Super-connectivity and super-edge-connectivity for some interconnection networks


Abstract

Let $G = (V, E)$ be a $k$-regular graph with connectivity $\kappa$ and edge connectivity $\lambda$. $G$ is maximum connected if $\kappa = k$, and $G$ is maximum edge connected if $\lambda = k$. Moreover, $G$ is super-connected if it is a complete graph, or it is maximum connected and every minimum vertex cut is $\{v, x \in E\}$ for some vertex $v \in V$; and $G$ is super-edge-connected if it is maximum edge connected and every minimum edge disconnecting set is $\{(v, x) \in E\}$ for some vertex $v \in V$. In this paper, we present three schemes for constructing graphs that are super-connected and super-edge-connected. Applying these construction schemes, we can easily discuss the super-connected property and the super-edge-connected property of hypercubes, twisted cubes, crossed cubes, m"{o}bius cubes, split-stars, and recursive circulant graphs.

Keywords: Connectivity; Edge connectivity; Super-connectivity; Super-edge-connectivity
1. Introduction

For the graph definitions and notations we follow [2]. $G = (V, E)$ is a simple graph if $V$ is a finite set and $E$ is a subset of $\{(a, b) \mid (a, b) \text{ is an unordered pair of } V\}$. We say that $V$ is the vertex set and $E$ is the edge set. The neighborhood of $v$, $N_G(v)$, is $\{x \mid (v, x) \in E\}$. The neighbor-edge of $v$, $NE_G(v)$, is $\{(v, x) \mid (v, x) \in E\}$. The degree of a vertex $v$, denoted by $\deg_G(v)$, is the number of vertices in $N_G(v)$. A graph $G$ is $k$-regular if $\deg_G(v) = k$, for every vertex $v \in V$.

A vertex cut of a graph $G$ is a set $S \subseteq V(G)$ such that $G - S$ has more than one connected component. It is known that only complete graphs do not have vertex cuts. The connectivity of $G$, written $\kappa(G)$, is defined as the minimum size of a vertex cut if $G$ is not a complete graph, and $\kappa(G) = |V(G)| - 1$ if otherwise. A graph $G$ is $k$-connected if $k \leq \kappa(G)$. Assume that $G$ is a $k$-regular graph with connectivity $\kappa$. We say that $G$ is maximum connected if $\kappa = k$; and $G$ is super-connected if it is a complete graph, or it is maximum connected and every minimum vertex cut is $N_G(v)$ for some vertex $v$.

An edge disconnecting set is a set $F \subseteq E(G)$ such that $G - F$ has more than one connected component. A graph is $k$-edge-connected if every disconnecting set has at least $k$ edges. The edge connectivity of $G$, written $\lambda(G)$, is the minimum size of an edge disconnecting set. Assume that $G$ is a $k$-regular graph with edge connectivity $\lambda$. A graph $G$ is $k$-edge-connected if $k \leq \lambda(G)$. We say that $G$ is maximum edge connected if $\lambda = k$; and $G$ is super-edge-connected if it is maximum edge connected and every minimum edge disconnecting cut is $NE_G(v)$ for some vertex $v$.

The architecture of an interconnection network is usually represented by a graph. There are numerous mutually conflicting requirements in designing the topology of interconnection networks. Network reliability is one of the major factors in designing the topology of an interconnection network. It has been shown that a network is more reliable if it is super-connected [3,4,9]. Some important families of interconnection networks have been proven to be super-connected [3,4,9]. In this paper, we present three schemes to construct super-connected and super-edge-connected graphs. With these construction schemes, we can easily discuss the super-connected property and the super-edge-connected property of hypercubes, twisted cubes, crossed cubes, möbius cubes, split-stars, and recursive circulant graphs.

2. The first constructing scheme

Assume that $t$ is a positive integer. Let $G_1$ and $G_2$ be two graphs with $t$ vertices, and $M$ be any arbitrary perfect matching between the vertices of $G_1$ and $G_2$; i.e., a set of $t$ edges with one endpoint in $G_1$, and the other endpoint in $G_2$. The graph $G(G_1, G_2; M)$ is defined as a graph with the vertex set
\[ V(G(G_1, G_2; M)) = V(G_1) \cup V(G_2), \] and edge set \[ E(G(G_1, G_2; M)) = E(G_1) \cup E(G_2) \cup M. \] We note that the cartesian product of a graph \( H \) and a complete graph \( K_2 \) can be viewed as a \( G(H, H; M) \) for some \( M \).

**Theorem 1.** Assume that \( t \) is a positive integer. Let \( G_1 \) and \( G_2 \) be two \( k \)-regular maximum connected graphs with \( t \) vertices, and \( M \) be any perfect matching between \( V(G_1) \) and \( V(G_2) \). Then, \( G(G_1, G_2; M) \) is \( (k+1) \)-regular super-connected if and only if (1) \( t > k + 1 \) or (2) \( t = k + 1 \) with \( k = 0, 1, 2 \).

