ABUNDANCE OF MOSAIC PATTERNS FOR CNN WITH SPATIALLY VARIANT TEMPLATES

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This work investigates the complexity of one-dimensional cellular neural network mosaic patterns with spatially variant templates on finite and infinite lattices. Various boundary conditions are considered for finite lattices and the exact number of mosaic patterns is computed precisely. The entropy of mosaic patterns with periodic templates can also be calculated for infinite lattices. Furthermore, we show the abundance of mosaic patterns with respect to template periods and, which differ greatly from cases with spatially invariant templates.

Keywords: Transition matrix; spatial entropy.

1. Introduction

Cellular neural networks (CNN) are large arrays of nonlinear circuits proposed by Chua and Yang [1988a, 1988b]. Since then much work has been done in the electrical engineering community, e.g. [Chua & Roska, 1993; Thiran et al., 1995]. Such systems occur as arrays of identical cells $C_i$ that are locally connected. If the cells are sitting on lattices $\mathbb{Z}_N := \{i \in \mathbb{Z} | -N \leq i \leq N\}$, then the equation describing cell $C_i$ at site $i$ is as follows:

$$\frac{dx_i}{dt} = -x_i + z + a_i f(x_i) + \sum_{0 < |k| \leq d} a_{i,k} f(x_{i+k}), \quad i \in \mathbb{Z}_N, \quad (1)$$

where $x_{i+k}$, $i + k \notin \mathbb{Z}_N$, satisfies certain boundary conditions described below. Here, $f(x)$ is a piecewise-linear output function defined by

$$f(x) = \frac{1}{2} (|x+1| - |x-1|), \quad (2)$$

and $y_i = f(x_i)$ is the output of cell at $i$. The quantity $z$ is called a threshold or bias term, and is related to independent voltage sources in electric circuits. The constant $d$ is a positive integer indicating the degree of interconnection among cells. The coupling coefficients of the output functions $a_{i,k}$ and $a_i$ are real constants called spatially variant A-templates and are denoted by

$$A_i \equiv [a_{i; -d}, \ldots, a_{i; -1}, a_i, a_{i; 1}, \ldots, a_{i; d}]. \quad (3)$$

Recently, theoretical and applied studies of the CNN model have focused on lattice dynamical systems [Hsu & Lin, 2000]. A basic and important class of solutions of (1) are the stable stationary solutions. Hence, related problems can be studied by...
examining two types of stationary solutions: mosaic and transitional. A mosaic solution \( x \) satisfies \( |x_i| > 1 \) for all \( i \in \mathbb{Z}^1 \), and a transitional solution satisfies \( |x_i| \geq 1 \) for all \( i \in \mathbb{Z}^1 \) and equality holds for some \( i \). Their corresponding patterns \( y \) are called respectively mosaic and transitional patterns. Two other types of stationary solutions exist: defective and linear. A defective solution \( x \) satisfies \( |x_i| > 1 \) for some \( i \in \mathbb{Z}^1 \) and \( |x_i| < 1 \) for \( k \in D \), where \( D \neq \emptyset \) and \( D \neq \mathbb{Z}^1 \). A linear solution \( x \) is a linear solution if \( |x_i| < 1 \) for all \( i \in \mathbb{Z}^1 \).

The outputs \( y = (f(x_i)) \), called patterns, are essential for understanding CNN systems. If the templates of (1) are spatially invariant, i.e. \( A_i = [r, p, s] \) for all \( i \), then many stationary solution spatial chaos results are obtained, see e.g. [Thiran, 1997; Juang & Lin 2000; Hsu, 2000]. In this work, we are interested in studying the numbers of stationary solutions of (1) on finite lattices and infinite lattices with spatially variant templates.

Since a practical CNN has finitely many cells, some boundary conditions (B.C.) must be imposed and realized. Here, we consider three types of boundary conditions: periodic, Neumann and Dirichlet. Below, conditions are illustrated for 1-D CNN with \( d = 1 \).

(I) **Periodic B.C.**
The cell at the right-most and the one at the left-most are connected to form a circular array. Specifically,
\[
x_{-N+1}(t) = x_N(t) \quad \text{(P.B.C.)}
\]
and
\[
x_{N+1}(t) = x_{-N}(t). \]

(II) **Neumann B.C.**
This is the zero-flux or reflective B.C., that is,
\[
x_{-N+1}(t) = x_{-N}(t) \quad \text{(N.B.C.)}
\]
and
\[
x_{N+1}(t) = x_N(t). \]

(III) **Dirichlet B.C.**
The absent cells \( x_{-N+1} \) and \( x_{N+1} \) are set to constants such that
\[
x_{-N+1}(t) = \bar{x}_{-N-1}, \quad x_{N+1}(t) = \bar{x}_{N+1}. \quad \text{(D.B.C)}
\]
where \( \bar{x}_{-N-1} \) and \( \bar{x}_{N+1} \) are constants with \( |\bar{x}_{-N-1}| > 1 \) and \( |\bar{x}_{N+1}| > 1 \).

