Cycle Embedding in Faulty Wrapped Butterfly Graphs

Chang-Hsiung Tsai
Department of Computer Science and Information Engineering, Dahan Institute of Technology, Hualien, Taiwan 971, Republic of China

Tyne Liang, Lih-Hsing Hsu
Department of Computer and Information Science, National Chiao Tung University, Hsinchu, Taiwan 300, Republic of China

Men-Yang Lin
Department of Information Science, National Taichung Institute of Commerce, Taichung, Taiwan, Republic of China

In this paper, we study the maximal length of cycle embedding in a faulty wrapped butterfly graph $BF_n$ with at most two faults in vertices and/or edges. When there is one vertex fault and one edge fault, we prove that the maximum cycle length is $n^{2^n} \cdot 2$ if $n$ is even and $n^{2^n} \cdot 1$ if $n$ is odd. When there are two faulty vertices, the maximum cycle length is $n^{2^n} \cdot 2$ for odd $n$. All these results are optimal because the wrapped butterfly graph is bipartite if and only if $n$ is even. © 2003 Wiley Periodicals, Inc.

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1. INTRODUCTION

The performance of a distributed system is significantly determined by its network topology. The hypercube (binary $n$-cube) is one of the most popular interconnection networks. It has been used to design various commercial multiprocessor machines. One basic drawback with hypercubes is that the vertex degree increases with the number of vertices. Among all networks with fixed degrees, the wrapped butterfly network is one of the most promising networks due to its nice topological properties. On the other hand, the cycle network contains several attractive properties such as simplicity, extensibility, and feasible implementation. Hence, embedding a cycle into a wrapped butterfly network has received many researchers’ efforts in recent years [1, 3, 6, 8].

To embed a cycle into a faulty butterfly network $BF_n$ with $n^{2^n}$ vertices, it is desirable to isolate those faulty components from the remaining ones so that a maximal-length cycle can be still embedded. Vadapalli and Srimani [6] proved that there exists a cycle of length $n^{2^n} \cdot 2$ when there is one vertex fault and there is a cycle of length $n^{2^n} \cdot 4$ when two vertex faults occur. In [3], Hwang and Chen showed that the maximal cycle of length $n^{2^n}$ can be embedded in a faulty wrapped butterfly graph which has two edge faults. For all integer $n \geq 3$, these results are optimal because the wrapped butterfly graph is bipartite if and only if $n$ is even.

In the previous two studies, faults are limited to either vertex faults or edge faults. However, faults in both vertices and edges may occur. Consequently, we are motivated to explore the embedding feasibility in the faulty wrapped butterfly graph. In this paper, when there is one vertex fault and one edge fault, we prove that the maximum cycle length is $n^{2^n} \cdot 2$ if $n$ is even and $n^{2^n} \cdot 1$ if $n$ is odd. When there are two faulty vertices, the maximum cycle length is $n^{2^n} \cdot 2$ for odd $n$. All these results are optimal because the wrapped butterfly graph is bipartite if and only if $n$ is even. In Table 1, we summarize all the results about faulty edges and/or vertices in $BF_n$.

In the following section, we discuss some properties of wrapped butterfly graphs. In Section 3, we prove that the faulty wrapped butterfly graph contains a cycle of length $n^{2^n} \cdot 2$ if it has one vertex fault and one edge fault. Finally, when $n$ is an odd integer, we prove that the wrapped butterfly graph contains a Hamiltonian cycle if it has at most two faults and at least one of them is a vertex fault.
2. WRAPPED BUTTERFLY GRAPHS AND THEIR PROPERTIES

An interconnection network can be modeled by an undirected graph \( G = (V, E) \) where the set of vertices \( V(G) \) represents the processing elements of the network and the set of edges \( E(G) \) represents the communication links. Throughout this paper, the graph theoretic definitions and notations in [4] are followed. Let \( F = V_1 \cup E_1 \) for \( E_1 \subseteq E \) and \( V_1 \subseteq V \). We use \( G - F \) to denote the graph \( G' = (V - V_1, (E - E_1) \cap ((V - V_1) \times (V - V_1))) \). A simple path (or path for short) is a sequence of adjacent edges \((v_0, v_1), (v_1, v_2), \ldots, (v_{m-1}, v_m)\), written as \( \langle v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_m \rangle \), in which all the vertices \( v_0, v_1, \ldots, v_m \) are distinct except possibly \( v_0 = v_m \). We also write the path \( \langle v_0 \rightarrow P_1 \rightarrow v_1 \rightarrow v_{i+1} \rightarrow \ldots \rightarrow v_j \rightarrow P_2 \rightarrow v_k \rightarrow v_{k+1} \rightarrow \ldots \rightarrow v_m \rangle \), where \( P_1 = \langle v_0 \rightarrow v_1 \rightarrow \ldots \rightarrow v_j \rangle \) and \( P_2 = \langle v_j \rightarrow v_{j+1} \rightarrow \ldots \rightarrow v_k \rangle \). A cycle is a path \( \langle v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_m \rightarrow v_0 \rangle \), where \( m \geq 2 \). A cycle is a Hamiltonian cycle if it traverses every vertex of \( G \) exactly once. A graph is Hamiltonian if it has a Hamiltonian cycle.

2.1. Wrapped Butterfly Graphs

The wrapped butterfly (butterfly for short) \( BF_n \) is a graph with \( n^2 \) vertices such that each vertex, at level \( i \), is labeled by \((a_i, \alpha_{i-1})\) with \( 0 \leq i \leq n - 1 \) and \( a_i \in \{0, 1\} \) for all \( 0 \leq j \leq n - 1 \). Edges of \( BF_n \) are described as follows: Vertex \((a_0, \alpha_{i-1})\) is adjacent to vertex \((a_0, \alpha_{i-1}), (i + 1) \mod n \) by a straight edge and adjacent to vertex \((a_0, \alpha_{i-1}), (i + 1) \mod n \) by a cross edge. Figure 1 illustrates \( BF_3 \).

In [5], Vadapalli and Srimani proposed a family of degree four Cayley graphs, \( G_n \). Later, Chen and Lau [2] pointed out that \( G_n \) is isomorphic to \( BF_n \). Thus, we can combine all the results of \( G_n \) and \( BF_n \). To prove our main result, we will describe some properties of \( BF_n \) proposed by [5]. Throughout the paper, the edges of \( BF_n \) are defined by the following four generators \( g, g^{-1}, f, \) and \( f^{-1} \) in the graph:

\[
g((a_0, \alpha_{i-1})k) = (a_0, \alpha_{i-1}, (k + 1) \mod n),
\]

\[
g^{-1}((a_0, \alpha_{i-1})k) = (a_0, \alpha_{i-1}, (k - 1) \mod n),
\]

\[
f((a_0, \alpha_{i-1})k) = (a_0, \alpha_{i-1}, a_{n-1}, k),
\]

\[
f^{-1}((a_0, \alpha_{i-1})k) = (a_0, \alpha_{i-1}, a_{n-1}, (k - 1) \mod n).
\]

Hence, the \( g \)-edges, \((u, g(u))\) or \((u, g^{-1}(u))\), and the \( f \)-edges, \((u, f(u))\) or \((u, f^{-1}(u))\), for \( u \in V(BF_n) \), represent the straight edges and cross edges, respectively. Consequently, we have Lemma 1:

**Lemma 1.** \( f^{-1}(g(u)) = g^{-1}(f(u)) \) for any vertex \( u \) in \( BF_n \).

2.2. \( g \)-Cycles and \( g \)-Edges

Let \( u \) be any vertex of \( BF_n \). We observe that \( g^n(u) = u \). Moreover, \((u \to g(u) \to g^2(u) \to \ldots \to g^n(u))\) forms a simple cycle \( C_g^n \) of length \( n \). We call such a cycle of \( BF_n \) a \( g \)-cycle at \( u \). Hence, \( C_g^n = C_g^m \) if and only if \( v \in V(C_g^m) \). As a result, all \( g \)-cycles form a partition of the straight edges of \( BF_n \). Meanwhile, any \( f \)-edge joins vertices from two different \( g \)-cycles. It can be seen that \((u, f(u))\) joins vertices from \( C_g^m \). Lemmas 2 and 3 were proved in [5].