**Proof.** Since \( G_1 \) and \( G_2 \) are \( k \)-regular connected graphs, \( t \geq k + 1 \). By definition, \( G(G_1, G_2; M) \) is a \((k+1)\)-regular graph. To prove \( G(G_1, G_2; M) \) is super-connected, we need to check if \( G \subset V(G(G_1, G_2; M)) \) such that \( |F| = k + 1 \) and \( F \neq N_{G(G_1, G_2; M)}(v) \) for any vertex \( v \in V(G(G_1, G_2; M)) \).

Suppose that \( t = k + 1 \). Obviously, \( G_1 \) and \( G_2 \) are isomorphic to the complete graph \( K_{k+1} \). Moreover, \( G(G_1, G_2; M) \) is isomorphic to the cartesian product of \( K_{k+1} \) and \( K_2 \). Without loss of generality, we assume that \( V(G_1) = \{a_0, a_1, \ldots, a_k\} \) and \( V(G_2) = \{b_0, b_1, \ldots, b_k\} \), where \( b_i \) is the vertex matched with \( a_i \) under \( M \) for every \( i \). By brute force, we can check that \( G(G_1, G_2; M) \) is super-connected for \( k = 0, 1, 2 \). When \( k \geq 3 \), we set \( F = \{a_0, a_1\} \cup \{b_i|2 \leq i \leq k\} \). It is easy to see that \( |F| = k + 1 \), \( F \neq N_{G(G_1, G_2; M)}(v) \), and \( F \) is a vertex cut of \( G(G_1, G_2; M) \). So \( G(G_1, G_2; M) \) is not super-connected.

Now, assume that \( t > k + 1 \). Since \( K_1 \) is the only connected 0-regular graph and \( K_2 \) is the only connected 1-regular graph, let \( k \geq 2 \). We set \( X_1 = F \cap V(G_1) \) and \( X_2 = F \cap V(G_2) \).

**Case 1.** \( |X_1| < k \) and \( |X_2| < k \). Thus, both \( G_1 - X_1 \) and \( G_2 - X_2 \) are connected. Since \( t = |M| > k + 1 \) and \( |F| = k + 1 \), there exists \( a \in V(G_1) - F \) and \( b \in V(G_2) - F \) such that \( (a, b) \in M \). Thus, \( G(G_1, G_2; M) - F \) is connected.

**Case 2.** Either \( k \leq |X_1| \leq k + 1 \) or \( k \leq |X_2| \leq k + 1 \). We assume without loss of generality that \( k \leq |X_1| \leq k + 1 \). Hence, \( |X_2| \leq 1 \). Since \( k \geq 2 \), \( G_2 - X_2 \) is connected. Let \( C \) be any connected component of \( G_1 - X_1 \). We will claim that there exists \( a \in C \) and \( b \in V(G_2) - F \) such that \( (a, b) \in M \). With this claim, \( G(G_1, G_2; M) - F \) is connected.

First, if \( C \) consists of only one vertex \( a \), then \( N_{G_1}(a) \subset F \). Let \( b \) be the vertex in \( G_2 \) with \( (a, b) \in M \). Since \( F \neq N_{G(G_1, G_2; M)}(a) \), \( b \in V(G_2) - F \). Thus, the claim holds. Now, if \( C \) contains at least two vertices \( a \) and \( a' \). Let \( b, b' \) be the matched vertices of \( a, a' \) in \( G_2 \), respectively. Since at most one vertex of \( G_2 \) is in \( F \), we may assume \( b \notin F \). Thus, our claim holds.

Therefore, the theorem is proved. \( \square \)
A similar argument leads to the following theorem for super-edge-connected, and the corollary.

**Theorem 2.** Assume that \( t \) is a positive integer. Let \( G_1 \) and \( G_2 \) be two \( k \)-regular maximum-edge-connected graphs with \( t \) vertices, and \( M \) is any perfect matching between \( V(G_1) \) and \( V(G_2) \). Then, \( G(G_1, G_2; M) \) is \((k + 1)\)-regular super-edge-connected if and only if (1) \( t > k + 1 \) or (2) \( t = k + 1 \) with \( k = 0 \).

**Corollary 1.** Assume that \( t \) is a positive integer. Let \( G_1 \) and \( G_2 \) be two \( k \)-connected and \( k' \)-edge connected graphs with \( t \) vertices, and \( M \) is any perfect matching between \( V(G_1) \) and \( V(G_2) \). Then, \( G(G_1, G_2; M) \) is \((k + 1)\)-connected and \((k' + 1)\)-edge connected.

