On the log-Sobolev constant for the simple random walk on the $n$-cycle: the even cases

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Abstract

Consider the simple random walk on the $n$-cycle $\mathbb{Z}_n$. For this example, Diaconis and Saloff-Coste (Ann. Appl. Probab. 6 (1996) 695) have shown that the log-Sobolev constant $\alpha$ is of the same order as the spectral gap $\lambda$. However the exact value of $\alpha$ is not known for $n \geq 4$. (For $n = 2$, it is a well known result of Gross (Amer. J. Math. 97 (1975) 1061) that $\alpha = \frac{1}{2}$. For $n = 3$, Diaconis and Saloff-Coste (Ann. Appl. Probab. 6 (1996) 695) showed that $\alpha = \frac{1}{2} \log 2 < \frac{1}{2} = 0.75$. For $n = 4$, the fact that $\alpha = \frac{1}{2}$ follows from $n = 2$ by tensorization.) Based on an idea that goes back to Rothaus (J. Funct. Anal. 39 (1980) 42; 42 (1981) 110), we prove that if $n \geq 4$ is even, then the log-Sobolev constant $\alpha$ and the spectral gap satisfy $\alpha = \frac{1}{2}$. This implies that $\alpha = \frac{1}{2} (1 - \cos \frac{2\pi}{n})$ when $n$ is even and $n \geq 4$.

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1. Introduction

Consider a finite state space $\mathcal{S}$ equipped with an irreducible Markov kernel $K(x, y)$, which is reversible with respect to a probability measure $\pi$ on $\mathcal{S}$ (i.e., $\pi(x)K(x, y) = \pi(y)K(y, x)$ for all $x, y \in \mathcal{S}$). Define an inner product on complex-valued functions on $\mathcal{S}$ by $\langle f, g \rangle = \sum_{s \in \mathcal{S}} f(s)g(s)\pi(s)$. The Dirichlet form associated with $(K, \pi)$ is then given by the formula

$$\mathcal{E}(f, g) = \mathcal{R} \langle (I - K)f, g \rangle,$$

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where $I$ is the identity matrix, $f$ and $g$ are two complex-valued functions, and $\Re z$ is the real part of a complex number $z$. Set

$$E_\pi f = \sum_{s \in \mathcal{S}} f(s) \pi(s)$$

and

$$\text{Var}_\pi (f) = \|f - E_\pi f\|_2^2.$$  

Here $\| \cdot \|_2$ is the usual $l^2$-norm with respect to the measure $\pi$. The spectral gap $\lambda$ of $(K, \pi)$ is defined by

$$\lambda = \min \left\{ \frac{\mathcal{E}(f, f)}{\text{Var}_\pi (f)} : \text{Var}_\pi (f) \neq 0 \right\}. \quad (1.1)$$

Since $(K, \pi)$ is reversible, it is easy to show that the spectral gap $\lambda$ is the smallest non-zero eigenvalue of $I - K$.

For every function $f$ on $\mathcal{S}$, consider the entropy-like quantity

$$\mathcal{L}(f) = \sum_{s \in \mathcal{S}} \left| f(s) \right|^2 \left( \log \frac{|f(s)|^2}{\|f\|_2^2} \right) \pi(s). \quad (1.2)$$

(Clearly we have $\mathcal{L}(f) \geq 0$ and $\mathcal{L}(f) = 0$ only if $f$ is a constant function.) A log-Sobolev inequality is an inequality of the type

$$\mathcal{L}(f) \leq C \mathcal{E}(f, f) \quad (1.3)$$

holding for all functions $f$. We say that $\alpha$ is the log-Sobolev constant of $K$ if $\frac{1}{\alpha}$ is the smallest constant $C$ such that inequality (1.3) holds. In other words,

$$\alpha = \inf \left\{ \frac{\mathcal{E}(f, f)}{\mathcal{L}(f)} : \mathcal{L}(f) \neq 0 \right\}. \quad (1.4)$$

(cf. (1.1)). Notice that $\mathcal{L}(f) = \mathcal{L}(|f|)$ and

$$\mathcal{E}(f, f) = \|f\|_2^2 - \Re(\langle Kf, f \rangle)$$

$$= \frac{1}{2} \sum_{x,y} \left( |f(x)|^2 - 2\Re(f(x)f(y)) + |f(y)|^2 \right) K(x,y) \pi(x)$$

$$= \frac{1}{2} \sum_{x,y} |f(x) - f(y)|^2 K(x,y) \pi(x)$$

$$\geq \frac{1}{2} \sum_{x,y} \|f(x)| - |f(y)||^2 K(x,y) \pi(x)$$

$$= \mathcal{E}(|f|, |f|).$$
Hence in the definition of the log-Sobolev constant $\alpha$ one can restrict $f$ to be real non-negative function. The following well-known result compares the log-Sobolev constant to the spectral gap. It is a special case of a result proved first by Simon and later independently by Rothaus by a different argument (see a survey paper of Gross [4] or [2]).

