Conclusion: In this note, the regulator problem of linear continuous-time systems with nonsymmetrical constrained control is studied. Necessary and sufficient conditions for domain $\mathcal{D}(F, q_1, q_2)$, which generates admissible control by feedback law, to be a positively invariant set w.r.t. system (6), are given. These conditions guarantee that system (1)-(7) is asymptotically stable for every motion emanating from domain $\mathcal{D}(F, q_1, q_2)$. A spectral analysis of equation $FA + FBH = HF$ is also given together with conditions on the existence of matrix $H$. The necessary condition of the main result is established by using an important property of the $\mathcal{D}(F, q_1, q_2)$: when domain $\mathcal{D}(F, q_1, q_2)$ is positively invariant w.r.t. system (6), $\mathcal{D}(F, q_2)$ is also positively invariant w.r.t. the system. Finally, the case of symmetrical constrained control is obtained easily by taking $q_1 = q_2 = p$.

REFERENCES

Optimal Periodic Control Implemented as a Generalized Sampled-Data Hold Output Feedback Control
Nie-Zen Yen and Yung-Chun Wu

Abstract—In this note, a conversion method to convert the analog linear quadratic regulation control to a generalized sampled-data hold output feedback control for a linear periodic system or a linear time-invariant system is presented. It is shown that by using such a conversion, one can implement the optimal periodic control scheme in the presence of incomplete and delayed state measurements.

I. INTRODUCTION
Consider the optimal control problem of a linear periodic system
\[
\dot{x}(t) = A(t)x(t) + B(t)u(t) \quad (1a)
\]
\[
y(t) = C(t)x(t) + D(t)u(t) \quad (1b)
\]
to minimize the following quadratic performance index:
\[
J = \int_0^T x^T(t)Q(t)x(t) + u^T(t)R(t)u(t) \, dt \quad (2)
\]
where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control, $y \in \mathbb{R}^p$ is the output measurement, and $A(t), B(t), C(t), D(t), Q(t)$, and $R(t)$ are continuous matrix functions which satisfy the periodicity property that $A(t) = A(t + T), B(t) = B(t + T), C(t) = C(t + T), D(t) = D(t + T), Q(t) = Q(t + T)$ and $R(t) = R(t + T)$, where $T$ is the periodic time, $Q(t) \in \mathbb{R}^{n \times n}$ is positive semi-definite, and $R(t) \in \mathbb{R}^{m \times m}$ is positive definite.

It is known (e.g., [2]–[5]) that the above linear quadratic regulation (LQR) control problem can be solved by the following periodic state feedback:
\[
u(t) = G(t)x(t) \quad (3a)
\]
where
\[
G(t) = -R(t)^{-1}B^T(t)P(t) \quad (3b)
\]
and $P(t) \in \mathbb{R}^{n \times n}$ is a periodic positive semi-definite matrix function solved from the following periodic Riccati equation (if the solution exists):
\[
-\dot{P}(t) = A(t)P(t) + P(t)A^T(t) - P(t)B(t)R(t)^{-1}B^T(t)P(t) + Q(t) \quad (3c)
\]
In general, to implement the analog optimal periodic control scheme (3.a) needs complete state measurement. In this note, one converts the control scheme into a generalized-sampled-data hold control using the output measurement (1.b) only.

II. DEVELOPMENT
A. Generalized Sampled-Data Hold Control
The periodic system (1) with the optimal periodic control (3.a) yields the following closed-loop system:
\[
\dot{x}(t) = (A(t) + B(t)G(t))x(t) \quad (4)
\]
Associated to the closed-loop system, one defines $\phi(t, \theta), \Psi(t, \theta)$, and $\Psi_p(t, \theta)$ as the state transition matrices satisfying the following three differential equations, respectively:
\[
\frac{d}{dt}\phi(t, \theta) = A(t)\phi(t, \theta); \quad \phi(\theta, \theta) = I_n \quad (5)
\]
\[
\frac{d}{dt}\Psi(t, \theta) = (A(t) + B(t)G(t))\Psi(t, \theta); \quad \Psi(\theta, \theta) = I_n \quad (6)
\]
\[
\frac{d}{dt}\Psi_p(t, \theta) = \left(\begin{array}{cc} A(t) & B(t)G(t) \\ O_n & A(t) + B(t)G(t) \end{array}\right)\Psi_p(t, \theta); \quad \Psi_p(\theta, \theta) = I_{2n} \quad (7)
\]
Where $I_n$ denotes the $n \times n$ identity matrix, $O_n$ denotes the $n \times n$ zero matrix. It is easy to check the following equality (see Lemma 1 in the Appendix):
\[
\Psi_p(t, \theta) = \left(\begin{array}{cc} \phi(t, \theta) & B_p(t, \theta) \\ O_n & \Psi(t, \theta) \end{array}\right) \quad (8)
\]
where
\[ B(t, \theta) = \int_0^t \phi(t, s)B(s)G(s)\Psi(s, \theta) \, ds = \Psi(t, \theta) - \phi(t, \theta). \] (9)