**Lemma 2** [5]. \((g(u), g^{-1}(f(u)))\) is an \( f \)-edge joining vertices of \( C_g^m \) and \( C_g^m \). Moreover, the path \((u \to f(u) \to g^{-1}(f(u)) \to g(u)) \) forms a cycle of length 4.

Any \( C_g^m \) contains exactly one vertex at each level. In particular, \( C_g^0 \) contains exactly one vertex at level 0, say \((a_0, a_{n-1})\). We use \( C_g^{(a_0, \ldots, a_{n-1})} \) as the name for \( C_g^m \). Now, we form a new graph \( BF_n^G \) with all the \( g \)-cycles of \( BF_n \) as vertices, where two different \( g \)-cycles are joined with an edge if and only if there exists at least one \( f \)-edge in the graph.
joining them. The vertex of \( BF_n^G \) corresponding to \( C_g^r \) is denoted by \( C_g^{(u)} \). We recall the definition of the hypercube as follows: An \( n \)-dimensional hypercube (abbreviated to an \( n \)-cube) consists of \( 2^n \) vertices which are labeled with the \( 2^n \) binary numbers from 0 to \( 2^n - 1 \). Two vertices are connected by an edge if and only if their labels differ by exactly one bit.

Lemma 3 [5]. \( BF_n^G \) is isomorphic to an \( n \)-dimensional hypercube. Moreover, the set of vertices which are adjacent to \( C_g^{(au_{a_n})} \) is \( \{ C_g^{(au_{a_1}), \ldots, a_{a_n})}, \ldots, C_g^{(au_{a_n})} \} \).

Let \( h = (C_g^{(au_{a_1}), \ldots, a_{a_n})}, \ldots, C_g^{(au_{a_n})} \) be any edge of \( BF_n^G \). We use \( X(h) \) to denote the set of edges in \( BF_n \) joining vertices from \( C_g^{(au_{a_1})} \) and \( C_g^{(au_{a_n})} \). Using standard counting techniques, we have the following two corollaries:

Corollary 1 [3]. If \((u, f(u)) \in X(h), then (g(u), f^{-1}(g(u))) \in X(h) \) and \( |X(h)| = 2 \).

Corollary 2 [5]. There is a unique cycle \( C \) such that edges of \( BF_n \) joining vertices between \( C_g \) and \( C_g \) are exactly \( (u, f(u)) \) and \( (g(u), f^{-1}(g(u))) \), and in that case, \( C \) is isomorphic to \( C_g^{(au_{a_n})} \).

According to Corollaries 1 and 2, any edge \( h = (C_g^{(au_{a_1}), \ldots, a_{a_n})}, \ldots, C_g^{(au_{a_n})} \) in \( BF_n^G \) induces a unique 4-cycle in \( BF_n \), with two \( f \)-edges and two \( g \)-edges. We use \( X(C_g^{(au_{a_1}), \ldots, a_{a_n})}, \ldots, C_g^{(au_{a_n})}) \) to denote the set of \( f \)-edges in this 4-cycle and \( X(C_g^{(au_{a_1}), \ldots, a_{a_n})}) \) to denote the set of \( g \)-edges in this cycle.

Lemma 4. Let \( T \) be any subtree of \( BF_n^G \) and let \( C_g^T \) be the subgraph of \( BF_n \) generated by

\[
\left( \bigcup_{C_g^T \in T} E(C_g^T) \bigcup \bigcup_{C_g^T \in T} X(C_g^T, C_g^T) \right) - \bigcup_{(C_g, C_g') \in T} X_g(C_g, C_g').
\]

Then, \( C_g^T \) is a cycle of length \( n \times |V(T)| \).

Proof. Let \( T \) be a subtree of \( BF_n^G \) and let \( (C_g^{(au_{a_n})}, \ldots, a_{a_n})}, \ldots, C_g^{(au_{a_n})} \) be an edge of \( T \). Hence, two cycles \( C_g \) and \( C_g \) in \( BF_n \) are joined by two \( f \)-edges \( X_g(C_g^T, C_g^T) \). It is easy to see that \( E(C_g^T) \cup E(C_g^T) \cup X_g(C_g^T, C_g^T) - X_g(C_g^T, C_g^T) \) forms a cycle of length \( 2n \) in \( BF_n \). Therefore, the cycle \( C_g^T \) of \( BF_n \) can be generated by

\[
\left( \bigcup_{C_g^T \in T} E(C_g^T) \bigcup \bigcup_{(C_g, C_g') \in T} X_g(C_g, C_g') \right) - \bigcup_{(C_g, C_g') \in T} X_g(C_g, C_g').
\]

At the same time, the length of \( C_g^T \) is \( n \times |V(T)| \). ■

In Figure 2(a), we have another layout of \( BF_3 \). The graph \( BF_3^G \) is shown in Figure 2(b). For example, \( X(C_g^{000}), C_g^{001} \) = \{ (000, 0), (001, 2), (000, 2), (001, 0) \} and \( X_g(C_g^{000}), C_g^{001} \) = \{ (000, 0), (000, 2), (001, 0), (001, 2) \}. Let \( T \) be the tree indicated by bold lines in Figure 2(b). The corresponding \( C_g^T \) is indicated by bold lines in Figure 2(a).

2.3. f-Cycles and f-Edges

Let \( u = (a_0a_1 \ldots a_{n-1}, k) \) be any vertex of \( BF_n \) and let \( \bar{u} \) denote the vertex \( (\bar{a}_0\bar{a}_1 \ldots \bar{a}_{n-1}, k) \). One can see that \( f^0(u) = \bar{u} \) and \( f^{2n}(u) = u \). Moreover, \( (u \rightarrow f(u) \rightarrow f^2(u) \rightarrow \ldots \rightarrow f^{2n}(u) \) forms a simple cycle of length \( 2n \), denoted by \( C_f \). So, all \( f \)-cycles form a partition of the cross edges of \( BF_n \). Meanwhile, any \( g \)-edge joins vertices from two different \( f \)-cycles. Also, then, \( (u, g(u)) \) joins vertices from \( C_f^T \) and \( C_g^{(au_{a_n})} \). Lemmas 5 and 6 were proved in [5].
Lemma 5 [5]. \((f(u), g^{-1}(f(u))), (\bar{u}, g(\bar{u})), (f(\bar{u})), g^{-1}(f(\bar{u})))\) are \(g\)-edges joining vertices of \(C_f^u\) and \(C_f^{\bar{u}}\). Moreover, the paths \((u \rightarrow f(u) \rightarrow g^{-1}(f(u)) \rightarrow g(u) \rightarrow u)\) and \((\bar{u} \rightarrow f(\bar{u}) \rightarrow g^{-1}(f(\bar{u})) \rightarrow g(\bar{u}) \rightarrow \bar{u})\) form two 4-cycles in \(B_{F_n}\).