**Theorem 1.** For any $K$ the log-Sobolev constant $\alpha$ and the spectral gap $\lambda$ satisfy $2\alpha \leq \lambda$.

The following theorem is a translation of a previous result of Rothaus [6,7]. For a simple proof in our setting, see [8, Theorem 2.2.3].

**Theorem 2.** Let $K$ be irreducible and $\pi$ be its stationary distribution. Then either $2\alpha = \lambda$ or there exists a positive non-constant function $f$ which is a solution of

$$2f \log f - 2f \log \|f\|_2 - \frac{1}{\alpha}(I - K)f = 0, \quad (1.5)$$

and such that $\alpha = \mathcal{E}(f, f)/\mathcal{L}(f)$.

Inequalities of Poincaré, Cheeger, Sobolev, Nash and log-Sobolev, are advanced techniques for bounded mixing times of finite irreducible reversible Markov chains. However, computing the log-Sobolev constant $\alpha$ exactly is difficult and it has been done only for a handful of examples. Diaconis and Saloff-Coste [2] gave the exact value of the log-Sobolev constant of the chain on a finite space with all rows of $K$ equal to $\pi$. (This includes all chains on a two-point space.) We refer to [1,2] for more examples.

In this paper we compute the exact value of the log-Sobolev constant for the simple random walk on the $n$-cycle. (The exact value of the log-Sobolev constant is well-known for $n \leq 4$ (see [2,3])). In Section 3 we prove that if $n$ is even and $n \geq 4$, then the log-Sobolev constant $\alpha$ and the spectral gap $\lambda$ satisfy $2\alpha = \lambda$ (see Theorem 3 below). This implies that $\alpha = \frac{1}{2}(1 - \cos \frac{2\pi}{n})$. Our main result (Theorem 3) follows from Theorems 1 and 2 by showing that if $2\alpha < \lambda$, then there is no positive non-constant function $f$ satisfying (1.5) and such that $\alpha = \mathcal{E}(f, f)/\mathcal{L}(f)$ (this approach was also used earlier in a different context by Mueller and Weissler [5]).

2. **The log-Sobolev constant for $n$-cycle**

Consider a simple random walk on the $n$-cycle $\mathbb{Z}_n$ and write $\mathbb{Z}_n = \{1, 2, \ldots, n\}$. Clearly the corresponding Markov kernel $K$ is given by $K(x, x \pm 1) = \frac{1}{2}$ and the uniform distribution on $\mathbb{Z}_n$ is its unique stationary distribution. (For $n = 2$, we have $K(1, 2) = K(1, 1) = K(2, 1) = K(2, 2) = \frac{1}{2}$. It is easy to check that the spectral gap of $K$ is 1. Also it follows from a result of Gross [3] that $\alpha = \frac{1}{2}$. Therefore we obtain that $\alpha = \frac{1}{2} = \frac{1}{2}$ in the case $n = 2$.) Throughout this paper we assume that $n \geq 3$. 
For every $l = 1, 2, \ldots, n - 1$, let

$$\theta_l = \frac{2\pi l}{n}$$

and

$$u_l = \begin{pmatrix} \sin \theta_l \\ \sin 2\theta_l \\ \vdots \\ \sin n\theta_l \end{pmatrix}.$$ 

Then $u_l \neq 0$ and direct computations imply that $Ku_l = (\cos \theta_l)u_l$ for $l = 1, 2, \ldots, n - 1$. Therefore the spectrum of $I - K$ is given by the set

$$\sigma(I - K) = \left\{ 1 - \cos \frac{2\pi l}{n} \mid l = 1, 2, \ldots, n \right\}.$$ 

Since $K$ is reversible, we observe that the spectral gap $\lambda$ of $K$ is $1 - \cos \frac{2\pi}{n}$.

Denote by $\alpha$ the log-Sobolev constant for the simple random walk on the $n$-cycle. Note that the log-Sobolev constant for the simple random walk on $\mathbb{Z}_3$ is $\frac{1}{2 \log 2}$ (see, e.g., [2]). Thus in this case we have $\alpha = \frac{1}{2 \log 2} \leq \frac{1}{2} = \frac{1}{2} (1 - \cos \frac{2\pi}{3}) = 0.75$. For $n = 4$, we obtain $\alpha = \frac{1}{2}$ from $n = 2$ by tensorization. For $n \geq 4$, Diaconis and Saloff-Coste [2] showed that $\alpha$ is of the same order as $\lambda$. In particular they proved that

$$\frac{8}{25} \frac{\pi^2}{n^2} \leq \alpha \leq \frac{2\pi^2}{n^2}.$$ 

By refining their arguments, we obtain

$$\frac{2}{5} \frac{\pi^2}{n^2} \leq \alpha \leq \frac{\pi^2}{n^2}.$$ 

The main result of this paper is as follows.