Network topology is always represented by a graph where vertices represent processors and edges represent links between processors. Among these topologies, the binary hypercube [7], \( Q_n \), is one of the most popular topology. The hypercube \( Q_n \) can be recursively defined as \( Q_1 = K_2 \) and \( Q_n \) is the cartesian product of \( Q_{n-1} \) and \( K_2 \). The super-connected property and the super-edge-connected property of hypercubes are discussed in [4,9]. Here, we reprove this result. Recursively applying Theorems 1 and 2, we can easily prove that \( Q_n \) is super-connected for every \( n \) and super-edge-connected if \( n \neq 2 \).

Twisted cubes [1], crossed cubes [6], and möbius cubes [5] are derived by changing the connection of some hypercube links according to some specified rules.

In [1], the twisted \( n \)-cube \( TQ_n \) is defined for odd values of \( n \). The vertex set of the twisted \( n \)-cube \( TQ_n \) is the set of all binary strings of length \( n \). Let \( u = u_{n-1}u_{n-2}\ldots u_1u_0 \) be any vertex in \( TQ_n \). For \( 0 \leq i \leq n-1 \), let the \( i \)th parity function be \( P_i(u) = u_i \oplus u_{i-1} \oplus \cdots \oplus u_0 \), where \( \oplus \) is the exclusive-or operation. We can recursively define \( TQ_n \) as follows: A twisted 1-cube, \( TQ_1 \), is a complete graph with two vertices 0 and 1. Suppose that \( n \geq 3 \). We can decompose the vertices of \( TQ_n \) into four sets, \( TQ_{n-2}^0, TQ_{n-2}^1, TQ_{n-2}^0, TQ_{n-2}^1 \) where \( TQ_{n-2}^i \) consists of those vertices \( u \) with \( u_{n-1} = i \) and \( u_{n-2} = j \). For each \( (i, j) \in \{(0, 0), (0, 1), (1, 0), (1, 1)\} \), the induced subgraph of \( TQ_{n-2}^{i,j} \) in \( TQ_n \) is isomorphic to \( TQ_{n-2} \). The edges that connect these four subtwisted cubes can be described as follows: Any vertex \( u_{n-1}u_{n-2}\ldots u_1u_0 \) with \( P_{n-3}(u) = 0 \) is connected to \( u_{n-1}u_{n-2}\ldots u_1u_0 \) and \( u_{n-1}u_{n-2}\ldots u_1u_0 \); and to \( u_{n-1}u_{n-2}\ldots u_1u_0 \) and \( u_{n-1}u_{n-2}\ldots u_1u_0 \) if \( P_{n-3}(u) = 1 \).

From the definition, both the subgraph induced by \( TQ_{n-2}^0 \cup TQ_{n-2}^1 \) and the subgraph induced by \( TQ_{n-2}^1 \cup TQ_{n-2}^1 \) are isomorphic to \( TQ_{n-2} \times K_2 \), where \( K_2 \) is the complete graph with two vertices. Moreover, the edges joining \( TQ_{n-2}^0 \cup TQ_{n-2}^1 \) and \( TQ_{n-2}^1 \cup TQ_{n-2}^1 \) form a perfect matching of \( TQ_n \). Recursively applying Theorems 1 and 2, we can easily prove that \( TQ_n \) is super-connected and super-edge-connected for every odd \( n \).
Two two-digit binary strings $x = x_1x_0$ and $y = y_1y_0$ are pair related, denoted by $x \sim y$, if and only if $(x, y) \in \{(00, 00), (10, 10), (01, 11), (11, 01)\}$. An $n$-dimension crossed cube $CQ_n$ [6] is a graph $CQ_n = (V, E)$ that is recursively constructed as follows: $CQ_1$ is a complete graph with two vertices labeled by 0 and 1. $CQ_n$ consists of two identical $(n-1)$-dimension crossed cubes, $CQ_{n-1}$ and $CQ_{n-1}$. The vertex $u = 0u_{n-2} \cdots u_0 \in V(CQ_{n-1})$ and vertex $v = 1v_{n-2} \cdots v_0 \in V(CQ_{n-1})$ are adjacent in $CQ_n$ if and only if (1) $u_{n-2} = v_{n-2}$ if $n$ is even; and (2) for $0 \leq i < [(n-1)/2]$, $u_{2i+1}u_{2i} \sim v_{2i+1}v_{2i}$.

From the definition, $CQ_n$ can be viewed as $G(CQ_{n-1}, CQ_{n-1}; M)$ for some perfect matching $M$. Recursively applying Theorems 1 and 2, we can easily prove that $CQ_n$ is super-connected for every $n$ and super-edge-connected if $n \neq 2$.

