of $\pi_2$ is a copy of $T^*$. Since $T^*$ is compact, $\pi_2^*(x, y)^{L_2} \, dg$ is integrable over each fiber of $\pi_2$. So by [12, Proposition 1.13, p. 95],

$$
\int_{G/G^*} \pi_2^*(x, y)^{L_2} \, dg = \int_{G/G^*} (x, y)^{L_2} \, \mu_x = \langle x, y \rangle^{L_2}.
$$

(6.15)

The proposition follows from (6.13), (6.14), and (6.15).

Observe that if we set $\phi = \varphi \equiv 1$ in the above proposition, then

$$
\langle \pi_2^* \mu_x, \pi_2^* \mu_y \rangle^L = \int_{A_2} \varphi^L \, da \langle x, y \rangle^{L_2}
$$

(6.16)

for all square-integrable $x, y \in \Omega^0 \eta(Y_\eta, L_\eta)$. Proposition 4.4 says that $\int_{A_2} \varphi^L \, da < \infty$ if and only if $\lambda \in \mathcal{U}_\eta$. So when this happens, $\pi_2^*$ preserves square-integrability.

Let $I_2^* \subset \Omega^0 \eta(X_\eta, L)$ denote the image of $\pi_2^*$. Since $\Omega^0 \eta(X_\eta, L)$ consists of differential forms which transform by $\lambda \in \mathcal{T}_\eta$ under the right $T_\eta$-action, we can write

$$
\Omega^0 \eta(X_\eta, L) = \bigoplus_{r \in \mathcal{U}_\eta} \left( I_2^* \bigotimes \left( \mathcal{C}^\infty(A_\eta) \bigotimes \bigwedge_h b_*^* \right) \right).
$$

(6.17)

Here $\{I_2^*\}_{q}$ and $\{\mathcal{C}^\infty(A_\eta) \bigotimes \bigwedge_h b_*^*\}_{q}$ are both subcomplexes under $\bar{\partial}$, so (6.17) is a tensor product of chain complexes. The subcomplex $\{\mathcal{C}^\infty(A_\eta) \bigotimes \bigwedge_h b_*^*\}_{q}$ can be identified with the $T_\eta$-invariant Dolbeault differential forms on $H_\eta$. Since $H_\eta$ is a Stein space, this subcomplex has trivial cohomology

$$
H^q \left( \left( \mathcal{C}^\infty(A_\eta) \bigotimes \bigwedge_h b_*^* \right)_{/q} \right) = \begin{cases} 
\mathbb{C} & q = 0, \\
0 & q \geq 1.
\end{cases}
$$

(6.18)

It also follows from (6.1) that whenever $(r, s) \neq (t, u),$

$$
\langle I_2^* \bigotimes \mathcal{C}^\infty(A_\eta) \bigotimes \bigwedge_h b_*^*, I_2^* \bigotimes \mathcal{C}^\infty(A_\eta) \bigotimes \bigwedge_h b_*^* \rangle^L = 0.
$$

(6.19)

Against the $L^2$-structure $\langle \cdot, \cdot \rangle^L$ on $\Omega^0 \eta(X_\eta, L)$, we define the formal adjoint $\bar{\partial}^*$ of $\bar{\partial}$. Namely, $\langle \bar{\partial}^* x, y \rangle^L = \langle x, \bar{\partial}^* y \rangle^L$ for all square-integrable
Proposition 6.3. \( \pi_\pm^* \delta^* = \delta^* \pi_\pm^* \).

Proof. To prove this proposition, we need to show that for arbitrary square-integrable \( x \in \Omega_{\pm}^0 (X_\alpha, L) \) and \( \beta \in \Omega_{\pm}^0 (Y_\gamma, L) \),
\[
\langle x, \pi_\pm^* \delta^* \beta \rangle^L = \langle x, \delta^* \pi_\pm^* \beta \rangle^L.
\]
(6.20)

We apply (6.17) to \( x \) and write \( x = \sum_{\pm} (\pi_\pm^* x_\gamma) \oplus y_\gamma \), up to linear combination. The same indices \( r, s \) apply to all \( \sum \) below. Note that \( \pi_\pm^* x_\gamma \in I_\pm^* \) and \( y_\gamma \in C^\infty (A_\gamma) \). Thus the LHS of (6.20) becomes
\[
\left\langle \sum (\pi_\pm^* x_\gamma) \oplus y_\gamma, \pi_\pm^* \delta^* \beta \right\rangle^L = \left\langle (\pi_\pm^* x_\gamma) \oplus y_0, \pi_\pm^* \delta^* \beta \right\rangle^L \quad \text{by (6.19)}
\]
\[
= \int_{A_\gamma} y_0 \alpha^2 e^{-F} \, da \left\langle x_\gamma, \delta^* \beta \right\rangle^L \quad \text{by Proposition 6.2}
\]
\[
= \int_{A_\gamma} y_0 \alpha^2 e^{-F} \, da \left\langle \delta x_\gamma, \beta \right\rangle^L.
\]
(6.21)