**Theorem 3.** Assume that $n$ is even. Then the log-Sobolev constant for the simple random walk on the $n$-cycle is just one half of its spectral gap: $\alpha = \frac{1}{2}$ (we will prove Theorem 3 in Section 3).

To compute the exact value of $\alpha$, we write functions $f$ on $\mathbb{Z}_n$ as vectors $(f(1), f(2), \ldots, f(n))$ in $\mathbb{R}^n$. For every function $f = (x_1, x_2, \ldots, x_n)$, we have

$$\mathcal{L}(f) = \frac{1}{n} \sum_{i=1}^{n} x_i^2 \log \frac{x_i^2}{\|f\|_2^2}$$

(2.1)
and

$$\mathcal{E}(f, f) = \frac{1}{2^n} (|x_1 - x_2|^2 + |x_2 - x_3|^2 + \cdots + |x_{n-1} - x_n|^2 + |x_n - x_1|^2). \quad (2.2)$$

Clearly function $\mathcal{E}$ is invariant if we permute the components of $f$, while function $\mathcal{L}$ is not. For a fixed function $f$, we investigate the extreme value of $\mathcal{E}$ over all permutations on the components of $f$.

Consider the function

$$F(x) = |x_1 - x_2|^2 + |x_2 - x_3|^2 + \cdots + |x_{n-1} - x_n|^2 + |x_n - x_1|^2, \quad (2.3)$$

where $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. Moreover to every $x = (x_1, x_2, \ldots, x_n)$ with $0 \leq x_1 \leq x_2 \leq \cdots \leq x_n$, there corresponds an element $\tilde{x} \in \mathbb{R}^n$ given by the formula

$$\tilde{x} = \begin{cases} (x_1, x_3, x_5, \ldots, x_{2k+1}, x_{2k}, \ldots, x_2, x_4, x_2) & \text{if } n = 2k + 1, \\ (x_1, x_3, x_5, \ldots, x_{2k-1}, x_{2k}, \ldots, x_2, x_4, x_2) & \text{if } n = 2k. \end{cases} \quad (2.4)$$

Denote by $S_n$ the set of all permutations on $\{1, 2, \ldots, n\}$ and write $\theta x = (x_{\theta(1)}, x_{\theta(2)}, \ldots, x_{\theta(n)})$ for $\theta \in S_n$ and $x \in \mathbb{R}^n$.

**Proposition 1.** For every $x = (x_1, x_2, \ldots, x_n)$ with $0 \leq x_1 \leq x_2 \leq \cdots \leq x_n$, we have $F(\theta x) \geq F(\tilde{x})$ for all $\theta \in S_n$.

**Proof.** We prove this by induction on $n$. Clearly there is nothing to prove in the case $n = 2$. Assume that it is also true for $n = k$. We consider the case $n = k + 1$ and fix $x = (x_1, x_2, \ldots, x_{k+1})$ where $0 \leq x_1 \leq x_2 \leq \cdots \leq x_{k+1}$.

**Step 1:** Set $y = (x_1, x_2, \ldots, x_k)$ and consider the corresponding $\tilde{y}$ given by (2.4). For every $i = 1, 2, \ldots, k - 2$, set

$$\tilde{y}_{i,i+2} = \begin{cases} (x_1, x_3, \ldots, x_i, x_{i+1}, x_{i+2}, \ldots, x_4, x_2) & \text{if } i \text{ is odd}, \\ (x_1, x_3, \ldots, x_{i+2}, x_{i+1}, x_i, \ldots, x_2, x_4, x_2) & \text{if } i \text{ is even}. \end{cases} \quad (2.5)$$

Thus $\tilde{y}_{i,i+2}$ is obtained by inserting $x_{k+1}$ in $\tilde{y}$ between $x_i$ and $x_{i+2}$. Also set $\tilde{y}_{1,2} = (x_1, x_3, \ldots, x_4, x_2, x_{k+1})$ and

$$\tilde{y}_{k-1,k} = \begin{cases} (x_1, x_3, \ldots, x_k, x_{k+1}, x_{k-1}, \ldots, x_4, x_2) & \text{if } k \text{ is odd}, \\ (x_1, x_3, \ldots, x_{k-1}, x_{k+1}, x_k, \ldots, x_4, x_2) & \text{if } k \text{ is even}. \end{cases} \quad (2.6)$$