Any \(C_f^u\) contains exactly two vertices at each level. Suppose that \(u\) is one of the vertices in \(C_f^u\) at level \(i\). Then, the other vertex in \(C_f^u\) at level \(i\) is \(\bar{u}\). Thus, \(C_f^u\) contains exactly one vertex at level 0, say \((a_0a_1 \ldots a_{n-1}, 0)\), with \(a_{n-1} = 0\). We use \(C_f^{a_0a_1 \ldots a_{n-1}}\) as the name for \(C_f^u\). Now, we form a new graph \(B_{F_n}\) with all the \(f\)-cycles of \(B_{F_n}\) as vertices, where two different \(f\)-cycles are joined with an edge if and only if there exists a \(g\)-edge joining them. The vertex of \(B_{F_n}\) corresponding to \(C_f^u\) is denoted by \(\bar{C}_f^u\). We recall the definition of the folded hypercube as follows: An \(n\)-dimensional folded hypercube is basically an \(n\)-cube augmented with \(2^{n-1}\) complement edges. Each complement edge connects two vertices whose labels are complements to each other.

Lemma 6 [5]. \(B_{F_n}\) is isomorphic to the \((n - 1)\)-dimensional folded hypercube. Moreover, the set of vertices which are adjacent to \(C_f^{a_0a_1 \ldots a_{n-1}}\) is \(\{C_f^{a_0a_1 \ldots a_{n-1}} \times \{0, 1\}, C_f^{a_0a_1 \ldots a_{n-1}} \times \{1, 0\}, \ldots, C_f^{a_0a_1 \ldots a_{n-1}} \times \{0, 1\}\} \cup \{\bar{C}_f^{a_0a_1 \ldots a_{n-1}}\}\).

Let \(h = (\bar{C}_f^u, \bar{C}_f^v)\) be any edge of \(B_{F_n}\). We use \(Y(h)\) to denote the set of edges of \(B_{F_n}\) joining vertices from \(C_f^u\) and \(C_f^v\). Using standard counting techniques, we have the following two corollaries:

Corollary 3 [3]. If \((u, g(u)) \in Y(h)\), then \((f(u), g^{-1}(f(u))), (\bar{u}, g(\bar{u})), (f(\bar{u})), g^{-1}(f(\bar{u})))\) are in \(Y(h)\) and \(|Y(h)| = 4\). Moreover, \((u, f(u), g(u), g^{-1}(f(u)))\) and \((\bar{u}, f(\bar{u}), g(\bar{u}), g^{-1}(f(\bar{u})))\) induce two 4-cycles in \(B_{F_n}\).

Corollary 4 [5]. There is a unique cycle \(C\) such that edges of \(B_{F_n}\) joining vertices between \(C_f^u\) and \(C\) are exactly \((u, g(u)), (f(u), g^{-1}(f(u))), (\bar{u}, g(\bar{u})), (f(\bar{u})), g^{-1}(f(\bar{u})))\), and in that case, \(C\) is isomorphic to \(C_f^u\).

According to Corollaries 3 and 4, any edge \(h = (\bar{C}_f^u, \bar{C}_f^v)\) induces two 4-cycles in \(B_{F_n}\). Let \(\alpha\) be an assignment of \((\bar{C}_f^u, \bar{C}_f^v) \in E(B_{F_n})\) such that \(\alpha(h)\) is the subset of \(Y(h)\) induced by the 4-cycles of \(B_{F_n}\). We use \(Y^\alpha(C_f^u, C_f^v)\) to denote the set of \(f\)-edges induced by \(\alpha(h)\) and \(Y_g^\alpha(C_f^u, C_f^v)\) to denote the set of \(g\)-edges induced by \(\alpha(h)\). Hence, \(|Y^\alpha(C_f^u, C_f^v)| = |Y_g^\alpha(C_f^u, C_f^v)| = 2\).

Lemma 7. Let \(T\) be any subtree of \(B_{F_n}\) and let \(C_f^{\alpha, T}\) be the subgraph of \(B_{F_n}\) generated by

\[
\bigcup_{C_f^{\alpha, T} \in \text{subgraph of } B_{F_n}} E(C_f^{\alpha, T}) = \bigcup_{(C_f^{\alpha, T}) \in \text{subgraph of } B_{F_n}} Y^\alpha(C_f^{\alpha, T}, C_f^{\alpha, T}) - \bigcup_{(C_f^{\alpha, T}) \in \text{subgraph of } B_{F_n}} Y_g^\alpha(C_f^{\alpha, T}, C_f^{\alpha, T}).
\]

Then, \(C_f^{\alpha, T}\) is a cycle of \(B_{F_n}\) of length \(2n \times |V(T)|\).

Proof. Let \(T\) be a subtree of \(B_{F_n}\) and let \((\bar{C}_f^u, \bar{C}_f^v)\) be an edge of \(T\). Hence, \((\bar{C}_f^u, \bar{C}_f^v)\) induces two 4-cycles in \(B_{F_n}\). Let \(\alpha\) be an assignment of \((C_f^u, C_f^v) \in E(B_{F_n})\) such that \(\alpha((\bar{C}_f^u, \bar{C}_f^v))\) induces a 4-cycle in \(B_{F_n}\). Consequently, two cycles \(C_f^u\) and \(C_f^v\) are joined by two \(g\)-edges in \(Y^\alpha(C_f^u, C_f^v)\). It is easy to see that \(E(C_f^{\alpha, T}) \cup E(C_f^{\alpha, T}) \cup Y_g^\alpha(C_f^{\alpha, T}, C_f^{\alpha, T}) - Y_g^\alpha(C_f^{\alpha, T}, C_f^{\alpha, T})\) forms a cycle of length \(4n\) in \(B_{F_n}\). Therefore, the subgraph \(C_f^{\alpha, T}\) of \(B_{F_n}\) generated by

\[
\bigcup_{C_f^{\alpha, T} \in \text{subgraph of } B_{F_n}} E(C_f^{\alpha, T}) = \bigcup_{(C_f^{\alpha, T}) \in \text{subgraph of } B_{F_n}} Y^\alpha(C_f^{\alpha, T}, C_f^{\alpha, T}) - \bigcup_{(C_f^{\alpha, T}) \in \text{subgraph of } B_{F_n}} Y_g^\alpha(C_f^{\alpha, T}, C_f^{\alpha, T}).
\]

is a cycle. Moreover, the length of \(C_f^{\alpha, T}\) is \(2n \times |V(T)|\).

In Figure 3(a), we have another layout of \(B_{F_3}\). The graph \(B_{F_3}\) is shown in Figure 3(b). For example, \(Y^\alpha(C_f^{(000), C_f^{(100)}}) = \{(000, 0), (000, 1), (100, 0), (100, 1)\}\) and \(Y_g^\alpha(C_f^{(000), C_f^{(100)}}) = \{(000, 0), (100, 1), (100, 0), (000, 1)\}\). Let \(T\) be the tree indicated by bold lines in Figure 3(a). The corresponding \(C_f^{\alpha, T}\) is indicated by bold lines in Figure 3(a).

3. CYCLE EMBEDDING IN A FAULTY WRAPPED BUTTERFLY GRAPH

Let \(F\) denote a faulty set in a wrapped butterfly graph \(B_{F_n}\) and assume that \(F\) has at most two elements. Then, we will determine the maximum length of a cycle in the graph \(B_{F_n} - F\). In [3], Hwang and Chen showed that the length of the cycle is \(n2^n\) when \(F\) consists of two edges.

Lemma 8 [3]. For any integer \(n \geq 3\), \(B_{F_n} - F\) is Hamiltonian if \(F\) consists of two edges.