Based on the optimal periodic control (3.a) and the closed-loop system (4), one can formulate a generalized sampled-data hold control (Kabamba [6]) as follows:
\[ u(kT + \theta) = G(\theta)\hat{x}(kT + \theta) = G(\theta)\Psi(\theta, 0)\hat{x}(kT), \] (10)
where \( k = 0, 1, 2, \ldots, \theta \in [0, T) \), \( \hat{x}(kT + \theta) = \Psi(\theta, 0)\hat{x}(kT) \), and \( \hat{x}(kT + \theta) \in R^n \). Notice that if \( \hat{x}(kT) = x(kT) \), then \( \hat{x}(kT + \theta) = x(kT + \theta) \) for all \( \theta \in [0, T) \) (see Lemma 2 in the Appendix), so that (10) is just equivalent to (3.a). Now, one defines
\[ \hat{\lambda}(kT) = \lim_{\theta \to T^-} \hat{x}((k - 1)T + \theta) = \Psi(T, 0)\hat{x}((k - 1)T) \] (11)
then one has
\[ u(kT - \sigma) = G(T - \sigma)\Psi(T - \sigma, 0)\hat{x}((k - 1)T) \]
\[ = [O_m \times n \ G(T - \sigma)]\Psi(T - \sigma, T) \left( x(kT) \right) \hat{\lambda}(kT), \] (12.a)
and
\[ x(kT - \sigma) - \phi(kT - \sigma, kT) x(kT) \]
\[ + \int_{kT}^{kT - \sigma} \phi(kT - \sigma, s)B(s)u(s) \, ds = \phi(T - \sigma, T)x(kT) \]
\[ + \int_{kT}^{T - \sigma} \phi(T - \sigma, s)B(s)G(s)\Psi \cdot (s, 0)x((k - 1)T) \, ds \]
\[ = \phi(T - \sigma, T)x(kT) \]
\[ + \int_{kT}^{T - \sigma} \phi(T - \sigma, s)B(s)G(s)\Psi \cdot (s, 0)x((k - 1)T) \, ds \]
\[ = [O_n \ G(T - \sigma)]\Psi(T - \sigma, T) \left( x(kT) \right) \hat{\lambda}(kT), \] (12.b)
for all \( \sigma \in (0, T) \). Thus by (12.a) and (12.b), it is concluded that
\[ y(kT - \sigma) = C(kT - \sigma)x(kT - \sigma) \]
\[ + D(kT - \sigma)u(kT - \sigma) \]
\[ - C_\lambda(-\sigma) \left( x(kT) \right) \hat{\lambda}(kT), \] (13)
where
\[ C_\lambda(-\sigma) = [C(kT - \sigma) \ D(kT - \sigma)G(T - \sigma)]\Psi(T, 0). \] (14)

**B. Conversion Algorithm**

Assume that \( \sigma_1, \sigma_2, \ldots, \sigma_l \) are positive real numbers which satisfy \( 0 < \sigma_1 < \cdots < \sigma_l < T \) and
\[ \begin{bmatrix} C_\lambda(-\sigma_1) \\ C_\lambda(-\sigma_2) \\ \vdots \\ C_\lambda(-\sigma_l) \end{bmatrix} \]
\[
\text{rank} \begin{bmatrix} C_\lambda(-\sigma_1) \\ C_\lambda(-\sigma_2) \\ \vdots \\ C_\lambda(-\sigma_l) \end{bmatrix} = 2n. \] (15)