We claim that

$$F(\tilde{y}_{1,2}) \geq F(\tilde{y}_{k-1,k}) \quad (2.7)$$

and

$$F(\tilde{y}_{i,i+2}) \geq F(\tilde{y}_{k-1,k}) \quad \text{for all } i = 1, 2, \ldots, k - 2. \quad (2.8)$$
Note that for every \( 1 \leq i \leq k - 2 \), we have
\[
F(\tilde{y}_{i+2}) = F(\tilde{y}) + (x_i - x_{k+1})^2 + (x_{k+1} - x_{i+2})^2 - (x_i - x_{i+2})^2. \quad (2.9)
\]
Therefore for \( 1 \leq i \leq k - 4 \), we have
\[
F(\tilde{y}_{i+2}) - F(\tilde{y}_{i+2,i+4}) = [(x_i - x_{k+1})^2 + (x_{k+1} - x_{i+2})^2 - (x_i - x_{i+2})^2]
- [(x_{i+2} - x_{k+1})^2 + (x_{k+1} - x_{i+4})^2 - (x_{i+2} - x_{i+4})^2]
= 2(x_{k+1} - x_{i+2})(x_{i+4} - x_i) \geq 0. \quad (2.10)
\]
Also we have
\[
F(\tilde{y}_{k-2,k}) - F(\tilde{y}_{k-1,k}) = [(x_{k+1} - x_{k-2})^2 + (x_{k+1} - x_{k})^2 - (x_{k-2} - x_{k})^2]
- [(x_{k+1} - x_{k-1})^2 + (x_{k+1} - x_{k})^2 - (x_{k-1} - x_{k})^2]
= 2(x_{k+1} - x_k)(x_{k-1} - x_{k-2}) \geq 0 \quad (2.11)
\]
and
\[
F(\tilde{y}_{k-3,k-1}) - F(\tilde{y}_{k-1,k}) = 2(x_{k+1} - x_{k-1})(x_k - x_{k-3}) \geq 0. \quad (2.12)
\]
Combining (2.10)–(2.12) gives (2.8). To prove (2.7), it suffices to show that \( F(\tilde{y}_{1,2}) \geq F(\tilde{y}_{1,3}) \). This follows easily from the fact that
\[
F(\tilde{y}_{1,2}) - F(\tilde{y}_{1,3}) = [(x_1 - x_{k+1})^2 + (x_{k+1} - x_2)^2 - (x_1 - x_2)^2]
- [(x_1 - x_{k+1})^2 + (x_{k+1} - x_3)^2 - (x_1 - x_3)^2]
= 2(x_{k+1} - x_1)(x_3 - x_2) \geq 0.
\]

**Step 2:** We prove that for every \( \theta \in S_{n+1} \), we have
\[
F(\theta x) \geq F(\tilde{y}_{k-1,k}) = F(\tilde{x}). \quad (13)
\]
Fix \( \theta \in S_{n+1} \) and set \( c = \theta x \). Write \( c = (\ldots, x_i, x_{k+1}, x_j, \ldots) \) for some \( i < j \) and let \( z = (\ldots, x_i, x_j, \ldots) \in \mathbb{R}^d \) be obtained by removing the component \( x_{k+1} \) from the vector \( c \). If \( 1 \leq j \leq k - 2 \), we have
\[
F(c) - F(\tilde{y}_{j,j+2}) = [F(z) + (x_i - x_{k+1})^2 + (x_j - x_{k+1})^2 - (x_i - x_j)^2]
- [F(\tilde{y}) + (x_j - x_{k+1})^2 + (x_{k+1} - x_{j+2})^2 - (x_j - x_{j+2})^2]
= F(z) - F(\tilde{y}) + 2(x_{k+1} - x_j)(x_{j+2} - x_i) \geq 0. \quad (2.14)
\]
(In the last inequality, we use the assumption that $F(z) \geq F(\tilde{y})$.) If $j = k - 1$, we have
\[
F(c) - F(\tilde{y}_{k-1,k}) = [F(z) + (x_i - x_{k+1})^2 + (x_{k-1} - x_{k+1})^2 - (x_i - x_{k-1})^2]

- [F(\tilde{y}) + (x_k - x_{k+1})^2 + (x_{k+1} - x_{k-1})^2 - (x_k - x_{k-1})^2]

= F(z) - F(\tilde{y}) + 2(x_k - x_i)(x_{k+1} - x_{k-1}) \geq 0. \tag{2.15}
\]
If $j = k$, we have
\[
F(c) - F(\tilde{y}_{k-1,k}) = [F(z) + (x_k - x_{k+1})^2 + (x_i - x_{k+1})^2 - (x_k - x_{k-1})^2]

- [F(\tilde{y}) + (x_k - x_{k+1})^2 + (x_{k+1} - x_{k-1})^2 - (x_k - x_{k-1})^2]

= F(z) - F(\tilde{y}) + 2(x_{k-1} - x_i)(x_{k+1} - x_k) \geq 0. \tag{2.16}
\]
Therefore (2.13) follows (2.14), (2.15), (2.16) and (2.8). \qed

Remark 1. Assume that the minimum $\alpha$ in (1.4) is attained at some positive non-constant function $f$. By the definition of the log-Sobolev constant and Proposition 1, there exists a minimizer of the form $f = (x_1, x_3, \ldots, x_4, x_2)$ while $0 \leq x_1 \leq x_2 \leq \cdots \leq x_n$. Moreover it is not hard to show that any minimizer of $E(f, f)$ must satisfy the non-linear equation (1.5).