By using the notation for transition matrices and boundary matrices (see Sec. 3), we can obtain the exact number of mosaic patterns formed by finite cells with spatially variant templates, and which generalize the work of [Ban et al., 2001b].

Let \( \{M_i\}_{i=-N}^N \), \( L^S \), and \( R^S \) be, respectively, the transition matrices of states \( \{x_i\}_{i=-N}^N \), boundary matrix of \( x_{-N} \) and boundary matrix of \( x_N \) (see Sec. 3). Here \( S = P, D, \text{ or, } N \) if the boundary conditions considered are, respectively, periodic, Dirichlet or Neumann. We also denote \( [M]_{i,j} \) as the \( ij \)-entry of matrix \( M \). The main results are as follows.

**Theorem 1** (Finite cells). Consider mosaic patterns of (1)-(3) on \( Z_N \) with various boundary conditions. Then the number \( \Gamma(Z_N) \) of mosaic patterns of (1) on \( Z_N \) is equal to
\[
\Gamma(Z_N) = \sum_{i,j=1}^4 \left[ L^S \cdot \prod_{k=1}^{N-1} M_k \cdot R^S \right]_{i,j}.
\]

Below, we use the symbols “+” and “−” to represent positive and negative saturated states as well as their output patterns, i.e. \( y_i = + \) or \( y_i = -1 \) if \( x_i > 1 \) or \( x_i < 1 \). Therefore, the elements in \( \{+,−\}^{2N} \), give all possible mosaic patterns on \( Z_N \). The results regarding the number of mosaic patterns with spatially invariant template in [Thiran, 1997] and [Ban et al., 2001b] can be recovered using the above theorem.

Notably, the patterns obtained on finite lattice \( Z_N \) can be considered restrictions of global mosaic patterns on \( Z^1 \), see [Shih, 2000]. Therefore, it is important to investigate the behavior or complexity of numbers of mosaic patterns on \( Z_N \) when \( N \) tends toward infinity. One quantity crucial to the study of behavior is spatial entropy, see Definition 4.1 [Chow & Mallet-Paret, 1995] or [Robinson, 1995]. According to Theorem 1, the entropy of mosaic patterns of (1) with spatially periodic templates (see Sec. 4) can be computed exactly as follows.

**Theorem 2** (Infinite cells). Let \( \ell \) and \( \tau \) be positive integers and \( \{A_i\}_{i=1}^{\ell \tau} \) be spatial periodic templates on finite lattice \( \{i\}_{i=1}^{\ell \tau} \) with period \( \tau \). Denote matrix \( M(\tau) = \prod_{j=1}^{\ell \tau} M_j \), the entropy of mosaic patterns of (1) will then be equal to \( 1/\tau \ln \lambda_\tau \). Here, \( \lambda_\tau \) is the largest positive maximal eigenvalue of \( M(\tau) \).

The rest of this paper is organized as follows. In Sec. 2, the conditions of the existence of feasible local patterns are given. According to these conditions, the parameters space of (1) can be partitioned.


into finite many regions such that feasible local patterns in each region can be determined. Section 3 constructs the transition matrices and boundary matrices in the case of spatially variant templates, and shows that the number of mosaic patterns on finite cells can be computed precisely. In Sec. 4, we obtain the entropy of mosaic patterns for spatially periodic templates and give some numerical results.

2. Partition of the Parameter Space

To construct mosaic patterns on $Z_N$, we need to determine the number of feasible local patterns for each cell.

Definition 2.1. Denote $N_d(0) = \{k| - d \leq k \leq d\}$.

(i) A set $\mathbb{S} \subset Z_N$ is called feasible if $\mathbb{S} = \mathbb{S}_d(i)$ $\equiv \{i + k|k \in N_d(0)\}$ for some $i \in Z_N$.

(ii) A feasible (mosaic) solution is a vector $\{x_{i+k}\}_{k \in N_d}$ defined on a feasible set $\mathbb{S}_d(i)$, with each $|x_{i+k}| > 1$, which satisfies the equations in (1).

(iii) The output patterns corresponding to feasible solutions are called feasible local patterns.

In this section, we partition the parameter space of (1) into finite many regions such that feasible local patterns can be determined in each region. Consider the general one-dimensional spatially variant templates,

$$A_i = [a_{i,-d} \cdots a_{i,-1} a_i a_{i,1} \cdots a_{i,d}]$$

For a given mosaic solution $x$, the state at cell $C_i$ is $+$, i.e. $x_i > 1$, if and only

$$\sum_{0<|k|\leq d} a_{i;k} y_{i+k} + a_i + z - 1 > 0 \quad (4)$$

Similarly, the state at cell $C_i$ is $-$, i.e. $x_i < -1$, if and only if

$$\sum_{0<|k|\leq d} a_{i;k} y_{i+k} - a_i + z + 1 < 0 \quad (5)$$

Herein, $y_{i+k}, i + k \notin Z_N$, are determined from the imposed boundary conditions. Therefore, the set of feasible local patterns for state $x_i$ with template $A_i$ and bias $z$ is