The möbius cube [5], $MQ_n = (V, E)$, of dimension $n$ has $2^n$ vertices. Each vertex is labeled by a unique $n$-bit binary string as its address and has connections to $n$ other distinct vertices. The vertex with address $X = x_{n-1}x_{n-2} \cdots x_0$ connects to $n$ other vertices $Y_i$, $0 \leq i \leq n - 1$, where the address of $Y_i$ satisfies (1) $Y_i = (x_{n-1} \cdots x_{i+1}1 x_i \cdots x_0)$ if $x_{i+1} = 0$; or (2) $Y_i = (x_{n-1} \cdots x_i1 x_{i+1} \cdots x_0)$ if $x_{i+1} = 1$.

From the above definition, $X$ connects to $Y_i$ by complementing the bit $x_i$ if $x_{i+1} = 0$, or by complementing all bits of $x_i \cdots x_0$ if $x_{i+1} = 1$. For the connection between $X$ and $Y_{n-1}$, we can assume that the unspecified $x_n$ is either 0 or 1, which gives slightly different topologies. If $x_n = 0$, we call the network generated the “0-möbius cube”, denoted by $0-MQ_n$; and if $x_n = 1$, we call the network generated the “1-möbius cube”, denoted by $1-MQ_n$.

According to the above definition, $0-MQ_{n+1}$ and $1-MQ_{n+1}$ can be recursively constructed from a $0-MQ_n$ and a $1-MQ_n$ by adding a perfect matching. Recursively applying Theorems 1 and 2, we can easily prove that every $0-MQ_n$ or $1-MQ_n$ is super-connected for every $n$ and super-edge-connected if $n \neq 2$.

Assume $n$ is a positive integer. The alternating graph $A_n$ [3] is an attractive interconnection graph topology. $V(A_n) = \{p|p = p_0p_1 \cdots p_{n-2}$ with $p_i \in \{1, 2, \ldots, n\}$ for $1 \leq i \leq n - 2$ and $p_i \neq p_j$ if $i \neq j\}$ and $E(A_n) = \{(p, q)|$ there exists a unique $i \in (n-2)$ such that $p_i \neq q_i\}$. In [3], split-star $S^2_n$ is proposed as an attractive interconnection network. $V(S^2_n) = \{p|p = p_0p_1 \cdots p_{n-2}$ with $p_0 \in \{0, 1\}$, $p_i \in \{1, 2, \ldots, n\}$ for $1 \leq i \leq n - 2$, and $p_i \neq p_j$ if $i \neq j \neq 0\}$ and $E(S^2_n) = \{(p, q)|$ there exists a unique $i$ with $0 \leq i \leq n - 2$ such that $p_i \neq q_i\}$.

It is pointed out in [3] that $S^2_n$ can be viewed as $G(A_n, A_n; M)$. Moreover, it is proved that $A_n$ is super-connected unless $n = 4$ and super-edge-connected for any $n$. Applying Theorems 1 and 2, we can easily reprove the result in [3] that $S^2_n$ is super-connected for any $n$ and super-edge-connected unless $n = 3$.

### 3. The second constructing scheme

Let $r$ and $t$ be positive integers with $r \geq 3$. Assume that $G_0, G_1, \ldots, G_{r-1}$ are graphs with $|V(G_i)| = t$ for $0 \leq i \leq r - 1$. We define $H = G(G_0, G_1, \ldots, G_{r-1}; M)$
with \( V(H) = V(G_0) \cup V(G_1) \cup \ldots \cup V(G_{r-1}) \) and \( E(H) = \mathcal{M} \cup \bigcup_{i=0}^{r-1} E(G_i) \), where \( \mathcal{M} = \bigcup_{i=0}^{r-1} M_{i,i+1}^{(mod\, r)} \) with \( M_{i,i+1}^{(mod\, r)} \) is any arbitrary perfect matching between \( V(G_i) \) and \( V(G_{i+1}^{(mod\, r)}) \).

**Theorem 3.** Let \( r \) and \( t \) be positive integers with \( r \geq 3 \). Assume that \( G_0, G_1, \ldots, G_{r-1} \) are \( k \)-regular maximum connected graphs with \( |V(G_i)| = t \) for \( 0 \leq i \leq r-1 \) and \( \mathcal{M} = \bigcup_{i=0}^{r-1} M_{i,i+1}^{(mod\, r)} \), where \( M_{i,i+1}^{(mod\, r)} \) is any arbitrary perfect matching between \( V(G_i) \) and \( V(G_{i+1}^{(mod\, r)}) \). Then, \( H = G(G_0, G_1, \ldots, G_{r-1}; \mathcal{M}) \) is \((k + 2)\)-regular super-connected if and only if (1) \( k \geq 1 \) or (2) \( k = 0 \) and \( r = 3, 4, 5 \).