3. Proof of the main result

Throughout this section we assume that $n$ is even and $n \geq 4$. We will argue by contradiction to verify that if $\alpha < \frac{\pi}{2}$, there is no positive non-constant function $f$ satisfying the non-linear equation (1.5) and such that $\alpha = \frac{\beta(f, f)}{\beta(f)}$. Then our main result (Theorem 3) follows from Theorems 1 and 2. Before proving the main result, we derive a series of lemmas by some combinatorial arguments.

Define the shift operator $\sigma$ by
\[
\sigma(x_1, x_2, \ldots, x_n) = (x_n, x_1, x_2, \ldots, x_{n-1}),
\]
where $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. Set $\sigma^j(x) = \sigma(\sigma^{j-1}(x))$ for $j \geq 2$ and write $\sigma^{-j}$ for the inverse of $\sigma^j$.

Lemma 1. Consider a vector of the form
\[
u = (x_1, x_3, \ldots, x_{2k-1}, x_{2k}, \ldots, x_4, x_2)
\]
where \( x_1 \leq x_2 \leq \cdots \leq x_{2k} \) and write \( \sigma^j(u) = ( (\sigma^j(u))_1, (\sigma^j(u))_2, \ldots, (\sigma^j(u))_{2k} ) \). Then for every \( 1 \leq j \leq k - 1 \), we have
\[
(\sigma^j(u))_i \leq (\sigma^j(u))_{2k-i+1} \quad \text{for } i = 1, \ldots, k 
\] (3.1)
and
\[
(\sigma^{-j}(u))_i \geq (\sigma^{-j}(u))_{2k-i+1} \quad \text{for } i = 1, \ldots, k.
\] (3.2)

**Proof.** Assume \( 1 \leq j \leq k - 1 \). Then we have
\[
(\sigma^j(u))_i = \begin{cases} 
  x_{2(j-i+1)} & \text{if } 1 \leq i \leq j, \\
  x_{2(i-j)-1} & \text{if } j + 1 \leq i \leq j + k, \\
  x_{2k-2j-(j+k+1)} & \text{if } j + k + 1 \leq i \leq 2k.
\end{cases}
\]

(Case \( 1 \leq i \leq j \wedge (k-j) \)) Since \( i \leq (k-j) \) we get \( 2k - i + 1 \geq k + j + 1 \) and \( (\sigma^j(u))_{2k-i+1} = x_{2(i+j)} \). Therefore we observe
\[
(\sigma^j(u))_i = x_{2(j-i+1)} \leq x_{2(i+j)} = (\sigma^j(u))_{2k-i+1}.
\]

(Case \( j \vee (k-j) < i \leq k \).) Note that \( (k-j) < i \leq k \) implies \( k + 1 \leq (2k - i + 1) \leq (k + j) \). We have
\[
(\sigma^j(u))_i = x_{2(i-j)-1}
\]
and
\[
(\sigma^j(u))_{2k-i+1} = x_{2(2k-i-j)+1}.
\]

Since \( 2(2k - i - j) + 1 \geq 2(i - j) - 1 \), we get \( (\sigma^j(u))_i \leq (\sigma^j(u))_{2k-i+1} \).

(Case \( j \wedge (k-j) < i \leq j \vee (k-j). \)) It is obvious that we only need to consider the situation that \( j \neq k-j \). We first consider the case that \( j < k-j \). Then we have \( j < i \leq (k-j) \) and \( 2k - i + 1 \geq j - k + 2k + 1 = k + j + 1 \). Therefore
\[
(\sigma^j(u))_i = x_{2(i-j)-1} \leq x_{2(i+j)} = (\sigma^j(u))_{2k-i+1}.
\]

On the other hand, if \( k - j < j \), then we have \( k - j < i \leq j \). This implies that
\[
(\sigma^j(u))_i = x_{2(j-i+1)} \leq x_{2(2k-i-j)+1} = (\sigma^j(u))_{2k-i+1}.
\]

This completes the proof of (3.1). The proof of (3.2) can be done by similar arguments. Here we omit it. \( \square \)

**Lemma 2.** Let \( u = (u_1, u_2, \ldots, u_{2k-1}, u_{2k}) \) be a vector with \( u_i > 0 \) for all \( 1 \leq i \leq 2k \). Assume further that there exist two positive constants, \( c \) and \( d \),
such that
\[ 2u_i - (u_{i-1} + u_{i+1}) = cu_i \log du_i^2 \]  
for all \( i = 1, \ldots, 2k \) (here we write \( u_0 = u_{2k} \) and \( u_{2k+1} = u_1 \)).