$$\Omega(A_i; z) := \Omega^+(A_i, z) \cup \Omega^-(A_i, z),$$

where

$$\Omega^+(A_i, z) = \left\{ (y_{i-d} \cdots y_{i-1} y_i y_{i+1} \cdots y_{i+d}) \right\} \sum_{0<|k|\leq d} a_{i;k} y_{i+k} + a_i + z - 1 > 0,$$

$$\Omega^-(A_i, z) = \left\{ (y_{i-d} \cdots y_{i-1} y_{i+1} \cdots y_{i+d}) \right\} \sum_{0<|k|\leq d} a_{i;k} y_{i+k} - a_i + z + 1 < 0.$$
In this section, we construct a scheme for obtaining the feasible patterns on $Z_N$. This scheme can be exactly implemented by formulating suitable transitions and boundary matrices. Consequently, the number of patterns on a finite lattice with certain boundary conditions can be computed precisely.

### 3. Transition and Boundary Matrices

In this section, we construct a scheme for obtaining the feasible patterns on $Z_N$. This scheme can be exactly implemented by formulating suitable transitions and boundary matrices. Consequently, the number of patterns on a finite lattice with certain boundary conditions can be computed precisely.

#### 3.1. Transition matrix

Firstly, we identify the indices $\{1, 2, 3, 4\}$ and the four $1 \times 2$ patterns $\{++, +-, -+, --\}$ using

\[
\begin{align*}
1 & \longleftrightarrow ++ , \\
2 & \longleftrightarrow +-, \\
3 & \longleftrightarrow -- , \\
4 & \longleftrightarrow -+ .
\end{align*}
\]

For simplicity, if $i \in \{1, 2, 3, 4\}$ then denote $i^+$ and $i^-$, respectively, as the left and right patterns of $i$ in (6). For example, $2^+ = -$ and $2^- = +$. As shown in Table 1, the transition matrix $M_i \equiv M_{i;m_k,n_k}$ of state $x_i$ with template $A_i \in [m_k,n_k]$, $0 \leq m_k, n_k \leq 4$ will be of the form

\[
M_i = \begin{bmatrix}
++ & +- & -+ & -- \\
++ & 0 & 0 & 0 \\
+ & m_{3,1} & m_{3,2} & 0 \\
- & 0 & 0 & m_{4,3} & m_{4,4}
\end{bmatrix}.
\]

The formation of feasible local patterns related to the transition matrix can be described as follows: the $(i,j)$-entry of $M_i$ is one if and only if the $j$th $1 \times 2$ pattern in (6) can be joined, with one site overlapped, to the right of the $i$th $1 \times 2$ pattern in (6) to form a $1 \times 3$ feasible local pattern in $\Omega(A_i,z)$. Some transition matrices are listed in Table 2.

One can use these transition matrices $M_i$ to generate patterns on lattices of lengths greater than three. For example, the nonzero $(1,2)$-entry of $M_2$: $M_3$, gives the patterns $(y_1,y_2,y_3,y_4) = (+,+,+,+)$ on a $1 \times 4$ lattice. Indeed, the $(i,j)$-th entry of $\prod_{k=-N+1}^{N-1} M_k$ gives the number of patterns on $Z_N$ with the $i$th $1 \times 2$ pattern in (6) at the two sites to the left-most of $Z_N$ and the $j$th $1 \times 2$ pattern in (6) at the two sites to the right-most of $Z_N$. For example, when $i = 2$, $j = 4$, the patterns on $Z_N$ are

<table>
<thead>
<tr>
<th>$m$</th>
<th>Feasible Local Patterns</th>
<th>$n$</th>
<th>Feasible Local Patterns</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>$+\oplus,-\oplus,+\oplus,-\oplus$</td>
<td>4</td>
<td>$-\oplus,-\oplus,-\oplus,+\oplus$</td>
</tr>
<tr>
<td>3</td>
<td>$+\oplus,-\oplus,+\oplus$</td>
<td>3</td>
<td>$-\oplus,+\oplus,-\oplus$</td>
</tr>
<tr>
<td>2</td>
<td>$+\oplus,-\oplus$</td>
<td>2</td>
<td>$-\oplus$</td>
</tr>
<tr>
<td>1</td>
<td>$+\oplus$</td>
<td>1</td>
<td>$+\oplus$</td>
</tr>
<tr>
<td>0</td>
<td>None</td>
<td>0</td>
<td>None</td>
</tr>
</tbody>
</table>

Fig. 2. Partition of $(z, p_i - 1)$ plane when $0 < r_i < s_i$. and two-dimensional CNN with spatially invariant templates. As Fig. 2 shows, there are various sets of parameters $(A_i,z)$ by which the corresponding sets of feasible local patterns are identical. In order to obtain the feasible patterns on $Z_N$, we need to glue such feasible local patterns cell by cell. Hence, some transition conditions for the interior cells and boundary conditions for the boundary cells are needed to match in the following section.
of the form,

\[+ - \cdots - -\]

If the states \(x_N\) and \(x_{N-1}\) satisfy the imposed B.C., defined in Sec. 1, then the above construction of transition matrices allow us to count the number of mosaic patterns on \(Z_{N-1}\). Therefore, it is necessary to describe the feasible patterns on the boundary sites with respect to the imposed B.C. using matrices called boundary matrices as shown below.