**Proof.** Because \( G_0, G_1, \ldots, G_{r-1} \) are \( k \)-regular connected graphs, \( t \geq k + 1 \), by definition, \( H \) is a \((k + 2)\)-regular graph. To prove \( H \) is super-connected, we need to check if \( H - F \) is connected for any vertex subset \( F \) of \( H \) such that \( |F| = k + 2 \) and \( F \neq N_H(v) \) for any vertex \( v \in V(H) \). We set \( X_i = F \cap V(G_i) \) for \( i = 0, 1, \ldots, r - 1 \).

**Case 1.** \( |X_i| < k \) for \( i = 0, 1, \ldots, r - 1 \). Then, \( G_i - X_i \) is connected for every \( i \).

Suppose that \( |X_i \cup X_{i+1}^{(mod\, r)}| \leq k \) for \( 0 \leq i \leq r - 1 \). By Theorem 1, there exist \( x_i \in V(G_i) - F \) and \( x_{i+1}^{(mod\, r)} \in V(G_{i+1}^{(mod\, r)}) - F \) such that \( (x_i, x_{i+1}^{(mod\, r)}) \in M_{i,i+1}^{(mod\, r)} \). Thus, \( H - F \) is connected.

Suppose that \( k + 1 \leq |X_i \cup X_{i+1}^{(mod\, r)}| \leq k + 2 \) for some \( i \). Without loss of generality, we may assume that \( k + 1 \leq |X_0 \cup X_1| \leq k + 2 \) and \( |X_0| \geq |X_1| \). Since \( |X_1| < k \), \( |X_0| \geq 2 \). Thus, \( |X_i \cup X_{i+1}^{(mod\, r)}| \leq k \) for \( 1 \leq i < r - 1 \). Suppose that \( k \leq 1 \). Then, \( |X_i| = 0 \) for every \( i \). Since \( F = \bigcup_{i=1}^{r-1} X_i, \ |F| = 0 \). This is impossible. Thus, \( k \geq 2 \). By Theorem 1, \( G(G_i, G_{i+1}^{(mod\, r)}; M_{i,i+1}^{(mod\, r)}) - (X_i \cup X_{i+1}^{(mod\, r)}) \) is connected for \( 1 \leq i < r - 1 \). Hence, \( H - F \) is connected.

**Case 2.** \( |X_i| \geq k \) for some \( i \). Without loss of generality, we assume that \( k \leq |X_0| \leq k + 2, |X_0| \geq |X_i| \) for \( 1 \leq i \leq r - 1 \), and \( |X_i| \geq |X_{i-1}| \).

**Subcase 2.1.** \( t > k + 1 \). Thus, \( G_i \) is not a complete graph. Hence, \( k > 1 \). By Theorem 1, \( G(G_i, G_{i+1}^{(mod\, r)}; M_{i,i+1}^{(mod\, r)}) \) is \((k + 1)\)-regular super-connected for \( 0 \leq i \leq r - 1 \).

Suppose that \( |X_0 \cup X_1| = k \). Then, \( |X_0| = k, |X_1| = |X_{r-1}| = 0 \), and \( |X_i \cup X_{i+1}| \leq k \) for \( 0 \leq i \leq r - 2 \). Then, \( G(G_i, G_{i+1}; M_{i,i+1}) - (X_i \cup X_{i+1}) \) is connected for \( 0 \leq i \leq r - 2 \). Hence, \( H - F \) is connected.

Suppose that \( |X_0 \cup X_1| = k + 1 \). Then, \( G(G_0, G_1; M_{0,1}) - (X_0 \cup X_1) \) is connected unless \( X_0 \cup X_1 = N_{G(G_0, G_1; M_{0,1})}(x) \) for some vertex \( x \in V(G_0) \cup V(G_1) \).

Suppose that \( G(G_0, G_1; M_{0,1}) - (X_0 \cup X_1) \) is connected. \( H - F \) is connected because \( G(G_i, G_{i+1}^{(mod\, r)}; M_{i,i+1}^{(mod\, r)}) - (X_i \cup X_{i+1}^{(mod\, r)}) \) is also connected for \( 1 \leq i \leq r - 1 \).
Suppose that $X_0 \cup X_1 = N_G(g_0;g_i;M_{0,1})$ for some vertex $x \in V(G_0) \cup V(G_i)$. Since $|X_0| \geq k, x \in V(G_0)$. Let $x_1$ be the vertex in $V(G_1)$ such that $(x, x_1) \in M_{0,1}$ and $x_{r-1}$ be the vertex in $V(G_{r-1})$ such that $(x_{r-1}, x) \in M_{r-1,0}$. Obviously, $x_1$ is the only vertex in $X_1$. Since $G_1$ is $k$-connected with $k > 1$, $G_1 - \{x_1\}$ is connected. Let $y$ be any vertex of $V(G_0)$ with $y \neq x$, and $y_1$ be the vertex in $V(G_1)$ such that $(y, y_1) \in M_{0,1}$. Hence, $y_1 \notin F$ since $y_1 \neq x_1$. Therefore, $G(G_0, G_1; M_{0,1}) - (X_0 \cup X_1 \cup \{x\})$ is connected. Since $F \neq N_H(x), x_{r-1} \notin F$. Since $|X_i| \leq 1$ for $1 \leq i \leq r - 1$, $G(G_i, G_{i+1}; M_{i,i+1}) - (X_i \cup X_{i+1})$ is connected for $2 \leq i \leq r - 2$ with $r > 3$ or $G_2 - X_2$ is connected with $r = 3$. Hence, $H = F$ is connected.