(a) If \( u_i \leq u_{2k-i+1} \) for all \( 1 \leq i \leq k \), then we have
\[ u_1^2 - u_{2k}^2 + u_k^2 - u_{k+1}^2 \geq c[(u_1^2 + \cdots + u_k^2) - (u_{k+1}^2 + \cdots + u_{2k}^2)]. \]

(b) If \( u_i \geq u_{2k-i+1} \) for all \( 1 \leq i \leq k \), then we have
\[ u_{2k}^2 - u_k^2 + u_{k+1}^2 - u_1^2 \geq c[(u_{k+1}^2 + \cdots + u_{2k}^2) - (u_1^2 + \cdots + u_k^2)]. \]

Proof. (a) Assume that \( u_i \leq u_{2k-i+1} \) for all \( 1 \leq i \leq k \). For every \( 1 \leq i \leq k \), rewrite Eq. (3.3) as
\[ 2 - \frac{u_{i-1} + u_{i+1}}{u_i} = c \log du_i^2. \]

Then we observe that
\[
\frac{u_{2k-i} + u_{2k-i+2}}{u_{2k-i+1}} \frac{u_{i-1} + u_{i+1}}{u_i} = \frac{u_i(u_{2k-i} + u_{2k-i+2})}{u_{2k-i+1}} u_{i-1} + u_{i+1} \\
= c \left( 2 \log \frac{u_i}{u_{2k-i+1}} \right) \geq c \left( \frac{u_i}{u_{2k-i+1}} - \frac{u_{2k-i+1}}{u_i} \right). \tag{3.4}
\]

(In the last inequality we use the fact that \( 2 \log t \geq t - \frac{1}{t} \) for every \( 0 < t \leq 1 \).) Inequality (3.4) implies that
\[ (u_i u_{2k-i+2} - u_i u_{2k-i+1}) + (u_i u_{2k-i} - u_i u_{2k-i+1}) \geq c(u_i^2 - u_{2k-i+1}^2) \]
for all \( i = 1, \ldots, k \). Our result follows by summing up the above \( k \) inequalities.

(b) Assume that \( u_i \geq u_{2k-i+1} \) for all \( 1 \leq i \leq k \). For every \( i \), set \( v_i = u_{2k-i+1} \). Then our result follows by applying (a) to the vector \( v = (v_1, v_2, \ldots, v_{2k}) \). □

Lemma 3. Consider the following \( k \times k \) matrices:

\[
A = \begin{bmatrix}
2 & 1 & 0 & \cdots & \cdots & 0 \\
1 & 2 & 1 & \ddots & \vdots & \vdots \\
0 & 1 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & 2 & 1 & 0 & \vdots \\
\vdots & \ddots & 1 & 2 & 1 & \vdots \\
0 & \cdots & \cdots & \cdots & 2 & 2
\end{bmatrix}
\]
and

\[
B = \begin{bmatrix}
2 & 1 & 0 & \cdots & \cdots & 0 \\
1 & 2 & 1 & \ddots & \ddots & \vdots \\
0 & 1 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & 2 & 1 & 0 & \vdots \\
\vdots & \ddots & 1 & 2 & 1 & \vdots \\
0 & \cdots & \cdots & 0 & 1 & 1
\end{bmatrix}.
\]

(a) If \( t < 2(1 - \cos \frac{\pi}{2k}) \), then \( P_A(t) = \det(A - tI) > 0 \).
(b) If \( t < 2(1 - \cos \frac{\pi}{2k + 1}) \), then \( P_B(t) = \det(B - tI) > 0 \).

**Proof.** (a) For every \( 1 \leq l \leq k \), let \( \theta_l = \frac{(2l-1)\pi}{2k} \) and

\[
v_l = \begin{bmatrix}
\sin \theta_l \\
\sin 2\theta_l \\
\vdots \\
\sin k\theta_l
\end{bmatrix}.
\]

Routine calculation shows that \( Av_l = 2(1 + \cos \theta_l)v_l \) for \( 1 \leq l \leq k \). Therefore \( \{2(1 + \cos \theta_l)|1 \leq l \leq k\} \) is the set of all real roots of the characteristic polynomial \( P_A(t) \). Note that \((-t)^k\) is the highest order term of \( P_A(t) \). This implies that \( \lim_{t \to -\infty} P_A(t) = \infty \). Since \( 2(1 - \cos \frac{\pi}{2k}) \) is the smallest real root of \( P_A(t) \), we observe that \( P_A(t) > 0 \) for all \( t < 2(1 - \cos \frac{\pi}{2k}) \).