On the other hand, the RHS of (6.20) becomes
\[
\left\langle \sum (\pi_\pm^* x_\gamma) \oplus y_\gamma, \delta^* \pi_\pm^* \beta \right\rangle^L = \left\langle \delta \sum (\pi_\pm^* x_\gamma) \oplus y_\gamma, \pi_\pm^* \beta \right\rangle^L
\]
\[
= \left\langle \sum (\delta \pi_\pm^* x_\gamma) \oplus y_\gamma + (-1)^r (\pi_\pm^* x_\gamma) \oplus \delta y_\gamma, \pi_\pm^* \beta \right\rangle^L.
\]
(6.22)

Since \( \{ C^\infty (A_\alpha) \otimes \Lambda^{0, q} b_{\gamma}^* \}_{q} \) is a subcomplex it contains \( \delta y_\gamma \), and therefore (6.19) says that \( \left\langle (\pi_\pm^* x_\gamma) \oplus \delta y_\gamma, \pi_\pm^* \beta \right\rangle^L = 0 \). Then (6.22) becomes
\[
\left\langle \sum (\delta \pi_\pm^* x_\gamma) \oplus y_\gamma, \pi_\pm^* \beta \right\rangle^L = \left\langle (\delta \pi_\pm^* x_\gamma) \oplus y_0, \pi_\pm^* \beta \right\rangle^L \quad \text{by (6.19)}
\]
\[
= \left\langle (\pi_\pm^* \delta x_\gamma) \oplus y_0, \pi_\pm^* \beta \right\rangle^L
\]
\[
= \int_{A_\gamma} y_0 \alpha^2 e^{-F} \, da \left\langle \delta x_\gamma, \beta \right\rangle^L \quad \text{by Proposition 6.2.}
\]
(6.23)

Thus (6.20) follows from (6.21), (6.22), and (6.23). Hence we have the proposition. \( \square \)
We have defined $\delta$ and $\delta^*$ on the square-integrable differential forms in $\Omega^0(X, L)$ and $\Omega^0(Y, L_\lambda)$. The differential forms which are annihilated by $\delta$ and $\delta^*$ are known as the harmonic forms. The Hilbert space of square-integrable harmonic forms are denoted by $H^0_{\omega} \subset \Omega^0(X, L)$ and $H^q(L_\lambda) \subset \Omega^q(Y, L_\lambda)$ and are called the $L^2$-cohomology. The next proposition considers $(H^q_{\omega})_2 = H^q_\omega \cap \Omega^q(X, L)$. Recall that $\pi^q_\omega$ is defined in (6.12).

**Proposition 6.4.** $(H^q_{\omega})_2$ lies in the image of $\pi^q_\omega$.

*Proof.* Let $I^q_\xi$ denote the image of $\pi^q_\omega$ as before, and let $E^q = (H^q_{\omega})_1 \cap I^q_\xi$. So $E^q$ is a closed subspace in the Hilbert space $(H^q_{\omega})_2$. This gives the direct sum and orthogonal projection

$$(H^q_{\omega})_2 = E^q \oplus E^q_\perp, \quad \theta : (H^q_{\omega})_2 \to E^q,$$

where $E^q_\perp$ is the orthogonal complement of $E^q$ in $(H^q_{\omega})_2$. Our goal is obviously to show that $E^q_\perp = 0$, so that $(H^q_{\omega})_2 \subset I^q_\xi$.

Since $\theta \delta = \delta \theta = 0$ on $(H^q_{\omega})_2$, the projection $\theta$ defines a map $\theta_\omega$ on the Dolbeault cohomology classes. Apply the Kunneth theorem to (6.17) and (6.18). It says that the natural inclusion $i : I^q_\xi \hookrightarrow \Omega^q(X, L)$ leads to an isomorphism $\tau^q_\omega$ in Dolbeault cohomology. Note that $\theta \cdot i$ is the identity map on $E^q$. So $\theta_\omega$ and $\tau^q_\omega$ are inverses of each other. In particular, for all $\tau \in (H^q_{\omega})_2$, $\tau_\omega$ and $\tau_\omega$ are cohomologous.

Pick $\tau \in (H^q_{\omega})_2$. Namely, $\tau$ is square-integrable and $\delta \tau = \delta^* \tau = 0$. Write $\tau = \zeta + \eta \in E^q \oplus E^q_\perp = (H^q_{\omega})_2$.