**Subcase 2.2.** $t = k + 1$. Thus, every $G_i$ is isomorphic to complete graph $K_{k+1}$. Since $K_{k+1}$ contains $k + 1$ vertices, $k \leq |X_0| \leq k + 1$.

Suppose that $k = 0$. Then, $H$ is isomorphic to the cycle $C_r$. It is easy to check $H$ is super-connected if and only if $r = 3, 4, 5$. Thus, we consider $k \geq 1$.

Suppose that $|X_0| = k + 1$. Thus, $|\bigcup_{i=0}^{r-1} X_i| = 1$. By Corollary 1, $G(G_i, G_{i+1}; M_{i,i+1})$ is $(k + 1)$-connected for $1 \leq i \leq r - 2$. Thus, $G(G_i, G_{i+1}; M_{i,i+1}) - (X_i \cup X_{i+1})$ is connected for $1 \leq i \leq r - 2$. Hence, $H = F$ is connected.

Suppose that $|X_0| = k$. Thus, there is only one vertex $a_0$ in $V(G_0) - F$.

Assume that $k = 1$. Then, $|X_i| \leq 1$ for $0 \leq i \leq r - 1$. Suppose that $|X_i \cup X_{i+1}| \leq 1$ for $0 \leq i \leq r - 2$. Thus, $G(G_i, G_{i+1}; M_{i,i+1}) - (X_i \cup X_{i+1})$ is connected for $0 \leq i \leq r - 2$. So $H = F$ is connected. Suppose $|X_i \cup X_{i+1}| = 2$ for some $i$. We may without loss of generality assume that $|X_0 \cup X_1| = 2$. In this case, we may label the vertices of $V(G_i)$ as $a_i, a'_i$ for $0 \leq i \leq r - 2$ with $(a_i, a_{i+1}) \in M_{i,i+1}$ and $(a'_i, a'_{i+1}) \in M_{i,i+1}$.

Suppose that $G(G_0, G_1; M_{0,i}) - (X_0 \cup X_1)$ is disconnected. Without loss of generality, we assume that $a'_0 \in F$ and $a_1 \in F$.

Suppose that $r = 3$. Since $F \neq N_H(a'_1), a'_2 \notin F$. Thus, $G(G_1, G_2; M_{1,2}) - (X_1 \cup X_2)$ is connected. $a_2 \in F$ since $|X_2| = 1$. So $a'_2 \in N_H(a_0)$ since $F \neq N_H(a_0)$. Thus, $G(G_0, G_2; M_{0,2}) - (X_0 \cup X_2)$ is connected. Hence, $H = F$ is connected.

Suppose that $r > 3$. Since $F \neq N_H(a_0)$ and $F \neq N_H(a'_1), a'_2 \notin F$ and $a_{r-1} \notin F$ where $a_{r-1}$ is the vertex such that $(a_0, a_{r-1}) \in M_{0,r-1}$. Thus, $G(G_1, G_2; M_{1,2}) - (X_1 \cup X_2)$ and $G(G_0, G_{r-1}; M_{0,r-1}) - (X_0 \cup X_{r-1})$ are connected. Obviously, $G(G_i, G_{i+1}; M_{i,i+1}) - (X_i \cup X_{i+1})$ is connected for $2 \leq i \leq r - 2$. Therefore, $H = F$ is connected.

Suppose that $G(G_0, G_1; M_{0,1}) - (X_0 \cup X_1)$ is connected. We may without loss of generality assume that $a'_0 \in F$ and $a'_1 \in F$. 