(b) The proof of (b) is the same as that of (a) where value of \( \theta_l \) is replaced by \( \frac{2\pi l}{2k+1} \). \( \square \)

**Lemma 4.** (a) Consider the following system of inequalities:

\[
\begin{cases}
A_j - A_{j+1} \geq 4t(A_1 + \cdots + A_j), & j = 1, \ldots, k - 1, \\
A_k \geq 2t(A_1 + \cdots + A_k).
\end{cases}
\] \hspace{1cm} (3.5)

If \( t < \frac{1}{2}(1 - \cos \frac{\pi}{2k}) \), then system (3.5) has no solution \( (A_1, A_2, \ldots, A_k) \) with \( A_1 < 0 \).

(b) Consider the following system of inequalities:

\[
\begin{cases}
A_j - A_{j+1} \geq 4t(A_1 + \cdots + A_j), & j = 1, \ldots, k - 1, \\
A_k \geq 4t(A_1 + \cdots + A_k).
\end{cases}
\] \hspace{1cm} (3.6)

If \( t < \frac{1}{2}(1 - \cos \frac{\pi}{2k+1}) \), then the system (3.6) has no solution \( (A_1, A_2, \ldots, A_k) \) with \( A_1 < 0 \).
Proof. (a) Let \( f_1(t) = 2 - 4t \) and \( g_1(t) = 4t \). For every \( 1 \leq l \leq k - 1 \), put
\[
f_{l+1}(t) = (1 - 4t)f_l(t) - g_l(t)
\]
and
\[
g_{l+1}(t) = 4tf_l(t) + g_l(t).
\]
Clearly (3.7)–(3.8) imply
\[
g_{l+1}(t) - g_l(t) = 4tf_l(t) = f_l(t) - g_l(t) - f_{i+1}(t).
\]
Hence we have \( f_l(t) = g_{l+1}(t) + f_{i+1}(t) \) for \( 1 \leq l \leq k - 1 \). Moreover for \( 2 \leq l \leq k - 1 \), we obtain
\[
f_{l+1}(t) = (2 - 4t)f_l(t) - (f_l(t) + g_l(t)) = (2 - 4t)f_l(t) - f_{l-1}(t).
\]
Note that \( f_1(t) = 2 - 4t, \ f_2(t) = (1 - 4t)f_1(t) - g_1(t) = (2 - 4t)^2 - 2 \). Therefore we observe
\[
f_l(t) = \det(M_l - 4tI_l), \quad 1 \leq l \leq k,
\]
where \( I_l \) is the \( l \times l \) identity matrix and \( M_l \) is the \( l \times l \) matrix of the same form as that in Lemma 3(a).

Assume that \( t < \frac{1}{4}(1 - \cos \frac{\pi}{k}) \) and \((A_1, A_2, \ldots, A_k)\) satisfies the system of inequalities (3.5). Since \( t < \frac{1}{4}(1 - \cos \frac{\pi}{k}) \) for \( 1 \leq l \leq k \), Lemma 3(a) and (3.9) imply that \( f_l(t) > 0 \) for all \( l = 1, 2, \ldots, k \).

For every \( 1 \leq i \leq k - 1 \), we have, by (3.5),
\[
A_{k-i} - A_{k-i+1} \geq 4t(A_1 + \cdots + A_{k-i}).
\]

For \( 1 \leq j \leq k \), we claim that
\[
f_j(t)A_{k-j+1} \geq g_j(t)(A_1 + \cdots + A_{k-j}).
\]
Clearly (3.10) holds for \( j = 1 \). Assume it also holds for some \( i \) with \( 1 \leq i \leq k - 1 \). Since \( f_i(t) > 0 \), we get
\[
f_i(t)A_{k-i} = f_i(t)(A_{k-i} - A_{k-i+1}) + f_{i+1}(t)A_{k-i+1} \geq (4tf_i(t) + g_i(t))(A_1 + \cdots + A_{k-i})
\]
\[
= g_{i+1}(t)(A_1 + \cdots + A_{k-i-1}) + (4tf_i(t) + g_i(t))A_{k-i}.
\]
The above inequality implies that (3.10) also holds for \( j = i + 1 \). Hence (3.10) is true for \( 1 \leq j \leq k \). Plugging \( j = k \) into (3.10) gives \( f_k(t)A_1 \geq 0 \). Since \( f_k(t) > 0 \), we observe that \( A_1 \geq 0 \). This completes the proof of (a).