Since $\zeta$ and $\theta \zeta = \zeta$ define the same Dolbeault cohomology class, there exists a $\beta$ such that $\delta \beta = \zeta - \theta \zeta = \eta$. Since $\delta^* \tau = 0$,

$$0 = \langle \beta, \delta^* \tau \rangle = \langle \delta \beta, \tau \rangle = \langle \eta, \zeta + \eta \rangle = \langle \eta, \eta \rangle.$$

Thus $\eta = 0$. This means that $E^q_\perp = 0$ and the proposition follows. 

The next proposition relates the $L^2$-cohomology spaces $(H^q_{\omega})_2$ and $H^q(L_\lambda)$. Recall that $U_F$ is the image of $\frac{1}{2} F'$.

**Proposition 6.5.** If $\lambda \in U_F$ then $\pi^q_{\lambda}$ defines an isomorphism $(H^q_{\omega})_2 \cong H^q(L_\lambda)$. If $\lambda \notin U_F$ then $(H^q_{\omega})_2 = 0$.

*Proof.* Suppose first that $\lambda \in U_F$. By Proposition 4.4, $\int_{A_\lambda} \rho(\alpha)^2 e^{-R(\alpha)} d\alpha < \infty$. Since $\pi^q_{\lambda}$ is injective, it suffices to prove that $\pi^q_{\lambda}(H^q(L_\lambda)) = (H^q_{\omega})_2$.

$$\pi^q_{\lambda}(H^q(L_\lambda)) = (H^q_{\omega})_2 \quad (6.24)$$
Clearly \( \pi^*_x \) commutes with \( \delta^* \), and Proposition 6.3 says that it commutes with \( \delta^* \) too. Further, by (6.16), \( \pi^*_x \) preserves square-integrability. These observations lead to the \( \Rightarrow \) part of (6.24).

It remains to prove the \( \Leftarrow \) part of (6.24). Let \( x \in (H^*_q)_\lambda \). By Proposition 6.4, \( x = \pi^*_x \beta \). The harmonic property of \( x \) and the injectivity of \( \pi^*_x \) imply that \( \beta \) is harmonic. Further, (6.16) says that the square-integrability of \( x \) implies the square-integrability of \( \beta \). Hence \( \beta \in H^q(L_\lambda) \), which implies the \( \Rightarrow \) part of (6.24). This proves the proposition for \( \lambda \in U_F \).

Next suppose that \( \lambda \notin U_F \). Let \( x \in D^0_q(X_\lambda, L) \). If \( x \in H^*_q \), then Proposition 6.4 and (6.16) say that \( x \) is not square-integrable unless \( x \equiv 0 \). We conclude that \( (H^*_q)_x = 0 \). Hence we have the proposition.

Proof of Theorem 3. By Proposition 6.5, it suffices to consider \( (H^*_q)_x \) for \( \lambda \in U_F \). Assuming this, Theorem 3 follows directly from Proposition 6.5 and the well-known results [16] on \( H^q(L_\lambda) \).

7. SYMPLECTIC REDUCTION

Let \( \omega = \sqrt{-1} \partial \bar{\partial} F \) be a \( (G \times T_\lambda) \)-invariant pseudo-Kähler form on \( X_\lambda \). In this section, we perform symplectic reduction [15] to the right \( T_\lambda \)-action. The moment map for this action is denoted

\[
\Phi_\lambda : X_\lambda \to t^*_\lambda
\]

and is called the right moment map. Recall that \( (t^*_\lambda)^{\text{reg}} \cong (a^*_\lambda)^{\text{reg}} \) is defined in (1.2).

**Proposition 7.1.** For all \( g a \in (G/G_{\lambda})_x = X_\lambda \), \( \Phi_\lambda(ga) = \frac{1}{2} F'(a) \in (t^*_\lambda)^{\text{reg}} \).

**Proof.** Since the right \( T_\lambda \)-action commutes with the \( G \)-action, it is clear that \( \Phi_\lambda \) is \( G \)-invariant. So it suffices to consider \( \Phi_\lambda(a) \) for \( a \in A_\lambda \).

Let \( v \in t^*_\lambda \) and let \( v^* \) and \( v' \) denote the infinitesimal vector fields on \( X_\lambda \) corresponding to the left and right actions respectively. Since \( T_\lambda A_\lambda \) is abelian, \( v^*_a = v'_a \) for all \( a \in A_\lambda \). Let \( \beta \) be the real \( G \times T_\lambda \)-invariant 1-form satisfying \( df = \omega \). Then

\[
(\Phi_\lambda(a), v) = -(\beta, v')_a \quad \text{by [1, Theorem 4.2.10]}
\]

\[
= -(\beta, v^*)_a
\]

\[
= (\Phi(a), v)
\]

\[
= \left( \frac{1}{2} F'(a), v \right) \quad \text{by Proposition 3.6.}
\]
Finally, Theorem 1 says that the image of $\frac{1}{2} F'$ lies in $(t^*_a)_{\text{reg}}$. Hence we have the proposition.