On the other hand, Vadapalli and Srimani [6] proposed the following lemma when a wrapped butterfly graph does not have any edge fault but has at most two vertex faults:

Lemma 9 [6]. For any integer \(n \geq 3\), \(B_{F_n} - F\) contains a cycle of length \(n2^n - 2\) if \(F\) consists of one vertex and a cycle of length \(n2^n - 4\) if \(F\) consists of two vertices.

In the remaining part of this section, we propose three lemmas: Lemma 10 proves that the cycle of length \(n2^n - 2\)
can be embedded in \( BF_n - F \) for any \( n \geq 3 \) when \( F \) consists of one vertex and one edge. Lemma 11 shows that the maximum cycle length is \( n2^n - 1 \) if \( n \) is odd when \( F \) consists of one vertex and one edge. Lemma 12 verifies that the maximum cycle length is \( 2n^2 - 2 \) for odd \( n \geq 3 \) when \( F \) consists of two vertices. To prove these three lemmas, we use three fundamental cycles, denoted by \( F_1, F_2, \) and \( F_3(j) \), respectively, to construct a larger cycle in a faulty wrapped butterfly graph.

The cycle \( B_1 \) shown in Figure 4(a) is constructed as follows: Let \( a_1 = \langle 00, \ldots, 0, 1 \rangle \) and \( P_1 \) be the path \( a_1 \rightarrow g(a_1) \rightarrow g^2(a_1) \rightarrow \ldots \rightarrow g^{n-2}(a_1) \). Hence, \( g^{n-2}(a_1) = a_2 = \langle 00, \ldots, 0, n - 1 \rangle \), \( f(a_2) = \langle 00, \ldots, 0, 1, 0 \rangle = a_3 \), and \( f(a_3) = \langle 100, \ldots, 1, 1, 0 \rangle = a_4 \). Let \( P_2 \) be the path \( a_4 \rightarrow g(a_4) \rightarrow g^2(a_4) \rightarrow \ldots \rightarrow g^{n-1}(a_4) \). Consequently, \( g^{n-1}(a_4) = a_5 = \langle 100, \ldots, 0, 0 \rangle \) and \( f^{-1}(a_5) = \langle 100, \ldots, 0, n - 1 \rangle \). Let \( P_3 \) be the path \( a_6 \rightarrow g^{-1}(a_6) \rightarrow g^{-2}(a_6) \rightarrow \ldots \rightarrow g^{-(n-1)}(a_6) \). Thus, \( g^{-(n-1)}(a_6) = a_7 = \langle 100, \ldots, 0, 0 \rangle \) and \( f(a_7) = a_1 \). Let \( B_1 \) be \( a_1 \rightarrow P_1 \rightarrow a_2 \rightarrow a_3 \rightarrow a_4 \rightarrow P_2 \rightarrow a_5 \rightarrow a_6 \rightarrow P_3 \rightarrow a_7 \rightarrow a_1 \).

Obviously, \( P_1 \) is a path in \( C^{a_1}_n \), \( P_2 \) is a path in \( C^{a_2}_n \), and \( P_3 \) is a path in \( C^{a_3}_n \). Since \( V(C^{a_u}_n) \cap V(C^{a_v}_n) = \emptyset \) for \( u \neq v \in \{a_1, a_4, a_6\} \) and \( n \geq 3 \), \( P_1, P_2, \) and \( P_3 \) are disjoint paths. Therefore, \( B_1 \) is a cycle of length \( 3n \).

Similarly, we can construct the basic cycle \( B_2 \) shown in Figure 4(b) as follows:

Let \( B_2 = \langle b_1 \rightarrow Q_1 \rightarrow b_2 \rightarrow b_3 \rightarrow Q_2 \rightarrow b_4 \rightarrow b_5 \rightarrow Q_3 \rightarrow b_6 \rightarrow b_7 \rightarrow Q_4 \rightarrow b_8 \rightarrow b_1 \rangle \), where

\[
\begin{align*}
b_1 &= \langle 00, \ldots, 0, 1 \rangle, & b_2 &= \langle 00, \ldots, 0, n - 1 \rangle, \\
b_3 &= \langle 00, \ldots, 0, 10, n - 2 \rangle.
\end{align*}
\]
are disjoint paths. Consequently, $P_3(j)$ is a cycle of length $2n + 4$. We even have $b_2 = b_4$ and $c_1 = c_6$ if and only if $n = 3.$ We have $V(P_2) \cap V(P_1) \neq \emptyset$ for $1 \leq l \neq k \leq 3$ and $n \geq 4$ because $P_1 = Q_1$ and $P_1 \cap R_1 \neq \emptyset$.

**Lemma 10.** For any integer $n \geq 3$, $BF_n - F$ contains a cycle of length $n2^n - 2$ if $F$ consists of one vertex and one edge.

**Proof.** Since $BF_n$ is vertex transitive, we assume that the faulty vertex is $x = (00\ldots0, 0)$ and the faulty edge is $e$.

**Case 1.** $e$ is a $g$-edge: Let $e = (u, g(u))$ for some $u \in V(BF_n)$. Since $n \geq 3$ and $BF_n$ is isomorphic to an $(n - 1)$-dimensional folded hypercube, $BF_n - \{C_g, C_g, C_g, C_{g(wo)}\}$ is connected. Let $T$ be any spanning tree of $BF_n - \{C_g, C_g, C_g(wo)\}$. By Lemma 7, $C_{g(wo)}$ for any $a$ forms an $n2^n - 2n$ cycle. Let $S_g = \{(y, g(y))|y = f^{2k}(x); 1 \leq k \leq n - 1\}$, $S_g = \{(y, g^{\nu}(f(y)))|y = f^{2k}(x); 1 \leq k \leq n - 1\}$, and $S_f = \{(g(y), f^{-1}(g(y)))|y = f^{2k}(x); 1 \leq k \leq n - 1\}$. Accordingly, $S_f \subseteq E(C_{g(x)})$ and $S_f \subseteq E(C_g(y))$.

**Case 1.1.** $e \notin S_g \cup S'_g$ (see Fig. 5): One can observe that $S_g \cup S'_g \cup S_f \cup S_f$ forms $n - 1$ disjoint $4$-cycles in $BF_n$. Meanwhile, $S'_f \subseteq E(C_{g(x)})$ and $S'_f \subseteq E(C_{g(y)})$. Hence,

\[(E(C_{g(x)}) \cup S_g \cup S'_g \cup S_f) - S'_f\]

forms a cycle in $BF_n - F$ and the cycle length is $n2^n - 2$.

**Case 1.2.** $e \in S_g \cup S'_g$: Constructing the cycle of length $n2^n - 2$ is similar to Case 1.1 except that $S_g = \{(y, g(y))|y = f^{2k}(x); 1 \leq k \leq n - 1\}$, $S'_g = \{(f(y), g^{-1}(f(y)))|y = f^{2k}(x); 1 \leq k \leq n - 1\}$, and $S_f = \{(y, f(y))|y = f^{2k}(x); 1 \leq k \leq n - 1\}$ and $S_f = \{(g(y), f^{-1}(g(y)))|y = f^{2k}(x); 1 \leq k \leq n - 1\}$. 

\[\text{FIG. 5. An illustration for Case 1.1 with } n = 3, S_g = \{(u_2, g(u_2)), (u_4, g(u_4))\}, \text{ and } S_f = \{(u_2, g^{-1}(u_2)), (u_4, g^{-1}(u_4))\}\]
CASE 2. e is an f-edge: Let $e = (u, f(u))$ for some $u \in V(BF_n)$.