By (13) and (15), one can obtain
\[ x(kT) = [I_n \ O_n] \left( x(kT) \right) \hat{\lambda}(kT) = L \begin{bmatrix} y(kT - \sigma_1) \\ y(kT - \sigma_2) \\ \vdots \\ y(kT - \sigma_l) \end{bmatrix}, \] (16)
where \( L \in R^{n 	imes l} \) is given by
\[ L = [I_n \ O_n], \]
\[ \begin{bmatrix} C_\lambda(-\sigma_1) \\ C_\lambda(-\sigma_2) \\ \vdots \\ C_\lambda(-\sigma_l) \end{bmatrix}. \] (17)

This means that \( x(kT) \) can be exactly predicted by \( y(kT - \sigma_1), y(kT - \sigma_2), \ldots, y(kT - \sigma_l) \) if the generalized sampled-data hold control (10) is valid in the interval \([k - 1)T, T)\), so that
\[ u(kT + \theta) = G(\theta)\Psi(\theta, 0)L \]
\[ \begin{bmatrix} y(kT - \sigma_1) \\ y(kT - \sigma_2) \\ \vdots \\ y(kT - \sigma_l) \end{bmatrix} \] (18)
is just an equivalent control of (3.a) for all \( k \geq 1 \) and \( \theta \in [0, T) \).

**C. Another Conversion**

An alternative conversion using less output measurements is also possible. To do so, one assumes
\[ \begin{bmatrix} C_\lambda(-\sigma_1) \\ C_\lambda(-\sigma_2) \\ \vdots \\ C_\lambda(-\sigma_l) \end{bmatrix} = [M_1 \ M_2] \] (19)
where \( g \leq f, M_1 \in R^{n \times l}, M_2 \in R^{l \times l}, \text{rank}[M_1] = n \). By (13) and (19), one obtains
\[ x(kT) = (M_1^T M_1)^{-1} M_1^T \]
\[ \begin{bmatrix} y(kT - \sigma_1) \\ y(kT - \sigma_2) \\ \vdots \\ y(kT - \sigma_l) \end{bmatrix}, \] (20)
Thus, an alternative conversion can be taken as follows:
\[ u(kT + \theta) = G(\theta)\Psi(\theta, 0)\hat{x}(kT) \] (21.a)
\[ \hat{x}(kT) = L_1 \begin{bmatrix} y(kT - \sigma_1) \\ y(kT - \sigma_2) \\ \vdots \\ y(kT - \sigma_l) \end{bmatrix} + L_2 \hat{x}((k - 1)T) \] (21.b)
where \( L_1 = (M_1^T M_1)^{-1} M_1^T, L_2 \in R^l \), \( l = (M_1^T M_1)^{-1} M_1^T M_2 \Psi(T, 0) \in R^l \), \( k = 1, 2, \ldots, \) and \( \theta \in [0, T) \).

Remark 1: Since \( u(\theta) \) for \( \theta \in [-T, 0) \) may not obey (10), a practical control scheme in the first periodic time can be taken as follows:
\[ u(\theta) = G(\theta)\Psi(\theta, 0)E(x(0)) \] (22)
for \( \theta \in [0, T] \), where the expectation \( E(x(0)) \) can be substituted by an estimated vector using any other approaches (e.g., the exact reconstruction method developed in [7]).

**Remark 2:** A necessary and sufficient condition to exist real numbers \( \sigma_1, \ldots, \sigma_n \) (\( 0 < \sigma_1 < \cdots < \sigma_n \leq T \)) for satisfying (15) is that the following Gramian matrix has full rank (see Lemma 3 in the Appendix).

\[
\int_0^T \Psi_x(\sigma, 0) [C(\sigma) \ D(\sigma) G(\sigma)]^T [C(\sigma) \ D(\sigma) G(\sigma)] \Psi_x(\sigma, 0) d\sigma.
\]  

(23)

Similarly, a necessary and sufficient condition to exist real numbers \( \sigma_1, \ldots, \sigma_n \) (\( 0 < \sigma_1 < \cdots < \sigma_n \leq T \)) for satisfying \( \text{rank}[M] = n \) is that the following Gramian matrix has full rank

\[
\int_0^T \phi^T(\sigma, 0) C(\sigma) C(\sigma) \phi(\sigma, 0) d\sigma.
\]  

(24)

**Remark 3:** It is interesting to compare the presented approach with a multirate output feedback control given as follows (e.g., (25)):

\[
u(kT + iT/f) + \theta) = L_i y(kT)
\]  

where \( f \) is a positive integer, \( \theta \in [0, T/f] \), and \( L_i \in \mathbb{R}^{m \times r} \). From the theoretical viewpoint, the converted generalized sampled-data hold control (18) is the optimal solution of the continuous-time LQR control problem, as defined in [7]. Besides, a multirate output feedback control scheme is easier for practical implementation because it only uses a zero-order hold and needs less output measurements.