Let $\lambda \in (t^*_a)_{\text{reg}}$ be in the image of $\Phi_r$. We consider the reduced space $R_\lambda = \Phi_r^{-1}(\lambda)/T_a$.

**Proposition 7.2.** Each connected component of $\Phi_r^{-1}(\lambda)/T_a$ is a copy of the flag domain $Y_a$.

*Proof.* Since $\omega$ is pseudo-Kähler, Theorem 1 says that $F$ is non-degenerate. By the inverse function theorem, $F'$ is a local diffeomorphism. So there exists a discrete set $I \subseteq A_\alpha$ such that $(\frac{1}{2} F')^{-1}(\lambda) = \Gamma$. By Proposition 7.1, $\Phi_r^{-1}(\lambda) = (G/G_a^*) \Gamma \subseteq (G/G_a^*) A_\alpha$. Consequently, 

$$\Phi_r^{-1}(\lambda)/T_a = (G/G^*) \Gamma.$$  
(7.1)

A typical connected component of this space is of the form $(G/G^*) a, a \in \Gamma$. Hence we have the proposition.

Consider the inclusion

$$i: \Phi_r^{-1}(\lambda) \to X_a$$  
(7.2)

and the fibration

$$\rho: \Phi_r^{-1}(\lambda) \to R_\lambda.$$  
(7.3)

The reduced form $\omega_3$ is defined to be the unique symplectic form on $R_\lambda$ such that $\rho^* \omega_3 = i^* \omega$. Since $i$ and $\rho$ commute with the $G$-action, it is clear that $\omega_3$ is $G$-invariant. Let

$$\psi: R_\lambda \to g^*$$

be the moment map of the $G$-action preserving $\omega_3$.

By (7.1), write a typical element of $R_\lambda$ as $ga$. If $g$ is the identity coset $eG^*$, we write $a = ga$ for simplicity.

**Proposition 7.3.** $\psi(a) = \lambda \in (t^*_a)_{\text{reg}}$.

*Proof.* Pick $x \in g$. By abuse of notation, let $x^g$ be the infinitesimal vector field for the $G$-action on $X_a$, $\Phi_r^{-1}(\lambda)$ or $R_\lambda$, depending on the context. Also, let $a$ denote the appropriate element in any of these three spaces. Since (7.2) and (7.3) commute with the $G$-action,

$$i(a) = a, \quad \rho(a) = a, \quad t_a(x^g_a) = x^g_a, \quad \rho_a(x^g_a) = x^g_a.$$
Since \(g\) is semisimple, up to linear combination \(x = [u, v]\). Then
\[
(\psi(a), x) = (\psi(a), [u, v]) = \omega_\lambda(u^\sharp, v^\sharp)_a = p^*\omega_\lambda(u^\sharp, v^\sharp)_u = \rho^*\omega_\lambda(u^\sharp, v^\sharp)_u = \omega(u^\sharp, v^\sharp)_u = (\Phi(a), [u, v]) = (\lambda, [u, v]) = (\lambda, x).
\] (7.4)

So \(\psi(a) = \lambda\) and the proposition follows.

Since \(Y_a\) is an open set of \(G^C/P\), it is a complex manifold. Consequently the reduced space \(R^*_\lambda\) is complex. Recall that \(\mathcal{E}\) is defined in (1.7).

**Proposition 7.4.** The reduced form \(\omega_\lambda\) is a \(G\)-invariant pseudo-Kähler form on \(R^*_\lambda\). In particular, it is Kähler if and only if \(\lambda \in (t_C^*)^\text{reg} \cap \mathcal{E}\).

**Proof.** The \(G\)-invariance of \(\omega_\lambda\) follows from the discussions in (7.2) and (7.3). So its pseudo-Kähler and Kähler properties remain to be checked.

Consider the elements \(\xi_i, \eta_i \in q\) from (2.3), indexed by the positive roots \(\pi_i\). Here \(\{\xi_i, \eta_i\}(\pi_i, t_a) \neq 0\) can be regarded as a basis of \(q/\mathfrak{g}^\mathbb{C}\). The almost complex structure inherited from \(G^C/P\) sends \(\xi_i\) to \(\eta_i\) and \(\eta_i\) to \(-\xi_i\). Substituting \(u = \xi_i\) and \(v = \eta_i\) in (7.4), we get
\[
\omega_\lambda(\xi_i^\sharp, \eta_i^\sharp)_u = \omega(\xi_i^\sharp, \eta_i^\sharp)_u. \tag{7.5}
\]

Since \(\omega\) is pseudo-Kähler, it follows from (7.5) that \(\omega_\lambda\) is pseudo-Kähler too.