### 3.2. Boundary matrices

Assume \(A_{-N} \in [m_k, n_k]_{-N}\). We use the symbols \(L^P(\text{or } L^P_{m_k, n_k})\), \(L^N(\text{or } L^N_{m_k, n_k})\), and \(L^D(\text{or } L^D_{m_k, n_k})\) to denote the left-most boundary matrices for state \(-N\) with periodic, Neumann and Dirichlet boundary conditions, respectively. Similarly, the symbols \(R^P(\text{or } R^P_{m_k, n_k})\), \(R^N(\text{or } R^N_{m_k, n_k})\) and \(R^D(\text{or } R^D_{m_k, n_k})\) denote the right-most boundary matrices for state \(x_N\) with periodic, Neumann and Dirichlet boundary conditions, respectively. The boundary matrices are defined below.

#### 3.2.1. Periodic boundary matrices

As defined in (6), let \(\{y_k\}_{k=-N}^{k=N}\) be a pattern on \(Z_N\) such that \(\{y_k\}_{k=-N-1}^{k=-N+1}\) is a feasible pattern on \(Z_{N-1}\) with \(y_{N-1}y_{N+1} = i\) and \(y_{N-1}y_N = j\), \(1 \leq i, j \leq 4\). If \(\{x_k\}_{k=-N}^{k=-N}\) satisfies the (P.B.C.), we need to check the feasibility of local patterns for cells at \(-N\) and \(N\) sites. According to Fig. 3, the right-most \(j\)th \(1 \times 2\) pattern can be joined to the left-most of the \(i\)th \(1 \times 2\) pattern if and only if \(y_{N-1}y_N y_{N-1}y_N = i\) is a feasible local pattern for cell \(-N\), and the left-most \(i\)th \(1 \times 2\) pattern can be joined to the right-most of the \(j\)th \(1 \times 2\) pattern if and only if \(y_{N-1}y_N y_{N-1}y_N = j\) is a feasible local pattern for cell \(N\). Hence, the left boundary matrix \(L^P\) and right boundary matrices \(R^P\) can be constructed as follows:

\[
L^P = \begin{bmatrix}
\ell_{1,1} & \ell_{1,2} & 0 & 0 \\
0 & 0 & \ell_{2,3} & \ell_{2,4} \\
\ell_{3,1} & \ell_{3,2} & 0 & 0 \\
0 & 0 & \ell_{4,3} & \ell_{4,4}
\end{bmatrix},
\]

\[
R^P = \begin{bmatrix}
\ell_{1,1} & \ell_{1,2} & 0 & 0 \\
0 & 0 & \ell_{2,3} & \ell_{2,4} \\
\ell_{3,1} & \ell_{3,2} & 0 & 0 \\
0 & 0 & \ell_{4,3} & \ell_{4,4}
\end{bmatrix}.
\]

Here,

\[
\ell_{k,i} = 1 \quad \text{for} \quad 1 \leq i, k \leq 4 \quad \text{if and only if} \quad [M_{-N}]_{k,i} \neq 0 \quad \text{and} \quad [M_{N}]_{j,k} \neq 0 \quad \text{for some} \quad 1 \leq j \leq 4
\]

(P.B.C.)

\[
r_{j,k} = 1 \quad \text{for} \quad 1 \leq j, k \leq 4 \quad \text{if and only if} \quad [M_{-N}]_{k,i} \neq 0 \quad \text{and} \quad [M_{N}]_{j,k} \neq 0 \quad \text{for some} \quad 1 \leq i \leq 4.
\]

#### 3.2.2. Neumann boundary matrices

Let \(\{y_k\}_{k=-N}^{k=N}\) be a pattern on \(Z_N\) such that \(\{y_k\}_{k=-N-1}^{k=-N+1}\) is a feasible pattern on \(Z_{N-1}\) with \(y_{N-1}y_N y_{N+1} = i\) and \(y_{N-1}y_N = j\). The Neumann boundary matrices can be similarly defined.
Fig. 3. Patterns with periodic boundary conditions.

Fig. 4. Patterns for Neumann boundary conditions.