CASE 2.1. $n$ is an even integer: Since $n \geq 3$ and $BF^G_n$ is isomorphic to the $n$-dimensional hypercube, $BF^G_n \setminus \{C^g, (C^u, C^w, \tilde{C}^u(w))\}$ is connected. Let $T$ be any spanning tree of $BF^G_n \setminus \{C^g, (C^u, C^w, \tilde{C}^u(w))\}$. By Lemma 4, $C^g$ forms a cycle spanning $V(BF^G_n) - V(C^g)$. Since $T$ does not contain the edge $(C^g, (C^u, \tilde{C}^u(w)))$ and $e \in X_g(C^u, C^w)$, $C^g$ does not contain the faulty edge $e$. Let $S = \{X_g(C^u, C^w(y))\} | y = g^{2k}(x); 1 \leq k < n/2\}. Hence, $S_1 \subset E(C^g)$ and $S_1 \subset \bigcup_{(u, v) \in S} X_g(C^u, C^w)$. Therefore,

$$\bigcup_{(u, v) \in S} X_g(C^u, C^w) \cup S \cup E(C^g) - S_1$$

forms a cycle of length $n2^n - 2$ in $BF_n - F$.

CASE 2.1.1. $e \notin S$: One can observe that $\bigcup_{(u, v) \in S} X_g(C^u, C^w) \cup S$ forms $(n/2) - 1$ disjoint 4-cycles in $BF_n - F$. Let $S_1 = \{(f(y), g^{-1}(f(y)))\} | y = g^{2k}(x); 1 \leq k < n/2\}. Hence, $S_1 \subset E(C^g)$ and $S_1 \subset \bigcup_{(u, v) \in S} X_g(C^u, C^w)$. Therefore,

$$\bigcup_{(u, v) \in S} X_g(C^u, C^w) \cup S \cup E(C^g) - S_1$$

forms a cycle of length $n2^n - 2$ in $BF_n - F$.

CASE 2.1.2. $e \in S$: The cycle can be constructed using the method of Case 2.1.1 except that $S = \{X_g(C^u, C^w(y))\} | y = g^{2k-1}(x); 1 \leq k < n/2\}.

CASE 2.2. $n$ is an odd integer: Let $S_1 = \{X_g(C^u, C^w(y))\} | y = g^{2k-1}(x); 1 \leq k \leq (n - 1)/2\}.

CASE 2.2.1. $e \notin S_1$ (see Fig. 6): Since $\tilde{C}^u \neq \tilde{C}^w$, we can choose $z \in \{u, f(u)\}$ according to the following rules:

1. If either $\tilde{C}^w = \tilde{C}^g$ or $\tilde{C}^w = \tilde{C}^r$, then we choose $z$ such that $\tilde{C}^g \neq \tilde{C}^r$.
2. Otherwise, we choose $z$ such that $(\tilde{C}^g, \tilde{C}^r) \notin E(BF^G_n)$.

Since $n \geq 3$ and $BF^G_n$ is isomorphic to an $n$-dimensional hypercube, $BF^G_n \setminus \{C^g, C^r\}$ is connected. Let $T$ be any spanning tree of $BF^G_n \setminus \{C^g, C^r\}$. By Lemma 4, $C^g$ is a cycle spanning $V(BF^G_n) - V(C^g) - V(C^r)$ and it does not contain $e$. Let $S_3 = \{X_g(C^u, C^w(y))\} | y = g^{2k-1}(z); 1 \leq k \leq (n - 1)/2\}. Hence, $S_1 \cap S_2 = \emptyset$, and $S_1 \cup S_2$ is fault free. We observe that $\bigcup_{(u, v) \in S} X_g(C^u, C^w) \cup S_3$ forms $(n - 1)/2$ disjoint 4-cycles and $\bigcup_{(u, v) \in S} X_g(C^u, C^w) \cup S_2$ forms $(n - 1)/2$ disjoint 4-cycles. Let $S_3 = \{(f(y), g^{-1}(f(y)))\} | y = g^{2k-1}(z); 1 \leq k \leq (n - 1)/2\}$. Hence, $S_3 \cap (S_1 \cup S_2) = \emptyset$, and $S_3 \cup S_1 \subset E(C^g)$. Meanwhile, $S_3 \cup S_4 \subset \bigcup_{(u, v) \in S} X_g(C^u, C^w)$. Therefore,

$$\bigcup_{(u, v) \in S} X_g(C^u, C^w) \cup S_1 \cup S_2 \cup E(C^g) - S_1 - S_4$$

forms a cycle in $BF_n - F$ and this cycle length is $n2^n - 2$.

CASE 2.2.2. $e \in S_1$ (see Fig. 7): Since $n \geq 3$, there exists $y \in V(BF_n)$ such that $f(y) \in V(C^g)$. So, both $y = f(y)$ and $(g(y), f^{-1}(g(y)))$ join vertices of $C^r$ and $C^w$. Let $W_1 = V(C^g) \cup V(C^w) \cup V(C^u) \cup V(C^r) \cup W_1 = \{C^g, C^w, C^u, C^r\}$. Hence, $C^g$ is not adjacent to any vertex in $W_1 = \{C^g\}$. Since $n \geq 3$ and $BF^G_n$ is isomorphic to an $n$-dimensional hypercube, $BF^G_n - W_1 - \{C^g\}$ is connected. Let $T$ be any spanning tree of $BF^G_n - W_1 - \{C^g\}$.

By Lemma 4, $C^G$ is a cycle spanning $V(BF_n) - W_1 - V(C^g)$. There exists $w \in V(BF_n)$ such that $X_g(C^u, C^w)$ is fault free and $(w, f(w))$ joins some vertex in both $B_1$ and $C^G$. Then,

$$C_w = (E(B_1) \cup E(C^g) \cup X_g(C^u, C^w)) - X_g(C^u, C^w)$$

forms a cycle length $n2^n - 2n$ spanning $(V(BF_n) - W_1 - V(C^g)) \cup V(B_1)$. Let $S_4 = \{X_g(C^u, C^w)(s) | s = g^{2k-1}(x); 1 \leq k \leq (n - 1)/2\}$. Then, $S_4 = \{X_g(C^u, C^w)(t) | t = g^{2k-1}(y); 1 \leq k \leq (n - 1)/2\}$.
\[ \leq (n - 1)/2. \] Hence, \( S_3 \cap S_4 = \emptyset \) and \( S_3 \cup S_4 \) is fault free. We observe that \( \bigcup_{(u,v) \in (S_3 \cup S_4)} X_h(C_{g^u}^n, C_{g^v}^n) \cup S_3 \cup S_4 \) forms \( n - 1 \) disjoint 4-cycles. Let \( S_5 = \{(f(s), \ g^{-1}(f(s)))| s = g^{2k-1}(a_3); 1 \leq k \leq (n - 1)/2 \} \) and \( S_6 = \{(f(t), \ g^{-1}(f(t)))| t = g^{2k-1}(y); 1 \leq k \leq (n - 1)/2 \} \). Hence, \( S_5 \cup S_6 = \emptyset \) and \( S_2 \cup S_6 \) is a subset of both \( E(C_e) \) and \( \bigcup_{(u,v) \in (S_2 \cup S_6)} X_h(C_{g^u}^n, C_{g^v}^n) \). Therefore,

\[
\bigcup_{(u,v) \in (S_2 \cup S_6)} X_h(C_{g^u}^n, C_{g^v}^n) \cup S_3 \cup S_4 \cup E(C_e) - S_5 - S_6
\]

forms a cycle of length \( n2^n - 2 \) in \( BF_n - F \).