In fact, \(\omega_\lambda\) is Kähler if and only if (7.5) is positive for all \((\pi_i, t_a) \neq 0\). Following the argument in (7.4), we see from (2.5) that
\[
\omega_\lambda(\xi_i^\sharp, \eta_i^\sharp)_u = \pm (\lambda, \pi_i). \tag{7.6}
\]

Here the sign \(\pm\) is positive when \(\pi_i\) is compact and negative when \(\pi_i\) is non-compact. So (7.6) is positive for all \((\pi_i, t_a) \neq 0\) if and only if \(\lambda \in (t_C^*)^\text{reg} \cap \mathcal{E}\), and this is the equivalent condition for \(\omega_\lambda\) to be Kähler.

For \(i = 1, 2\), consider the reduced spaces \((R^*_\lambda, (\omega_\lambda)_{\lambda_i})\), with moment maps \(\psi: R^*_\lambda \to \mathfrak{g}^\mathbb{C}\). By the previous proposition, these reduced spaces are pseudo-Kähler. So we can compare them under the notions of \(\sim\) and \(=\) introduced in (1.12).

**Proposition 7.5.** Suppose that \(R^*_\lambda\) have the same number of connected components. Then \((\omega_1)_{\lambda_1} \sim (\omega_2)_{\lambda_2}\) if and only if \(\lambda_1 \sim \lambda_2\), and \((\omega_1)_{\lambda_1} = (\omega_2)_{\lambda_2}\) if and only if \(\lambda_1 = \lambda_2\).

**Proof.** Suppose that this proposition has been proved for all connected reduced spaces. Let \(R^*_\lambda\) be a reduced space, possibly non-connected. For \(i = 1, 2\), let \(Y^*_a\) be connected components of \(R^*_\lambda\). By Proposition 7.3, their
moment maps satisfy $\psi_i(a_i) = \lambda_i$. So by the present proposition for connected reduced spaces, $Y_{a_i}$ and $Y_{a_2}$ are isomorphic pseudo-Kähler manifolds. We conclude that all connected components of $R_\lambda$ are isomorphic to one another, and so the present proposition holds for non-connected reduced spaces too.

From this observation, we only have to prove the proposition for connected reduced spaces. So assume that $R_\lambda$ are connected for $i=1,2$. Write $R_\lambda = (G/G^\ast) a_i$ for some $a_i \in A_g$.

Suppose that $\lambda_1 \sim \lambda_2$. Thus there is a coadjoint orbit $\mathcal{O} \in \mathfrak{g}^\ast$ which contains $\lambda_1$ and $\lambda_2$. By Proposition 7.3, $\psi_i(a_i) = \lambda_i$. By Theorem 1 and Proposition 7.1, $\lambda_i \in (\mathfrak{t}_1^\ast)^\text{reg} \subset \mathfrak{t}^\ast$, so the isotropy subgroup of $\lambda_i$ in $G$ is $G^\ast$. Hence $\mathcal{O} = G/G^\ast$. So $\psi_i$ is a diffeomorphism from $(G/G^\ast) a_i$ onto the elliptic orbit $\mathcal{O}$. In fact, since $\psi_i$ is $G$-equivariant, it identifies $(\omega_1)_{\lambda_i}$ with the Kirillov–Kostant symplectic form $\omega_{KK}$ on $\mathcal{O}$. We conclude that $(\omega_1)_{\lambda_1} \sim \omega_{KK} \sim (\omega_2)_{\lambda_2}$.

Conversely, if $(\omega_1)_{\lambda_1} \sim (\omega_2)_{\lambda_2}$, then $\psi_i$ have the same image $\mathcal{O}$. By Proposition 7.3, $\psi_i(a_i) = \lambda_i \in \mathcal{O}$, so $\lambda_1 \sim \lambda_2$.

The last part of this proposition remains to be proved, where $\sim$ is replaced with $\equiv$. Suppose that $\lambda_1 = \lambda_2$. By (7.4), for all $u, v \in \mathfrak{g}$,

$$
(\omega_1)_{\lambda_1}(u^\ast, v^\ast)_{\lambda_1} = (\lambda_1, [u, v]) = (\omega_2)_{\lambda_2}(u^\ast, v^\ast)_{\lambda_2}.
$$

(7.7)

Consider the $G$-equivariant biholomorphic map

$$
\kappa: Y_{a_1} \to Y_{a_2}, \quad \kappa(ga_1) = ga_2.
$$

(7.8)

By (7.7), $\kappa((\omega_2)_{\lambda_2})$ and $(\omega_1)_{\lambda_1}$ agree on $a_1$. By $G$-invariance, they agree everywhere. So $\kappa$ preserves the pseudo-Kähler structures and $(\omega_1)_{\lambda_1} = (\omega_2)_{\lambda_2}$.