According to Fig. 4, the left boundary matrix $L^N$ and right boundary matrices $R^N$ are defined as follows:

$$L^N = + + \begin{bmatrix} \ell_{1,1} & \ell_{1,2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \ell_{4,3} & \ell_{4,4} \end{bmatrix}, \quad R^N = + + \begin{bmatrix} r_{1,1} & 0 & 0 & 0 \\ 0 & 0 & 0 & r_{2,4} \\ r_{3,1} & 0 & 0 & 0 \\ 0 & 0 & 0 & r_{4,4} \end{bmatrix}.$$

Here,

$$\ell_{1,i} = 1, \ i = 1, 2 \quad \text{if and only if} \quad [M_{-N}]_{1,i} = 1,$$
$$\ell_{4,i} = 1, \ i = 3, 4 \quad \text{if and only if} \quad [M_{-N}]_{4,i} = 1,$$
$$r_{i,1} = 1, \ i = 1, 3 \quad \text{if and only if} \quad [M_N]_{i,1} = 1,$$
$$r_{i,4} = 1, \ i = 2, 4 \quad \text{if and only if} \quad [M_N]_{i,4} = 1.$$ (N.B.C.)

Some Neumann boundary matrices are listed in Table 3.

### 3.2.3. Dirichlet boundary matrices

Let $\{y_k\}_{k=-N+1}^{N+1}$ be a pattern on $Z_{N+1}$ such that $\{y_k\}_{k=-N}^{N}$ is a feasible pattern on $Z_N$ with $\bar{y}_{-N-1} \bar{y}_{-N} = i$ and $\bar{y}_N \bar{y}_{N+1} = j$. According to Fig. 5, it is obvious that $\{y_k\}_{k=-N}^{N}$ satisfies (D.B.C.) if and only if $i^-$ and $j^+$ are equal to the output patterns of $\bar{y}_{-N-1}$ and $\bar{y}_{N+1}$, respectively.

**Table 3. Neumann boundary matrices.**

<table>
<thead>
<tr>
<th>$L_{m,n}$, $n \geq 3$</th>
<th>$L_{2,3}^N$</th>
<th>$L_{3,2}^N$</th>
<th>$L_{m,n}^N$, $2 \geq m$, $n \geq 1$</th>
<th>$L_{1,0}^N$</th>
<th>$L_{0,1}^N$</th>
<th>$L_{0,0}^N$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\ell_{1,1}$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\ell_{1,2}$</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\ell_{4,3}$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\ell_{4,4}$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>
The left boundary matrix $L^D$ and right boundary matrices $R^D$ are defined as follows:

$$L^D = \begin{bmatrix} ++ & +- & -+ & -- \\ +- & \ell_{1,1} & \ell_{1,2} & 0 & 0 \\ -- & 0 & 0 & \ell_{2,3} & \ell_{2,4} \\ -+ & \ell_{3,1} & \ell_{3,2} & 0 & 0 \\ -- & 0 & 0 & \ell_{4,3} & \ell_{4,4} \end{bmatrix}, \quad R^D = \begin{bmatrix} ++ & +- & -+ & -- \\ +- & r_{1,1} & r_{1,2} & 0 & 0 \\ -- & 0 & 0 & r_{2,3} & r_{2,4} \\ -+ & r_{3,1} & r_{3,2} & 0 & 0 \\ -- & 0 & 0 & r_{4,3} & r_{4,4} \end{bmatrix}.$$  

Here

$$\ell_{i,j} = 1 \text{ if and only if } i^- = \bar{y}_{-N-1} \text{ and } [M_{-N}]_{i,j} \neq 0,$$

$$r_{i,j} = 1 \text{ if and only if } j^+ = \bar{y}_{N+1} \text{ and } [M_N]_{i,j} \neq 0.$$  

![Fig. 5. Patterns with Dirichlet boundary conditions.](image)

Using the preceding arguments, we prove Theorem 1 as follows.

**Proof of Theorem 1.** Proof is given only for the case of periodic boundary conditions; the other cases can be proved in similar fashion.

According to Fig. 3, let $\{y_k\}_{k=-N}^{k=N}$ be a pattern on $Z_N$ such that $\{y_k\}_{k=-N+1}^{k=N-1}$ is a feasible pattern on $Z_{N-1}$ with $y_{-N}y_{-N+1} = i$ and $y_{N-1}y_N = j$. Then the number of such feasible patterns on $Z_{N-1}$ is equal to $[\prod_{-N+1}^{N-1} M_k]_{i,j}$. Hence, $\{y_k\}_{k=-N}^{k=N}$ is a feasible pattern on $Z_N$ equivalent to that $\bar{y}_{-N}y_{-N+1}$ is a feasible local pattern for cell $-N$, and $\bar{y}_{N-1}y_Ny_{-N}$ is a feasible local pattern for cell $N$. Therefore, according to the construction (P.B.C.) of periodic boundary matrices, the results obviously follow. The proof is complete. 

If the templates considered are spatially invariant, i.e. $A_i = [r, p, s]$ for all cells, then the results of Theorem 1 replicate the results obtained by [Ban et al., 2001b; Thiran, 1997]. For details see Table 4.