(b) The proof of (b) follows word by word that of (a) while replacing \( f_i(t) \) by \( 1 - 4t \). \( \square \)

**Proof of Theorem 3.** By Theorems 1 and 2, it suffices to show that if \( \alpha < \frac{4}{\pi} \), then there is no positive non-constant function \( f \) satisfying the non-linear equation (1.5) and such that \( \alpha = \frac{\epsilon(f,f)}{2f(f)} \). We argue by contradiction. Suppose that \( \alpha < \frac{4}{\pi} = \frac{1}{2}(1 - \cos \frac{2\pi}{n}) \) and there exists a positive non-constant unit function \( f \) satisfying the non-linear equation (1.5) and such that \( \alpha = \frac{\epsilon(f,f)}{2f(f)} \). By Remark 1, we can assume further that 

\[
 f = (x_1, x_2, \ldots, x_{n-1}, x_n, \ldots, x_4, x_2),
\]

where \( 0 < x_1 \leq x_2 \leq \cdots \leq x_n \) and \( x_1 < x_n \). Moreover the function \( f \) satisfies the equations:

\[
 2x_i - (x_i^{(1)} + x_i^{(2)}) = 2xx_i \log nx_i^2, \quad 1 \leq i \leq n,
\]

where \( x_i^{(1)} \) and \( x_i^{(2)} \) are the two nearest neighbors of \( x_i \).

Recall that \( \sigma \) is the shift operator and \( \sigma^j = \sigma^{(j-1)} \) for \( j \geq 2 \). Write \( n = 4k \) or \( n = 4k + 2 \). For \( j = 1, \ldots, k \), we have

\[
 \sigma^j(f) = (x_{2j}, \ldots, x_2, x_1, \ldots, x_{n-2j-1}, x_n-x_{2j+1}, \ldots, x_{n-1}, x_n, \ldots, x_{2j+2})
\]

and

\[
 \sigma^{-j}(f) = (x_{2j+1}, \ldots, x_n-x_1, x_n, \ldots, x_{n-2j+2}, x_{n-2j}, \ldots, x_2, x_1, \ldots, x_{2j-1}).
\]

By Lemmas 1 and 2(a), we get

\[
 (x_{2j}^2 - x_{2j+2}^2 + x_{n-2j-1}^2 - x_{n-2j+1}^2) \\
 \geq 2\alpha[(x_1^2 + x_2^2 + \cdots + x_{2j-1}^2 + x_1^2 + x_3^2 + \cdots + x_{n-2j-1}^2) \\
 - (x_{n-2j+1}^2 + x_{n-2j+3}^2 + \cdots + x_{n-1}^2 + x_{2j+2}^2 + x_{2j+4}^2 + \cdots + x_n^2)].
\]

Similarly Lemmas 1 and 2(b) imply that

\[
 (x_{2j-1}^2 - x_{2j+1}^2 + x_{n-2j}^2 - x_{n-2j+2}^2) \\
 \geq 2\alpha[(x_1^2 + x_2^2 + \cdots + x_{2j-1}^2 + x_2^2 + x_4^2 + \cdots + x_{n-2j}^2) \\
 - (x_{2j+1}^2 + x_{2j+3}^2 + \cdots + x_{n-1}^2 + x_{n-2j+2}^2 + x_{n-2j+4}^2 + \cdots + x_n^2)].
\]
Note that \( n - 2j - 1 \geq 2j + 1 \) and \( n - 2j \geq 2j + 2 \) for \( 1 \leq j \leq k \). Summing up the above two inequalities gives

\[
(x_{2j-1}^2 + x_{2j}^2 - x_{2j+1}^2 - x_{2j+2}^2) + (x_{n-2j-1}^2 + x_{n-2j}^2 - x_{n-2j+1}^2 - x_{n-2j+2}^2) 
\geq 4\alpha [(x_1^2 + x_2^2 + \cdots + x_{2j}) - (x_{n-2j+1}^2 + x_{n-2j+2}^2 + \cdots + x_n^2)].
\]

Let \( A_i = x_{2i-1}^2 + x_{2i}^2 - x_{n-2i+1}^2 - x_{n-2i+2}^2 \) for \( 1 \leq i \leq k \). If \( n = 4k \), then we have

\[
\begin{cases}
A_j - A_{j+1} \geq 4\alpha (A_1 + A_2 + \cdots + A_j), & j = 1, \ldots, k - 1, \\
A_k \geq 2\alpha (A_1 + A_2 + \cdots + A_k).
\end{cases}
\]

If \( n = 4k + 2 \), then we observe that

\[
\begin{cases}
A_j - A_{j+1} \geq 4\alpha (A_1 + A_2 + \cdots + A_j), & j = 1, \ldots, k - 1, \\
A_k \geq 4\alpha (A_1 + A_2 + \cdots + A_k).
\end{cases}
\]

Note that \( \alpha < \frac{1}{2}(1 - \cos \frac{2\pi}{n}) \) and \( A_1 = x_1^2 + x_2^2 - x_{n-1}^2 - x_n^2 \leq x_1^2 - x_n^2 < 0 \). By Lemma 4, we get a contradiction. This completes the proof. \( \square \)

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References