Conversely, suppose that $\lambda_1 \not\sim \lambda_2$. If $\lambda_i$ are in different coadjoint $G$-orbits, then the first part of the proposition says that $(\omega_1)_{\lambda_i}$ are not symplectomorphic, so in particular $(\omega_1)_{\lambda_1} \not\sim (\omega_2)_{\lambda_2}$. Hence we may assume that $\lambda_i$ are in the same orbit. Each connected component of $(\mathfrak{t}_1^\ast)^\text{reg} \subset \mathfrak{g}^\ast$ intersects a $G$-orbit at most once. From $\lambda_i \in (\mathfrak{t}_1^\ast)^\text{reg}$, $\lambda_1 \not\sim \lambda_2$ and $\lambda_1 \sim \lambda_2$ we conclude that $\lambda_i$ are in different connected components of $(\mathfrak{t}_1^\ast)^\text{reg}$. The holomorphic map (7.8) fails to preserve the pseudo-Kähler structures because (7.6) shows that there is a sign problem arising from (2.5). Other symplectomorphisms between $(\omega)_{\lambda_1}$ have to permute the connected components of $(\mathfrak{t}_1^\ast)^\text{reg}$, so they cannot be holomorphic. We conclude that $(\omega_1)_{\lambda_1} \not\sim (\omega_2)_{\lambda_2}$. This proves the proposition.

**Proof of Theorem 4.** The theorem follows directly from Propositions 7.1 through 7.5.
8. FLAG DOMAINS

In this section, we study the $G$-invariant pseudo-Kähler structures on the flag domain $Y = G/G^*$ and their relations to symplectic reduction (1.11). These will lead to Theorem 5.

If $G$ is compact, then $Y$ is compact and obviously the pseudo-Kähler forms on $Y$ are not exact. But when $G$ is not compact some work is needed to show that the $G$-invariant pseudo-Kähler forms on $Y$ are not exact. This is done by the next proposition.

**Proposition 8.1.** A $G$-invariant pseudo-Kähler form on $Y$ cannot be exact.

Proof. Recall from (3.1) that $g^* \subseteq \mathfrak{l}$, and define the relative exterior algebra $\Lambda^1(\mathfrak{l}, g^*)^*$ from (2.12). We first claim that

$$1 \Lambda^1(\mathfrak{l}, g^*)^* = 0. \quad (8.1)$$

Pick $\beta \in \Lambda^1(\mathfrak{l}, g^*)^*$. Since $\mathfrak{l}$ is compact, $\mathfrak{l} = \mathfrak{l}_s + \mathfrak{z}$, where $\mathfrak{l}_s$ and $\mathfrak{z}$ are respectively the semisimple commutator subalgebra and the center of $\mathfrak{l}$. From $\beta \in \Lambda^1(\mathfrak{l}, g^*)^*$ and $\mathfrak{z} \subseteq \mathfrak{g}$, it is necessary that $\beta \in \mathfrak{l}_s^* \subseteq \mathfrak{l}^*$. Consider $\mathfrak{l}_s = (\mathfrak{l}_s \cap V) + (\mathfrak{l}_s \cap V)$, where $V$ is the space from (2.6). Since $(\mathfrak{l}_s \cap V) \subset \mathfrak{g}^*$, we get $\beta \in (\mathfrak{l}_s \cap V)^*$.

Suppose that $\beta \neq 0$. Since $\mathfrak{l}_s$ is semisimple, there exists $x$ in its Cartan subalgebra $\mathfrak{l}_s \cap \mathfrak{t}$ such that $\text{ad}_x^* \beta \neq 0$. Since $x \in (\mathfrak{l}_s \cap \mathfrak{t}) \subseteq \mathfrak{g}^*$, the condition $\text{ad}_x^* \beta \neq 0$ contradicts $\beta \in \Lambda^1(\mathfrak{l}, g^*)^*$. So $\beta$ has to vanish and (8.1) follows.

Consider the restriction map $r^*: \mathcal{O}(G/G^*) \rightarrow \mathcal{O}(K/G^*)$. Let $\omega$ be a $G$-invariant pseudo-Kähler form on $G/G^*$. Also, let $\zeta, \gamma, \eta$ be the vectors from (2.6) with $\sigma, \zeta, \gamma, \eta \neq 0$. It is clear from earlier discussions that $(r^*\omega)(\zeta, \gamma, \eta) \neq 0$, so $r^*\omega \neq 0$. Since $r^*$ commutes with the $K$-action, $r^*\omega$ is $K$-invariant. Consequently,

$$0 \neq r^*\omega \in 2 \Lambda^1(\mathfrak{l}, g^*)^*. \quad (8.2)$$

If $\omega$ is exact, then so is $r^*\omega$. But this is impossible due to (8.1) and (8.2).

The proposition follows.

As before, we let $\psi: Y \rightarrow \mathfrak{g}$ be the moment map and let $e \in Y$ be the identity coset. Recall the notion of $\sim$ in (1.12).