**III. EXAMPLE**

Consider the optimal control problem of the following linear periodic system

\[
\begin{align*}
\dot{x}(t) &= \begin{bmatrix} \cos(2\pi t) & \sin(2\pi t) \\ -\sin(2\pi t) & \cos(2\pi t) \end{bmatrix} u(t) \\
y(t) &= \begin{bmatrix} \cos(2\pi t) & \sin(2\pi t) \end{bmatrix} x(t)
\end{align*}
\]  

(26)

to minimize a quadratic performance index as follows:

\[
J = \int_0^T x^T(t)x(t) + u^T(t)u(t) dt.
\]  

(27)

By solving the periodic Riccati equation (3c), one obtains the optimal periodic control as follows:

\[
u(t) = \begin{bmatrix} -\cos(2\pi t) & \sin(2\pi t) \\ -\sin(2\pi t) & -\cos(2\pi t) \end{bmatrix} x(t).
\]  

(28)

This control scheme yields the following closed-loop system:

\[
\dot{x}(t) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} x(t).
\]  

(29)

Now, let \( T = 1 \), \( \sigma_1 = 0.4 \), \( \sigma_2 = 0.6 \), \( \sigma_3 = 0.8 \), and \( \sigma_4 = 1.0 \). By (14), one obtains

\[
\begin{bmatrix} C_4(-0.4) \\ C_4(-0.6) \\ C_4(-0.8) \\ C_4(-1.0) \end{bmatrix} = \begin{bmatrix} -0.8090 & -0.5878 & -0.3979 & -0.2891 \\ -0.8090 & 0.5878 & 0.6651 & 0.4832 \\ 0.3909 & 0.9511 & 0.7878 & 1.1656 \\ 1.0000 & 0.0000 & 1.7183 & -0.0000 \end{bmatrix}
\]  

(30)

and by (17), one obtains

\[
L = \begin{bmatrix} -0.7196 & -1.3087 & 0.3641 & -0.7535 \\ -1.7058 & 2.0658 & -1.2796 & 0.6866 \end{bmatrix}
\]  

(31)

thus, the converted generalized sampled-data hold control (18) is given by

\[
u(kT + \theta) = \begin{bmatrix} -\cos(2\pi\theta) & \sin(2\pi\theta) \\ -\sin(2\pi\theta) & -\cos(2\pi\theta) \end{bmatrix}
\]  

(32)

This converted control scheme is checked by simulation as shown in Fig. 1.

**IV. CONCLUSIONS**

In this note, a conversion method to convert the analog optimal periodic control to a generalized sampled-data hold output feedback control for a linear periodic system is developed. Such a conversion enables us to implement the optimal periodic control scheme in the presence of incomplete state measurements. Besides, the converted control scheme can use the delayed output feedback to offer a leisure time for online computation, so that it can provide ability to tolerate the time delay (such as: measurement delay, computation time lag, etc). Such a conversion algorithm is also applicable to a linear time-invariant system just by considering the system as a periodic model with an arbitrary periodic time.

**APPENDIX**

**Lemma 1:** We only have to show that

\[
B_i(t, \theta) = \int_0^\theta \phi(t, s)B(s)G(s)\Psi(s, \theta) ds
\]

\[
= \Psi(t, \theta) - \phi(t, \theta).
\]

**Proof:** One has

\[
\frac{d}{dt} \left( \int_0^\theta \phi(t, s)B(s)G(s)\Psi(s, \theta) ds \right)
\]

\[
= A(t) \left( \int_0^\theta \phi(t, s)B(s)G(s)\Psi(s, \theta) ds \right) + B(t)G(t)\Psi(t, \theta)
\]

(A.1)
Thus, the equality can be obtained by checking the differential equation (7) directly.

Lemma 2: If \( \dot{x}(kT) = x(kT) \), then the control (10) equals to (3.a) on \([kT, (k+1)T)\).