### 4. Results of Entropy

Obtaining the numbers of feasible patterns for various boundary conditions was shown in the preceding section. It is also important and interesting to study the behavior of such number with respect to the length of $Z_N$. One way to investigate this behavior is to compute the entropy (see [Chow & Mallet-Paret, 1995]) of stationary solutions of (1). In one-dimensional cases, the entropy function $h$ on $Z^1$ can be defined as

**Definition 4.1.** Let $\Gamma(Z_N)$ be the number of feasible patterns on $Z_N$ that satisfy the imposed boundary conditions. Thus,

$$h_N(Z_N) \equiv \frac{\ln \Gamma(Z_N)}{2N+1}$$

and

$$h(Z^1) \equiv \lim_{N \to \infty} \frac{\ln \Gamma(Z_N)}{2N+1}.$$  

$h(Z^1)$ is called the entropy of stationary solutions of (1). Furthermore, if $h(Z^1) > 0$ or $h(Z^1) = 0$ then

<table>
<thead>
<tr>
<th>$R_{m,n}^N$, $n \geq 2$</th>
<th>$R_{1,2}^N$</th>
<th>$R_{2,1}^N$</th>
<th>$R_{1,1}^N$</th>
<th>$R_{1,0}^N$</th>
<th>$R_{0,1}^N$</th>
<th>$R_{0,0}^N$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_{1,1}$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$r_{2,4}$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$r_{3,1}$</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$r_{4,4}$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

**Table 3. (Continued).**
Table 4. Exact number of mosaic patterns for a spatially invariant template.

<table>
<thead>
<tr>
<th>Region in $r - s$ Plane</th>
<th>Region in $z - p$ Plane</th>
<th>P.B.C.</th>
<th>N.B.C.</th>
</tr>
</thead>
<tbody>
<tr>
<td>[4, 4]</td>
<td>$2^H$</td>
<td>$2^H$</td>
<td></td>
</tr>
<tr>
<td>(I), (II)</td>
<td>[3, 3]</td>
<td>$\Gamma_2(H)$</td>
<td>$\Gamma_1(H)$</td>
</tr>
<tr>
<td>[2, 2]</td>
<td>2</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>(III), (VIII)</td>
<td>[3, 3]</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>[2, 2]</td>
<td>0</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>(IV), (VII)</td>
<td>[4, 4]</td>
<td>$2^H$</td>
<td>$2^H$</td>
</tr>
<tr>
<td>[3, 3]</td>
<td>3 + $(-1)^H$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>[2, 2]</td>
<td>1 + $(-1)^H$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(V), (VI)</td>
<td>[4, 4]</td>
<td>$2^H$</td>
<td>$2^H$</td>
</tr>
<tr>
<td>[3, 3]</td>
<td>$\Gamma_2(H)$</td>
<td>$\Gamma_1(H)$</td>
<td></td>
</tr>
<tr>
<td>[2, 2]</td>
<td>2</td>
<td>1 + $(-1)^H$</td>
<td></td>
</tr>
</tbody>
</table>

$H = 2N + 1.$

\[
\Gamma_1(H) = \left[ \left(\frac{1 + \sqrt{5}}{2}\right)^H + \left(\frac{1 - \sqrt{5}}{2}\right)^H \right] + 2 \cos \left(\frac{2\pi}{3} H\right),
\]

\[
\Gamma_2(H) = \frac{2}{\sqrt{5}} \left[ \left(\frac{1 + \sqrt{5}}{2}\right)^{H+1} - \left(\frac{1 - \sqrt{5}}{2}\right)^{H+1} \right].
\]

(1) is said to be spatial chaos or pattern formation, respectively.

Shih [2000] proved that boundary conditions have negligible influence on pattern formation and spatial chaos of mosaic patterns of (1) with spatially invariant templates. If fact, by the same arguments, the results also hold for the case of spatially variant templates.

If the templates are spatially invariant, according to [Juang & Lin, 2000], the entropy $h$ of mosaic patterns on $\mathbb{Z}^1$ is

\[
h = \begin{cases} 
\ln 2 & \text{if } (z, p) \in [4, 4], \\
\ln \lambda & \text{if } (z, p) \in [4, 3] \text{ or } [3, 4], \\
\ln \frac{1 + \sqrt{5}}{2} & \text{if } (z, p) \in [3, 3], \\
0 & \text{otherwise.}
\end{cases}
\]

On the other hand, if the templates are spatially variant, the structure of the entropy function is quite different. Some partial entropy will appear for different combinations of templates. For instance, if we assume one template for odd cells and another template for even cells, then the entropy $h$ on $\mathbb{Z}^1$ is computed as follows.