**Proposition 8.2.** The $G$-invariant pseudo-Kähler forms on $Y$ are classified by $(\mathfrak{l}_s^*)_{\text{reg}}$ via $\psi(e) \in (\mathfrak{l}_s^*)_{\text{reg}}$. If we ignore the complex structures, then the
G-invariant symplectic forms on \( Y_\alpha \) are classified by \((t^*_\alpha)_{\text{reg}}/\sim\) up to G-symplectomorphisms.

**Proof.** Let \( \omega \) be a G-invariant pseudo-Kähler form on \( Y_\alpha \) with moment map \( \psi \). We want to show that \( \psi(e) \in (t^*_\alpha)_{\text{reg}} \).

Choose \( u, v \in V_i \) from (2.6) such that \( 0 \neq [u, v] \in V_i \). For all \( g \in G^* \), \( ge = e \in Y_\alpha \). Since \( \psi \) is G-equivariant,

\[
(\psi(e), [u, v]) = (\psi(ge), [u, v]) = (\psi(e), Ad_g[u, v]).
\]  

(8.3)

Let \( g \) vary in \( T \subset G^* \), so that \( Ad_g[u, v] \in Ad_T V_i \). Let \( \theta_i : g^* \to V_i^* \) be the projection with respect to (2.8). Since \( [u, v] \) and \( Ad_g[u, v] \) are in \( V_i \), (8.3) can be written as

\[
(\theta_i \psi(e), [u, v]) = (\theta_i \psi(e), Ad_g[u, v]).
\]  

(8.4)

But as \( g \) varies in \( T \), \( Ad_g[u, v] \) traces out a circle in the two-dimensional space \( V_i \). So for (8.4) to hold we need \( 0 = \theta_i \psi(e) \in V_i^* \). This happens for all \( V_i \), so \( \psi(e) \in t^* \).

Pick \( u \in V_i, v \in V_j \) where \( i \neq j \). By (2.7), \( [u, v] \in V \). Since \( \psi(e) \in t^* \), it follows that

\[
\omega(u^*, v^*)_e = (\psi(e), [u, v]) \in (t^*, V) = 0.
\]  

We conclude that

\[
\omega(V_i^*, V_j^*)_e = 0, \quad i \neq j.
\]  

(8.5)

We still have to go from \( \psi(e) \in t^* \) to the sharper \( \psi(e) \in (t^*_\alpha)_{\text{reg}} \). Consider \( \zeta_i, \gamma_i \) from (2.3). By (2.5),

\[
\omega(\zeta_i^*, \gamma_j^*)_e = (\psi(e), [\zeta_i, \gamma_j]) = \mp (\psi(e), \eta_i).
\]  

(8.6)

Consider \( (\zeta_i, t_a) = 0 \) so that \( V_i \subset g^* \) by (2.11). Since \( G^* \) fixes \( e \in Y_\alpha \), \( (\zeta_i^*)_e = (\zeta_i^*)_e = 0 \). So (8.6) vanishes whenever \( (\zeta_i, t_a) = 0 \), which implies that \( \psi(e) \in (t^*_\alpha)_{\text{reg}} \).

On the other hand, consider \( (\zeta_i, t_a) \neq 0 \). By (2.11), \( V_i \not\subset g^* \). Then \( (\zeta_i^*)_e, (\gamma_j^*)_e \neq 0 \). Since \( \omega \) is non-degenerate, by (8.5) \( \omega(\zeta_i^*, \gamma_j^*)_e \neq 0 \). So by (8.6) \( (\psi(e), \eta_i) \neq 0 \) whenever \( (\zeta_i, t_a) \neq 0 \). We conclude that \( \psi(e) \in (t^*_\alpha)_{\text{reg}} \).

We have proved that the moment map of a G-invariant pseudo-Kähler form on \( Y_\alpha \) satisfies \( \psi(e) \in (t^*_\alpha)_{\text{reg}} \). The rest of the proposition on the classifications of the pseudo-Kähler and symplectic forms follows essentially from arguments in Proposition 7.5. Hence we have the proof.
We now show that every $G$-invariant pseudo-Kähler form on $Y_a$ can be obtained via symplectic reduction from $X_{a'}$. In view of Theorem 4, the reduction process (1.11) simplifies to (1.13).