Theorem 4. Let $r$ and $t$ be positive integers with $r \geq 3$. Assume that $G_0, G_1, \ldots, G_{r-1}$ are $k$-regular maximum edge connected graphs with $|V(G_i)| = t$ for $0 \leq i < r$ and $M = \bigcup_{i=0}^{r-1} M_{i, i+1}$ with $M_{i, i+1}$ is any arbitrary perfect matching between $V(G_i)$ and $V(G_{i+1})$. Then, $H = G(G_0, G_1, \ldots, G_{r-1}; M)$ is $(k + 2)$-regular super-edge-connected if and only if (1) $k = 2$, (2) $k = 1$ and $r \neq 3$, or (3) $k = 0$ and $r = 3$.

Corollary 2. Let $r$ and $t$ be positive integers with $r \geq 3$. Assume that $G_0, G_1, \ldots, G_{r-1}$ are connected and $k'$-edge connected graphs with $|V(G_i)| = t$ for $0 \leq i < r - 1$ and $M = \bigcup_{i=0}^{r-1} M_{i, i+1}$, where $M_{i, i+1}$ is any arbitrary perfect matching between $V(G_i)$ and $V(G_{i+1})$. Then, $H = G(G_0, G_1, \ldots, G_{r-1}; M)$ is $(k + 2)$-connected and $(k' + 2)$-edge-connected.

4. The third constructing scheme

Assume that $t$ is an integer with $t \geq 2$. Let $G_1$ and $G_2$ be two graphs with $t$ vertices such that $V(G_1) = \{a_i|0 \leq i < t\}$ and $V(G_2) = \{b_i|0 \leq i < t\}$. Let $E$ be a set of edges given by $E = \{(a_i, b_j)|0 \leq i < t\} \cup \{(b_i, a_{i+1})|0 \leq i < t\}$. The graph $G(G_1, G_2; E)$ is defined to be the graph with the vertex set $V(G(G_1, G_2; E)) = V(G_1) \cup V(G_2)$, and edge set $E(G(G_1, G_2; E)) = E(G_1) \cup E(G_2) \cup E_C$. 

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The theorem is proved. □

With a similar argument as above, we have the following results.
Theorem 6. Assume that $t$ is an integer with $t \geq 2$. Let $G_1$ and $G_2$ be two $k$-regular maximum connected graphs with $t$ vertices such that $V(G_1) = \{a_i|0 \leq i < t\}$ and $V(G_2) = \{b_i|0 \leq i < t\}$. Let $\mathcal{E}$ be a set of edges given by $\mathcal{E} = \{(a_i,b_i)|0 \leq i < t\} \cup \{(b_i,a_{i+1} \text{mod } t)|0 \leq i < t\}$. Then, $G(G_1,G_2;\mathcal{E})$ is $(k+2)$-regular super-edge-connected if (1) $t > k + 1$ with $k \geq 3$ or (2) $t = k + 1$ with $k = 1, 2, 3$.

Proof. Since $G_1$ and $G_2$ are $k$-regular connected graphs, $t \geq k + 1$. By definition, $G(G_1,G_2;\mathcal{E})$ is $(k+2)$-regular. To prove $G(G_1,G_2;\mathcal{E})$ is super-connected, we need to check if $G(G_1,G_2;\mathcal{E}) - F$ is connected for any $F \subset V(G(G_1,G_2;\mathcal{E}))$ such that $|F| = k + 2$ and $F \neq N_{G(G_1,G_2;\mathcal{E})}(v)$ for any vertex $v \in V(G(G_1,G_2;\mathcal{E}))$.

Suppose that $t = k + 1$. Obviously, $G_1$ and $G_2$ are isomorphic to the complete graph $K_{k+1}$. By brute force, we can check that $G(G_1,G_2;\mathcal{E})$ is super-connected for $k = 1, 2, 3$. When $k \geq 4$, set $F = \{a_0,a_{i-2},a_{i-1}\} \cup \{b_i|0 \leq i \leq t - 3\}$. It is easy to see that $|F| = k + 2, F \neq N_{G(G_1,G_2;\mathcal{E})}(v)$, and $F$ is a vertex cut of $G(G_1,G_2;\mathcal{E})$. Hence, $G(G_1,G_2;\mathcal{E})$ is not super-connected.

Now, assume that $t > k + 1$ with $k \geq 3$. We set $X_1 = F \cap V(G_1)$ and $X_2 = F \cap V(G_2)$.

Case 1. $|X_1| < k$ and $|X_2| < k$. $G(G_1,G_2;\mathcal{E}) - F$ is connected with the same argument in Theorem 1.

Case 2. Either $k \leq |X_1| \leq k + 2$ or $k \leq |X_2| \leq k + 2$. We assume without loss of generality that $k \leq |X_1| \leq k + 2$, then $|X_2| \leq 2$ and $G_2 - X_2$ is connected. Let $C$ be any connected component of $G_1 - X_1$. We will claim that there exists $a_i \in C$ such that at least one of $(a_i,b_{i-1} \text{mod } t), (a_i,b_i)$ is in $\mathcal{E}$. With this claim, $G(G_1,G_2;\mathcal{E}) - F$ is connected.