**Example 4.1.** Let $N \geq 3$ be a positive integer and

\[
A_k = [r_k, p_k, s_k] \in \begin{cases} 
[4, 1], & \text{if } k \text{ is odd,} \\
[1, 4], & \text{if } k \text{ is even,}
\end{cases}
\]

on $\mathbb{Z}_N$. If we consider the mosaic patterns of (1) on $\mathbb{Z}_N$ with different boundary conditions imposed as in previous section, then the entropy functions are equal. Moreover, we have

\[
h(\mathbb{Z}^1) = \ln \frac{1 + \sqrt{5}}{2}.
\]

**Proof.** Without lost of generality, we may assume $\mathbb{Z}_N = \{i\}_{i=1}^N$. First, let $\{S(K)\}_{K=1}^\infty$ be a Fibonacci sequence, i.e.

\[
S(1) = 1, \quad S(2) = 2,
\]

and

\[
S(K + 2) = S(K + 1) + S(K), \quad \text{if } K \geq 1.
\]
It is well known that

$$S(K) = \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^K - \left( \frac{1 - \sqrt{5}}{2} \right)^K \right]. \quad (9)$$

Using (8), if $K \geq 3$ then it is not difficult to compute that

$$\prod_{k=2}^{2K} M_k = \begin{bmatrix}
S(2K-4) & 0 & S(2K-3) & S(2K-3) \\
S(2K-3) & 0 & S(2K-2) & S(2K-2) \\
0 & 0 & 0 & 0 \\
S(2K-3) & 0 & S(2K-2) & S(2K-2)
\end{bmatrix},$$

and

$$\prod_{k=2}^{2K-1} M_k = \begin{bmatrix}
S(2K-4) & S(2K-4) & 0 & S(2K-5) \\
S(2K-3) & S(2K-3) & 0 & S(2K-4) \\
0 & 0 & 0 & 0 \\
S(2K-3) & S(2K-3) & 0 & S(2K-4)
\end{bmatrix},$$

by (10), we have

$$\Gamma(Z_{2K+1}) = S(2K+2), \quad K \geq 3,$$

and

$$\Gamma(Z_{2K}) = S(2K+3), \quad K \geq 3.$$ 

**Case (II) (N.B.C.).** Elementary computation yields

$$L^N = \begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}, \quad R^N = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},$$

Hence, using Theorem 1 and (9), we obtain

$$\Gamma(Z_{2K+1}) = S(2K+2), \quad K \geq 3,$$

and

$$\Gamma(Z_{2K}) = S(2K+1), \quad K \geq 3.$$ 

**Case (III) (D.B.C.).** In this case, we consider $f(\tilde{x}_0) = 1$ and $f(\tilde{x}_{N+1}) = 1$. The other cases can also be investigated in similar fashion. Using Theorem 1, it is easy to determine that

$$L^D = \begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad R^D = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},$$

and

$$R^D = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.$$ 

Therefore,

$$\Gamma(Z_{2K+1}) = \Gamma(Z_{2K}) = S(2K), \quad K \geq 3.$$ 

Consequently, according to Definition 4.1 and (9) and the computations above, we have $h(Z^1) = \ln(1 + \sqrt{5})/2$. The proof is complete.
Remark 4.1.

(i) The preceding example shows the influence of boundary conditions on pattern formation and spatial chaos is negligible. In general, this is also true for any type of spatially variant templates. For details, see [Shih, 2000].

(ii) According to (7), entropy is equal to zero if the template is spatially invariant in region $[4, 1]$ or $[1, 4]$. However, the entropy obtained in Example 4.1 is equal to the entropy obtained by [Juang & Lin, 2000] in region $[3, 3]$ although the patterns are quite different. Therefore, the structure of entropy is more complicated when templates are spatially variant.

Motivated by Example 4.1, we introduce a definition of periodic templates and prove Theorem 2 below. For simplicity, we consider the finite lattices $Z_N = \{k\}_{k=1}^N$ and $\Gamma(Z_N) = \sum_{i,j=1}^{4} [\prod_{k=1}^{N} M_k]_{i,j}$ hereinafter.

**Definition 4.2.** Let $\ell$ and $\tau$ be positive integers. The family of templates $A_i = [r_i, p_i, s_i] \in [m_i, n_i]$, $i = 1$ to $\ell \tau$ are called spatially periodic on $Z_{\ell \tau}$ with period $\tau$ if

$$m_i = m_j \text{ and } n_i = n_j, \text{ for } i = j \text{ (mod } \tau).$$

Here, $1 \leq j \leq \tau$, $1 \leq i \leq \ell \tau$ and $0 \leq m_i, n_i \leq 4$.

**Proof of Theorem 2.** When $\tau = 1$, according to [Chow & Mallet-Paret, 1995], we have

$$\lim_{\ell \to \infty} h_{\ell}(Z_{\ell}) = \lim_{\ell \to \infty} \frac{1}{\ell} \sum_{i,j=1}^{4} \left[ \prod_{k=1}^{\ell} M(1) \right]_{i,j} = \ln \lambda_1,$$

where $\lambda_1$ is the largest positive maximal eigenvalue of $M(1)$. From Theorem 1, we obtain

$$\Gamma(Z_{\ell \tau}) = \sum_{i,j=1}^{4} \left[ \prod_{k=1}^{\ell \tau} M(\tau) \right]_{i,j}$$

![Fig. 6. Structure of $h(Z^1)$ when $\tau = 2$.](image)
Table 5. Characteristic equations, exact entropy and numerical results.