**Lemma 11.** For any odd integer \( n \geq 3 \), \( BF_n - F \) is Hamiltonian if \( F \) consists of one vertex and one edge.

**Proof.** Since \( BF_n \) is vertex transitive, we may assume that the faulty vertex is \( x = \langle 00 \ldots 0, 0 \rangle \) and the faulty edge is \( e \). Let \( W_1 = V(C_{g^u}^n) \cup V(C_{g^v}^n) \cup V(C_{g^z}^n) \cup V(C_{g^e}^n) \) and \( \bar{W}_1 = \{C_{g^u}^n, C_{g^v}^n, C_{g^z}^n, C_{g^e}^n\} \).

**Case 1.** \( e \) is an \( f \)-edge: Let \( e = (u, f(u)) \) for some \( u \in V(BF_n) \).

**Case 1.1.** \( \{u, f(u)\} \cap (V(C_{g^u}^n) \cup V(C_{g^v}^n)) = \emptyset \) (see Fig. 8): Since \( n \geq 3 \) and \( BF_n \) is isomorphic to an \( n \)-dimensional hypercube, \( BF_n - W_1 = \{(C_{g^u}^n, C_{g^v}^n)\} \) is connected. Let \( T \) be any spanning tree of \( BF_n - W_1 = \{(C_{g^u}^n, C_{g^v}^n)\} \). By Lemma 4, \( C_{g^z}^n \) is a cycle spanning \( BF_n - V(C_{g^u}^n) \). Since \( S \) is fault free, \( \bigcup_{(u,v) \in S} X_h(C_{g^u}^n, C_{g^v}^n) \cup S \) forms \( n - 1/2 \) disjoint 4-cycles. Let \( S_1 = \{(f(y), g^{-1}(f(y)))| y = g^{2k-1}(a_3); 1 \leq k \leq (n - 1)/2 \} \). Consequently, \( S_1 \) is a subset of both \( \bigcup_{(u,v) \in S} X_h(C_{g^u}^n, C_{g^v}^n) \) and \( E(C_{g^z}^n) \). Therefore,

\[
C_e = \bigcup_{(u,v) \in S} X_h(C_{g^u}^n, C_{g^v}^n) \cup S \cup E(C_{g^z}^n) - S_1
\]

forms a cycle of length \( 2n^2 - 3n - 1 \) spanning \( V(BF_n) - V(B_1) - \{x\} \) and it does not contain \( e \). Since the length of \( B_1 \) is \( 3n \), there exists \( z \in V(B_1) \) such that \( X_h(C_{g^u}^n, C_{g^z}^n) \) is fault free and \((z, f(z)) \) joins some vertex in both \( C_e \) and \( B_1 \). Therefore,

\[
(E(C_e) \cup E(B_1) \cup X_h(C_{g^u}^n, C_{g^z}^n)) - X_h(C_{g^u}^n, C_{g^e}^n)
\]

forms a Hamiltonian cycle of \( BF_n - F \).

**Case 1.2.** \( \{u, f(u)\} \cap (V(C_{g^u}^n) \cup V(C_{g^v}^n)) \neq \emptyset \) (see Fig. 9): Let \( S = \{X_h(C_{g^u}^n, C_{g^v}^n)| y = g^{2k-1}(x); 1 \leq k \leq (n - 1)/2 \} \). Hence, \( e \notin S \). Since \( n \geq 3 \) and \( BF_n \) is isomorphic to an \( n \)-dimensional hypercube, \( BF_n - \{C_{g^z}^n\} \) is connected. Let \( T \) be any spanning tree of \( BF_n - \{C_{g^z}^n\} \). By Lemma 4, \( C_{g^z}^n \) is a cycle spanning \( BF_n - V(C_{g^u}^n) \). Since \( S \) is fault free, \( \bigcup_{(u,v) \in S} X_h(C_{g^u}^n, C_{g^v}^n) \cup S \) forms \( n - 1/2 \) disjoint 4-cycles. Let \( S_1 = \{(f(y), g^{-1}(f(y)))| y = g^{2k-1}(x); 1 \leq k \leq (n - 1)/2 \} \). Hence, \( S_1 \) is a subset of both \( \bigcup_{(u,v) \in S} X_h(C_{g^u}^n, C_{g^v}^n) \) and \( E(C_{g^z}^n) \). Therefore,

\[
\bigcup_{(u,v) \in S} X_h(C_{g^u}^n, C_{g^v}^n) \cup S \cup E(C_{g^z}^n) - S_1
\]

forms a Hamiltonian cycle of \( BF_n - F \).

**Case 2.** \( e \) is a \( g \)-edge: Let \( e = (u, g(u)) \) for some \( u \in V(BF_n) \).

**Case 2.1.** \( \tilde{C}_g^u = \tilde{C}_g^v \): Since \( n \geq 3 \) and \( BF_n \) is isomorphic to an \( n \)-dimensional hypercube, \( BF_n - \bar{W}_1 \) is connected. Let \( T \) be any spanning tree of \( BF_n - \bar{W}_1 \). Constructing a Hamiltonian cycle for this case is very similar to Case 1.1 except that the chosen vertex \( z \) is vertex \( u \) when \( e \in E(B_1) \).

**Case 2.2.** \( \tilde{C}_g^u \neq \tilde{C}_g^v \) and \( \tilde{C}_g^v \) is not connected with \( \tilde{C}_g^e \) in \( BF_n \). Since \( n \geq 3 \) and \( BF_n \) is isomorphic to an \( n \)-dimensional hypercube, \( BF_n - \{C_{g^z}^n\} \) is connected. Let \( T \) be a spanning tree of \( BF_n - \{C_{g^z}^n\} \) such that \( (\tilde{C}_g^u, \tilde{C}_g^v) \)
\( \epsilon \in E(T) \). Then, the Hamiltonian cycle of \( BF_n - \epsilon \) can be constructed by the same method used in Case 1.2.

**CASE 2.3.** \((C^u_g, \tilde{C}_g) \in E(BF^G_n)\) (i.e., \( \tilde{C}_g \) and \( C^u_g \) are connected):

**CASE 2.3.1.** \( \tilde{C}_g \neq \tilde{C}_g^{(1)} \) and \( \tilde{C}_g \neq \tilde{C}_g^{(2)} \) (see Fig. 10). Since \( n \geq 3 \) and \( BF^u_n \) is isomorphic to an \( n \)-dimensional hypercube, \( BF^u_n - \tilde{W}_1 - \{ C^u_g \} \) is connected. Let \( T \) be any spanning tree of \( BF^u_n - \tilde{W}_1 - \{ C^u_g \} \). By Lemma 4, \( C^u_g \) is a cycle spanning \( V(BF^u_n) - \tilde{W}_1 - V(C^u_g) \). Let \( S = \{ X(C^u_g \cap C^u_g) \} \). Then, \( S \) forms \( n - 1 \) 2 disjoint 4-cycles. Let \( S_i = \{ (f(y), g^{-1}(f(y))) \} \). By Lemma 4, \( C^u_g \) is a cycle spanning \( V(BF^u_n) - \tilde{W}_2 \). Let \( S \) be any spanning tree of \( BF^u_n - \tilde{W}_2 \). Then, \( S \) forms \( n - 1 \) 2 disjoint 4-cycles. Let \( S_i = \{ (f(y), g^{-1}(f(y))) \} \). By Lemma 4, \( C^u_g \) is a cycle spanning \( V(BF^u_n) - \{ C^u_g \} \) and it does not contain \( \epsilon \). Since the length of \( \tilde{W}_2 \) is 3, we can choose a vertex \( z \in V(C^u_g) \) such that \( f(z) \in V(C^u_g) \). Therefore, \( (E(C^u_g) \cup E(\tilde{W}) \cup E(C^u_g) \cup X_1(C^u_g, C^u_g) \cup X_2(C^u_g, C^u_g)) \)

forms a Hamiltonian cycle of \( BF_n - \epsilon \).