**Proposition 8.3.** Every $G$-invariant pseudo-Kähler form on $Y_a$ can be obtained via symplectic reduction (1.13).

**Proof.** Let $\Omega$ be a $G$-invariant pseudo-Kähler form on $Y_a$ with moment map satisfying $\psi(e) = \lambda$. By Proposition 8.2, $\lambda \in (t^*_a)_{\text{reg}}$. Note that $(t^*_a)_{\text{reg}}$ consists of connected components which are open cones. Let $D$ be the connected component containing $\lambda$. Let $\lambda_1, \ldots, \lambda_r \in t^*$ be on the edges of $D$ so that $D$ consists of positive linear combinations of $\lambda_1, \ldots, \lambda_r$. We can normalize them so that $\lambda = \sum_1^r \lambda_k$. Define $F \in C^\infty(a_\omega)$ by $F(y) = 2 \sum_1^r \exp(\lambda_k, y)$. By (1.5), we identify it with $F \in C^\infty(A_a)$. By Proposition 5.1 $F$ is strictly convex and the image of $F$ is $D$. So by Theorem 1 $\omega = \sqrt{-1} \partial \bar{\partial} F$ is pseudo-Kähler. Also, Propositions 5.1 and 7.1 say that $8(\epsilon^a) = \frac{1}{2} F'(0) = \lambda$. Since $F$ is strictly convex $F$ is injective, so the reduced space $R_{\lambda}$ is connected. By Proposition 7.2 $R_{\lambda} = Y_a$. By Proposition 7.3 the moment map of $(R_{\lambda}, \omega_{\lambda})$ sends $e$ to $\lambda$. Since the moment maps of $\Omega$ and $\omega_{\lambda}$ agree on $e$, Proposition 8.2 says that $\Omega = \omega_{\lambda}$. Hence we have the proposition.

**Proof of Theorem 5.** By Propositions 8.1 and 8.2 the $G$-invariant pseudo-Kähler forms on $Y_a$ are not exact and are classified by $(\epsilon^a)(t^*_a)_{\text{reg}}$. By Propositions 7.3 and 8.3 all of them can be obtained by symplectic reduction and the one with $(\epsilon^a) = \lambda$ is obtained from $(X_{a'}, \lambda)$.

### 9. QUANTIZATION COMMUTES WITH REDUCTION

The main purpose of this section is to prove Theorem 6. For convenience, the integral weights in $t^*_a$ are denoted by $Z(t^*_a)$. Let $\omega$ be a $(G \times T_a)$-invariant pseudo-Kähler form on $X_{a'}$. By Theorem 1 $\omega = \sqrt{-1} \partial \bar{\partial} F$. In this section we assume that $F$ is strictly convex. Let $\lambda \in Z(t^*_a)$. In Theorem 3, we prove that

$$
(H^*_q)_\lambda = \begin{cases} 
\Theta_{\lambda + \rho} & \text{if } \lambda \in U_F \cap Z(t^*_a), \lambda + \rho \in (t^*), \text{and } q = \|\lambda\|, \\
0 & \text{otherwise.}
\end{cases}
$$

By Theorem 1 and Proposition 7.1, we know that $U_F$ is also the image of the right moment map $\Phi_a$. Define $R_{\lambda} = \Phi_a^{-1}(\lambda)/T_a$ as in Section 7 and let $\omega_{\lambda}$ be the reduced form on $R_{\lambda}$. We want to quantize [14] the $G$-action on $(R_{\lambda}, \omega_{\lambda})$ and construct the $G$-representation $H^*_q$. Clearly we need $\lambda \in U_F$, for otherwise $R_{\lambda} = \emptyset$. For $\lambda \in U_F$, we know that $(\frac{1}{2} F')^{-1}(\lambda)$ has exactly one element because $F$ being strictly convex implies that $\frac{1}{2} F'$ is...
injective. So by Proposition 7.2 $R_\lambda = Y_\sigma$. The next step is to quantize the $G$-action on the reduced space $(Y_\sigma, \omega_{12})$. By Proposition 8.1, $\theta \neq [\omega_{12}] \in H^2(Y_\sigma, \mathbb{R})$. We can find a line bundle $L_\lambda \rightarrow Y_\sigma$ with Chern class $[\theta]_1$ if and only if $[\omega_{12}] \in H^2(Y_\sigma, \mathbb{Z})$ or equivalently $\lambda \in \mathbb{Z}(t^*_\mathbb{R})$. We have thus shown that $\lambda \in U_F \cap \mathbb{Z}(t^*_\mathbb{R})$ is necessary for quantization. Assuming this, $L_\lambda \rightarrow Y_\sigma$ is the homogeneous line bundle corresponding to the character $e^{\lambda} : H_\mathbb{R} \rightarrow \mathbb{C}$.

The harmonic forms on $Y_\sigma$ with coefficients in $L_\lambda$ are denoted by $H^q((\omega_{12}))$. It is a unitary $G$-representation and we conclude from [16] that

$$H^q((\omega_{12})) = \begin{cases} 0 & \text{if } \lambda \in U_F \cap \mathbb{Z}(t^*_\mathbb{R}), \lambda + \rho \in (t^*)_\text{reg}, \text{ and } q = \ell(\lambda), \\ \Theta_{\lambda + \rho} & \text{otherwise.} \end{cases}$$

(9.2)

**Proof of Theorem 6.** This follows directly from (9.1) and (9.2). □

### References