First, if $C$ consists of only one vertex $a_i$, then $N_{G_1}(a_i) \subset F$. Since $F \neq N_{G(G_1,G_2;\mathcal{E})}(a_i)$, at most one of $b_i, b_{i-1} \text{mod } t$ lies in $F$. Thus, the claim holds.

Now, if $C$ contains at least two vertices $a_i$ and $a_j$. Since $|X_2| \leq 2$, we may assume $b_{i-1} \text{mod } t \notin F$. Thus, our claim holds.

The theorem is proved. □

A similar theorem leads to the following results.

Theorem 6. Assume that $t$ is an integer with $t \geq 2$. Let $G_1$ and $G_2$ be two $k$-regular maximum edge connected graphs with $t$ vertices such that $V(G_1) = \{a_i|0 \leq i < t\}$ and $V(G_2) = \{b_i|0 \leq i < t\}$. Let $\mathcal{E}$ be a set of edges given by $\mathcal{E} = \{(a_i,b_i)|0 \leq i < t\} \cup \{(b_i,a_{i+1} \text{mod } t)|0 \leq i < t\}$. Then, $G(G_1,G_2;\mathcal{E})$ is $(k+2)$-regular super-edge-connected if (1) $t > k + 1$ with $k \geq 2$ or (2) $t = k + 1$ with $k \geq 1$. 


Assume that $c$, $d$, $r$ are integers with $r \geq 0$, $d > 1$, and $1 \leq c < d$. It is proposed in [8] that the recursive circulant graph $RC(c,d,r)$ as the circulant graph $G(c d' ; \{ 1 , d , \ldots , d' [ \log_d c d' ] - 1 \} )$. For $0 \leq i < d$, let $V_i^r$ denote the set $\{ j | 0 \leq j < c d' , j = i ( \text{mod} \ d ) \}$. We use $RC_i(c,d,r)$ to denote the subgraph of $RC(c,d,r)$ induced by $V_i^r$. For a positive integer $N$, let $Z_N$ be the additive group of residue classes modulo $N$. We can recursively describe $RC(c,d,r)$ as follows: Assume that $r = 0$. Then $RC(c,d,0)$ is the graph with $V( RC(c,d,0) ) = \{ 0 \}$ and $E( RC(c,d,0) ) = \emptyset$ if $c = 1$, $V( RC(c,d,0) ) = \{ 0,1 \}$ and $E( RC(c,d,0) ) = \{ (0,1) \}$ if $c = 2$, and $V( RC(c,d,0) ) = Z_r$ and $E( RC(c,d,0) ) = \{ (i,i+1) | i \in Z_r \}$ if $c \geq 3$. Assume that $r \geq 1$. The induced subgraph $RC_i(c,d,r)$ is isomorphic to $RC(c,d,r-1)$ for $0 \leq i < d$. More precisely, let $f_i^r$ be the function from $Z_{cd' - 1}$ into $V_i^r$ defined by $f_i^r(x) = dx + i$. Then, $f_i^r$ induces an isomorphism from $RC(c,d,r-1)$ into $RC_i(c,d,r)$. Let $H(c,d,r)$ denote the set of edges of $RC(c,d,r)$ not in $\cup_{i=0}^{d-1} E( RC(c,d,r) )$. Then, $H(c,d,r) = \{ (i,i+1) | i \in Z_{cd'} \}$.

Suppose $d \geq 3$. Then, $RC(c,d,r)$ can be expressed as $G(H,H,\ldots,H;\mathcal{C})$, with $d$ $H$’s, where $H$ is $RC(c,d,r-1)$ for some $\mathcal{C} = \cup_{i=1}^{d-1} M_i i+1 ( \text{mod} \ d )$. Suppose $d = 2$. Then, $RC(c,d,r)$ can be expressed as $G(H,H;\mathcal{C})$, where $H$ is $RC(c,d,r-1)$. Recursively applying Theorems 3–6, we can easily prove that $RC(c,d,r)$ is super-connected unless (1) $r = 0$ and $c \geq 5$ or (2) $r = 1$, $c = 1$, and $d \geq 5$; and super-edge-connected unless (1) $r = 0$ and $c \geq 4$, (2) $r = 1$, $c = 1$, and $d \geq 4$, or (3) $r = 1$, $c = 2$, and $d = 3$.

**Corollary 3.** Let $t$ be an integer with $t \geq 2$. Assume that $G_1$ and $G_2$ are two $k$-connected and $k'$-edge connected graphs with $t$ vertices. Then, $G(G_1,G_2;\mathcal{C})$ is $(k+2)$-connected and $(k'+2)$-edge connected.

### References