**CASE 2.3.2.** \( \tilde{C}_g = \tilde{C}_g^{(1)} \) (see Fig. 11): Let \( S = \{ X(C^u_g, C^u_g) \} \). Assume that \( \epsilon \not\in S \). Since \( n \geq 3 \) and \( BF^u_n \) is isomorphic to an \( n \)-dimensional hypercube, \( BF^u_n - \tilde{W}_1 \) is connected. Let \( T \) be any spanning tree of \( BF^u_n - \tilde{W}_1 \). Then, the Hamiltonian cycle of \( BF_n - \epsilon \) can be constructed by the same method used in Case 1.1. Assume that \( \epsilon \in S \). Let \( W_2 = V(C^u_g) \cup V(C^u_g) \cup V(C^u_g) \cup V(C^u_g) \). Let \( W_2 = \{ C^u_g, \tilde{C}_g, C^u_g, C^u_g \} \). Since \( n \geq 3 \) and \( BF^u_n \) is isomorphic to an \( n \)-dimensional hypercube, \( BF^u_n - \tilde{W}_2 \) is connected. Let \( T \) be any spanning tree of \( BF^u_n - \tilde{W}_2 \). Then, \( S \) forms a cycle spanning \( V(BF^u_n) - \{ C^u_g \} \) and it does not contain \( \epsilon \). Since the length of \( \tilde{W}_2 \) is 3, we can choose a vertex \( z \in V(C^u_g) \) such that \( f(z) \in V(C^u_g) \). Therefore, \( (E(C^u_g) \cup E(\tilde{W}) \cup E(C^u_g) \cup X_1(C^u_g, C^u_g) \cup X_2(C^u_g, C^u_g)) \)

forms a Hamiltonian cycle of \( BF_n - \epsilon \).
isomorphic to an n-dimensional hypercube, $B^G_n - 1$ is connected. Let $T$ be any spanning tree of $B^G_n - 1$ and $e = (u, v(u))$ for some $u \in V(B^G_n)$. Constructing a Hamiltonian cycle for this case is very similar to Case 1 except that the chosen vertex $z$ is vertex $u$ when $e \in E(B_1)$. ■

Lemma 12. For any odd integer $n \geq 3$, $B^G_n - F$ is Hamiltonian if $F$ consists of two vertices.

Proof. Since $B^G_n$ is vertex transitive, we may assume that one faulty vertex is $x = (00 \ldots 0, 0)$ and the other is $y$. Let $\overline{B}_3 = \{C^{(1)}_g, C^{(2)}_g, C^{(3)}_g, C^{(4)}_g\}.$

Case 1. $\overline{C}_g = \overline{C}_g$ (see Fig. 12): Let $y = (00 \ldots 0, j)$ for some $1 \leq j \leq n - 1$. Since $n$ is an odd integer, without loss of generality, we may assume that $j$ is an even integer. Since $n \geq 3$ and $B^G_n$ is isomorphic to an $n$-dimensional hypercube, $B^G_n - 1$ is connected. Let $T$ be any spanning tree of $B^G_n - 1$. By Lemma 4, $C^G_m$ is a cycle spanning $V(B^G_n) - 1$. Let

$$S_1 = \left\{ X(C^G_{m}, C^{(j)}_g) \mid y = g^{2k-1}(c_2); 1 \leq k \leq \frac{n - j - 1}{2} \right\},$$

$$S_2 = \left\{ X(C^G_{m}, C^{(j)}_g) \mid y = g^{2k-1}(c_2); 1 \leq k \leq \frac{n - j - 1}{2} \right\},$$

$$S_3 = \left\{ X(C^G_{m}, C^{(j)}_g) \mid y = g^{2k-1}(c_2); 1 \leq k \leq \frac{j}{2} - 1 \right\},$$

$$S_4 = \left\{ X(C^G_{m}, C^{(j)}_g) \mid y = g^{2k-1}(c_2); 1 \leq k \leq \frac{j}{2} - 1 \right\}.$$

One can observe that $S_1 \cap S_2 = \emptyset$ for $1 \leq i \neq j \leq 4$ and $\cup_{(u,v) \in S_1 \cup S_2 \cup S_3 \cup S_4} X_u(C^G_{m}, C^{(j)}_g) \cup S_1 \cup S_2 \cup S_3 \cup S_4$ forms $n - 3$ disjoint 4-cycles. Let

$$S_5 = \left\{ (f(y), g^{-1}(f(y))) \mid y = g^{2k}(c_2); 1 \leq k \leq \frac{n - j - 1}{2} \right\},$$

$$S_6 = \left\{ (f(y), g^{-1}(f(y))) \mid y = g^{2k-1}(c_2); 1 \leq k \leq \frac{n - j - 1}{2} \right\},$$

$$S_7 = \left\{ (f(y), g^{-1}(f(y))) \mid y = g^{2k-1}(c_2); 1 \leq k \leq \frac{j}{2} - 1 \right\},$$

$$S_8 = \left\{ (f(y), g^{-1}(f(y))) \mid y = g^{2k-1}(c_2); 1 \leq k \leq \frac{j}{2} - 1 \right\}.$$

Consequently, $S_5 \cap S_6 \cap S_7 \cap S_8 = \emptyset$, and $S_5 \cup S_6 \cup S_7 \cup S_8$ is a subset of both $E(C^G_{m})$ and $\cup_{(u,v) \in S_1 \cup S_2 \cup S_3 \cup S_4} X_u(C^G_{m}, C^{(j)}_g).$ Therefore,

$$C_\epsilon = \cup_{(u,v) \in S_1 \cup S_2 \cup S_3 \cup S_4} X_u(C^G_{m}, C^{(j)}_g) \cup S_1 \cup S_2 \cup S_3 \cup S_4 \cup E(C^G_{m}) - S_5 - S_6 - S_7 - S_8$$

forms a cycle of length $n2^n - 2n - 6$ spanning $V(B^G_n) - V(B_3(j)) - \{x, y\}$. Since the length of $B_3(j)$ is $2n + 4$, there exists $z \in V(B_3(j))$ such that $X_z(C^G_{m}, C^{(j)}_g)$ is a fault-free vertex and $(z, f(z))$ joins some vertex in both $C_\epsilon$ and $B_3(j)$. Then,

$$E(C^G_{x}) \cup E(B_3(j)) \cup X_u(C^G_{m}, C^{(j)}_g) \cup S_1 \cup S_2 \cup S_3 \cup S_4 \cup E(C^G_{m}) - S_5 - S_6 - S_7 - S_8$$

forms a Hamiltonian cycle of $B^G_n - F$.