<table>
<thead>
<tr>
<th>Class</th>
<th>$P(2)$</th>
<th>$h(Z^\infty) = \frac{1}{2} \ln \lambda_2$</th>
<th>$h_{1000}(Z_{1000})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$\lambda^4 - 4\lambda^3$</td>
<td>0.6931</td>
<td>0.6945</td>
</tr>
<tr>
<td>B</td>
<td>$\lambda^4 - 3\lambda^3 - 2\lambda^2$</td>
<td>0.6350</td>
<td>0.6366</td>
</tr>
<tr>
<td>C</td>
<td>$\lambda^4 - 3\lambda^3 - \lambda^2 - \lambda$</td>
<td>0.6904</td>
<td>0.6109</td>
</tr>
<tr>
<td>D</td>
<td>$\lambda^4 - 2\lambda^3 - 3\lambda^2 - \lambda$</td>
<td>0.5624</td>
<td>0.5640</td>
</tr>
<tr>
<td>E</td>
<td>(i) $\lambda^4 - 2\lambda^3 - 3\lambda^2$</td>
<td>0.5493</td>
<td>0.5510</td>
</tr>
<tr>
<td></td>
<td>(ii) $\lambda^4 - 3\lambda^3$</td>
<td>0.6350</td>
<td>0.6366</td>
</tr>
<tr>
<td>F</td>
<td>$\lambda^4 - 2\lambda^3 - 2\lambda^2 - \lambda$</td>
<td>0.5203</td>
<td>0.5220</td>
</tr>
<tr>
<td>G</td>
<td>(i) $\lambda^4 - 2\lambda^3 - \lambda^2 - 2\lambda + 1$</td>
<td>0.4812</td>
<td>0.4830</td>
</tr>
<tr>
<td></td>
<td>(ii) $\lambda^4 - 3\lambda^3 + \lambda^2$</td>
<td>0.366</td>
<td>0.3486</td>
</tr>
<tr>
<td>H</td>
<td>$\lambda^4 - 2\lambda^3 - \lambda^2$</td>
<td>0.4407</td>
<td>0.4426</td>
</tr>
<tr>
<td>I</td>
<td>$\lambda^4 - 2\lambda^3 - \lambda$</td>
<td>0.3955</td>
<td>0.3975</td>
</tr>
<tr>
<td>J</td>
<td>$\lambda^4 - 2\lambda^3$</td>
<td>0.3466</td>
<td>0.3486</td>
</tr>
<tr>
<td>K</td>
<td>$\lambda^4 - 2\lambda^3 + \lambda^2 - \lambda$</td>
<td>0.2812</td>
<td>0.2834</td>
</tr>
<tr>
<td>L</td>
<td>$\lambda^4 - \lambda^3 - \lambda^2$</td>
<td>0.2466</td>
<td>0.2428</td>
</tr>
</tbody>
</table>

Table 6. Numerical results for $h(Z^1)$ with $3 \leq \tau \leq 6$.

<table>
<thead>
<tr>
<th>$\tau$</th>
<th>$T(\tau)$</th>
<th>$P(\tau)$</th>
<th>$DP(\tau)$</th>
<th>$C(\tau)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>25$^1$</td>
<td>4</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>25$^2$</td>
<td>152</td>
<td>14</td>
<td>12</td>
</tr>
<tr>
<td>3</td>
<td>25$^3$</td>
<td>4420</td>
<td>33</td>
<td>27</td>
</tr>
<tr>
<td>4</td>
<td>25$^4$</td>
<td>121456</td>
<td>90</td>
<td>84</td>
</tr>
<tr>
<td>5</td>
<td>25$^5$</td>
<td>3209404</td>
<td>214</td>
<td>195</td>
</tr>
<tr>
<td>6</td>
<td>25$^6$</td>
<td>82014632</td>
<td>577</td>
<td>497</td>
</tr>
</tbody>
</table>

and

$$\lim_{\ell \to \infty} h_\ell(Z_{\ell \tau}) = \lim_{\ell \to \infty} \frac{\Gamma(Z_{\ell \tau})}{\ell \tau} = \frac{1}{\tau} \ln \lambda_\tau.$$ 

The proof is complete. \[\square\]

The characteristic equations, $h(Z^1)$ and $h_N$ for $N = 1000$ are also listed in Table 5.

Remark 4.2. Some numerical results are listed in Table 6 for the case of $3 \leq \tau \leq 6$ with:

$T(\tau)$ $\equiv$ total combination of templates,

$P(\tau)$ $\equiv$ all combinations of templates with positive entropy,

$DP(\tau)$ $\equiv$ distinct characteristic polynomials with positive entropy,

$C(\tau)$ $\equiv$ distinct positive entropies.

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