Proof: With the control (10), the state of the periodic system (1) becomes

\[
\begin{align*}
\frac{d}{dt} (\Psi(t, \theta) - \phi(t, \theta)) &= A(t)(\Psi(t, \theta) - \phi(t, \theta)) \\
&\quad + B(t)G(t)\Psi(t, \theta).
\end{align*}
\]

(10)

Fig. 1. The response of the periodic system (26) with the converted control scheme (32), where one assumes \( x(0) - [x(0)x(0)] = [2 3]' \).

Thus, if \( \dot{x}(kT) = x(kT) \), then \( x(kT + \theta) = \phi(\theta, 0)x(kT) + \int_0^\theta \phi(\theta, s)B(s)w(kT + s) \, ds \\
- \phi(\theta, 0)x(kT) + \int_0^\theta \phi(\theta, s)B(s)G(s)\Psi(s, 0)\dot{x}(kT) \, ds \\
= \phi(\theta, 0)x(kT) + B(\theta, 0)\dot{x}(kT) \\
= \phi(\theta, 0)(x(kT) - \dot{x}(kT)) + \Psi(\theta, 0)\dot{x}(kT) \quad \text{(A.3)}
\]

Thus, if \( \dot{x}(kT) = x(kT) \), then \( x(kT + \theta) - \dot{x}(kT + \theta) \) for all \( \theta \in [0, T] \).

Lemma 3: Assume \( H(t) \) is a continuous matrix function from \([0, T]\) into \( \mathbb{R}^{n \times n} \), then the following Gramian matrix

\[
\Omega_T = \int_0^T H(t)H^*(t) \, dt \quad \text{(A.4)}
\]

is positive definite, if and only if there exist finite points \( \sigma_1, \sigma_2, \ldots, \sigma_T \) of \([0, T]\), such that

\[
\{H(\sigma_1); H(\sigma_2); \ldots; H(\sigma_T)\}
\]

has full row rank.

Proof: If there exist finite points \( \sigma_1, \sigma_2, \ldots, \sigma_T \) of \([0, T]\), such that the given matrix (A.5) has full row rank, then for any nonzero vector \( \xi \in \mathbb{R}^n \), one can find at least a point \( \sigma_i \in \{\sigma_1, \sigma_2, \ldots, \sigma_T\} \), such that \( \xi^*H(\sigma_i) \neq 0 \). Since \( H(t) \) is continuous, this implies

\[
\xi^*\Omega_T\xi = \int_0^T \xi^*H(t)H^*(t)\xi \, dt > 0
\]

(6)

so that \( \Omega_T \) is positive definite.

(Only if) Assume \( \Omega_T \) is positive definite, and consider arbitrary finite points \( \sigma_1, \sigma_2, \ldots, \sigma_T \) of \([0, T]\), if the given matrix (A.5) has full row rank, then one can find \( \xi \in \mathbb{R}^n \) and \( \sigma_{T+1} \in (0, T] \), such that

\[
\xi^*H(\sigma_{T+1}) \neq 0 \quad \text{(A.7)}
\]

and

\[
\xi^*[H(\sigma_1); H(\sigma_2); \ldots; H(\sigma_T)] = 0. \quad \text{(A.8)}
\]

This implies that some columns of \( H(\sigma_{T+1}) \) cannot be expressed as a linear combination of columns of matrix (A.5), so that it is true that

\[
\text{rank} \{H(\sigma_1); H(\sigma_2); \ldots; H(\sigma_T)\} \\
\geq \text{rank} \{H(\sigma_1); H(\sigma_2); \ldots; H(\sigma_T)\} + 1. \quad \text{(A.9)}
\]

By giving the extending procedure at most \( n_1 \) times, one can finally find \( \sigma_1, \sigma_2, \ldots, \sigma_T, \sigma_{T+1}, \ldots, \sigma_{T+n_1} \subset (0, T] \), such that

\[
\text{rank} \{H(\sigma_1); H(\sigma_2); \ldots; H(\sigma_T); \ldots; H(\sigma_{T+n_1})\} = n_1.
\]

\[
\square
\]

On Discrete Spectral Factorizations—A Unify Approach

M. C. Tsai

Abstract—This note summarizes state-space formulae for all key spectral factorizations appearing in the discrete-time \( H^*/H \) optimization. The factorization problems are categorized into three groups. The construction of solutions is formulated into finding special coprime factors of a given transfer matrix by the associated discrete algebraic Riccati equation. Solution procedures for the three groups are in general the same, and under that we may lead to yield a unify approach.

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