Case 2. $\overline{C}_g \neq \overline{C}_g$ and $\overline{C}_g$ is not connected with $\overline{C}_g$ in $B^G_n$. (See Fig. 13): Since $n \geq 3$ and $B^G_n$ is isomorphic to an $n$-dimensional hypercube, $B^G_n - \{\overline{C}_g, \overline{C}_g\}$ is connected. Let $T$ be any spanning tree of $B^G_n - \{\overline{C}_g, \overline{C}_g\}$. By Lemma 4, $C^G_m$ is a cycle spanning $V(B^G_n) - V(C^G_{m}) - V(C^G_{m})$. Let $S_1 = \{X_u(C^G_{m}, C^{(j)}_g) \mid y = g^{2k-1}(x); 1 \leq k \leq \frac{n - 1}{2} \}$ and $S_2 = \{X_u(C^G_{m}, C^{(j)}_g) \mid y = g^{2k-1}(x); 1 \leq k \leq \frac{n - 1}{2} \}$. Since $S_1 \cap S_2 = \emptyset$ and $S_1 \cup S_2$ is fault free, $\cup_{(u,v) \in S_1 \cup S_2 \cup S_3 \cup S_4} X_u(C^G_{m}, C^{(j)}_g) \cup S_1 \cup S_2 \cup S_3 \cup S_4$ forms $n - 1$ disjoint 4-cycles. Let

$$S_3 = \{ (f(s), g^{-1}(f(s))) \mid y = g^{2k-1}(x); 1 \leq k \leq \frac{n - 1}{2} \} \text{ and } S_4 = \{ (f(t), g^{-1}(f(t))) \mid y = g^{2k-1}(y); 1 \leq k \leq \frac{n - 1}{2} \}.$$
forms a Hamiltonian cycle of $BF_n - F$.

**Case 3.** $(\tilde{C}_g^x, \tilde{C}_g^y) \in E(BF_n^G)$ (i.e., $\tilde{C}_g^x$ and $\tilde{C}_g^y$ are connected):

**Case 3.1.** $\tilde{C}_g^y \neq \tilde{C}_g^{x_i}$ and $\tilde{C}_g^y \neq \tilde{C}_g^{x_j}$ (see Fig. 7): Since $BF_n^G$ is an $n$-dimensional hypercube with $n \geq 3$, $\tilde{C}_g^x$, $\tilde{C}_g^y$, and $C_g^y$ are fault free. As a result, $\tilde{C}_g^x$ is not adjacent to $\tilde{C}_g^x$ in $BF_n^G$. Since $BF_n^G - W_1 - \{C_g^x\}$ is connected, let $T$ be any spanning tree of $BF_n^G - W_1 - \{C_g^x\}$. The Hamiltonian cycle of the graph $BF_n - F$ can be constructed by the same method used in Case 2.2.2 of Lemma 10.

**Case 3.2.** $\tilde{C}_g^y = \tilde{C}_g^{x_i}$ or $\tilde{C}_g^y = \tilde{C}_g^{x_j}$. Let $S_1 = \{\langle0, \ldots, 0, 1, 2k \rangle | 1 \leq k \leq (n-1)/2\}$ and $S_2 = \{\langle0, \ldots, 0, 2k\rangle | 1 \leq k \leq (n-1)/2\}$. Therefore, there does not exist any edge in $X_i(C_g^x, C_g^{\alpha(i)})$ which joins vertices of $C_g^x$ and $C_g^{\alpha(i)}$ for all $i \in S_1 \cup S_2$.

**Case 3.2.1.** $y \notin S_1 \cup S_2$ (See Fig. 13): Since $n \geq 3$ and $BF_n^G$ is isomorphic to an $n$-dimensional hypercube, $BF_n^G - \{C_g^x, \tilde{C}_g^y\}$ is connected. Let $T$ be any spanning tree of $BF_n^G - \{C_g^x, \tilde{C}_g^y\}$. With the same method used in Case 2, the Hamiltonian cycle of the graph $BF_n - F$ can be constructed.

**Case 3.2.2.** $y \in S_1 \cup S_2$:

Given an integer $k$ with $0 \leq k < n$, the mapping $\sigma_k$ from $V(BF_n)$ into $V(BF_n)$ can be defined by $\sigma_k((a_0a_1 \ldots a_{n-1}, l)) = (a_0a_1 \ldots a_{k+1} \ldots a_{n-1}a_0a_1 \ldots a_{k-1}, (l - k) \mod n))$. Similarly, we can define the mapping $\varphi_c$ from $V(BF_n)$ into $V(BF_n)$ as $\varphi_c((a_0a_1 \ldots, a_0a_{i+1} \ldots, a_{n-1}, l)) = (a_0a_1 \ldots a_{i+1} \ldots a_{n-1}, l)$, where $0 \leq i < n$. Hence, $\sigma_k$ and $\varphi_c$ are two automorphisms of $BF_n$.

Suppose that $y \in S_1$. Then, $y = \langle0, \ldots, 0, 1, l\rangle$ for some odd $l$, $1 \leq l \leq n - 2$. Accordingly, $\varphi_{n-1-l} \circ \sigma_l$ is an automorphism of $BF_n$ such that $\varphi_{n-1-l} \circ \sigma_l(y) = x$ and $\varphi_{n-1-l} \circ \sigma_l(z) = \langle0, \ldots, 0, 1, 00, \ldots, 0, n - l\rangle$. Consequently, $z \notin S_1 \cup S_2$. Therefore, we can construct a Hamiltonian cycle $C$ in $BF_n - \{x, z\}$ by using the same method as in Case 3.2.1. Finally, $(\varphi_{n-1-l} \circ \sigma_l)^{-1}(C)$ also forms a Hamiltonian cycle of $BF_n - \{x, y\}$.

Suppose that $y \in S_2$. Then, $y = \langle1, 00, \ldots, 0, l\rangle$ for some even $l$, $2 \leq l \leq n - 1$. Hence, $\varphi_{n-1-l} \circ \sigma_l$ is an automorphism of $BF_n$ such that $\varphi_{n-1-l} \circ \sigma_l(y) = x$ and $\varphi_{n-1-l} \circ \sigma_l(z) = \langle0, \ldots, 0, 1, 100, \ldots, 0, n - l\rangle$. Consequently, $z \notin S_1 \cup S_2$. With the same method used in Case 3.2.1, we can construct a Hamiltonian cycle $C$ in $BF_n - \{x, z\}$. Hence, $(\varphi_{n-1-l} \circ \sigma_l)^{-1}(C)$ also forms a Hamiltonian cycle of $BF_n - \{x, y\}$.

Combining Lemmas 8–12, we have the following theorem:

**Theorem 1.** For any integer $n$ with $n \geq 3$, let $F \subseteq V(BF_n) \cup E(BF_n)$, $f_w = |F \cap V(BF_n)|$, and $|F| \equiv 2$. $BF_n - F$ contains a cycle of length $n \times 2^n - 2f_w$. In addition, $BF_n - F$ contains a Hamiltonian cycle if $n$ is an odd integer.

**4. CONCLUSIONS**

It is known that wrapped butterfly graphs are very suitable for VLSI implementation because they are regular and have a constant degree. In this paper, the properties of wrapped butterfly graphs are described and investigated. Unlike the previous studies [3, 6] which considered faults on either vertices or edges, this paper showed that there exists a cycle of length $n2^n - 2$ in the faulty wrapped butterfly graph when there is one vertex and one edge fault. When $BF_n$ is not a bipartite graph, we show that it contains a Hamiltonian cycle if it has at most two faults containing at least one vertex fault. These results are optimal because wrapped butterfly graphs are bipartite if and only if $n$ is even. We say two vertices $u$ and $v$ have the same color, black or white, if and only if $u$ and $v$ are in the same partite set. When a wrapped butterfly graph $BF_n$ is a bipartite graph, we have two conjectures:

1. $BF_n - F$ contains a Hamiltonian cycle if $F$ consists of two vertices with different colors.
2. $BF_n - F$ is Hamiltonian if $F$ consists of two black colored vertices and two white colored vertices.
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REFERENCES