On some super fault-tolerant Hamiltonian graphs

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Abstract

A k-regular Hamiltonian and Hamiltonian connected graph G is super fault-tolerant Hamiltonian if G remains Hamiltonian after removing at most k – 2 nodes and/or edges and remains Hamiltonian connected after removing at most k – 3 nodes and/or edges. A super fault-tolerant Hamiltonian graph has a certain optimal flavor with respect to the fault-tolerant Hamiltonicity and Hamiltonian connectivity. In this paper, we investigate a construction scheme to construct super fault-tolerant Hamiltonian graphs. In particular, twisted-cubes, crossed-cubes, and Möbius cubes are all special cases of this construction scheme. Therefore, they are all super fault-tolerant Hamiltonian graphs.

Keywords: k-regular; k-Hamiltonian; k-Hamiltonian connected; Fault-tolerant; Super fault-tolerant Hamiltonian

1. Introduction

For the graph definitions and notations we follow [4]. G = (V, E) is a graph if V is a finite set and E is a subset of \{ (a, b) | (a, b) is an unordered pair of V \}. We say that V is the node set and E is the edge set. The degree of a node v, denoted by \text{deg}(v), is the number of edges incident to v. A graph G is k-regular if \text{deg}(v) = k for every node in G. Two nodes a and b are adjacent if \( (a, b) \in E \).
A path is a sequence of adjacent edges \((v_0, v_1), (v_1, v_2), \ldots, (v_{m-1}, v_m)\), written as \(v_0v_1v_2\ldots v_m\), in which all the nodes \(v_0, v_1, \ldots, v_m\) are distinct. We also write the path \(v_0, v_1, v_2, \ldots, v_m\) as \(P(v_0, v_1, v_2, \ldots, v_m)\) where \(P(v_0, v_i) = (v_0, v_1, \ldots, v_i)\) and \(P(v_j, v_i) = (v_j, v_{j+1}, \ldots, v_i)\). For our purpose in this paper, a path may contain only one node. A path is a Hamiltonian path if its nodes are distinct and they span \(V\). A cycle is a path with at least three nodes such that the first node is the same as the last one. A cycle is a Hamiltonian cycle if it traverses every node of \(G\) exactly once. A graph \(G\) is Hamiltonian if it has a Hamiltonian cycle, and \(G\) is Hamiltonian connected if there exists a Hamiltonian path joining any two nodes of \(G\).

The architecture of an interconnection network is usually represented by a graph. There are a lot of mutually conflicting requirements in designing the topology of interconnection networks. It is almost impossible to design a network which is optimum for all conditions. One has to design a suitable network depending on the requirements of their properties. The Hamiltonian property is one of the major requirements in designing the topology of networks. Fault tolerance is also desirable in massive parallel systems that have relatively high probability of failure. There are many researches on the ring embedding problems in faulty interconnection networks [2,10,12,14,15].

Since node faults and edge faults may happen when a network is used, it is practically meaningful to consider faulty networks. A graph \(G\) is called \(l\)-fault-tolerant Hamiltonian (\(l\)-fault-tolerant Hamiltonian connected respectively) or simply \(l\)-Hamiltonian (\(l\)-Hamiltonian connected respectively) if it remains Hamiltonian (Hamiltonian connected respectively), after removing at most \(l\) nodes and/or edges. The fault-tolerant Hamiltonicity, \(\mathcal{H}_l(G)\), is defined to be the maximum integer \(l\) such that \(G - F\) remains Hamiltonian for every \(F \subset V(G) \cup E(G)\) with \(|F| \leq l\) if \(G\) is Hamiltonian, and undefined if otherwise. Obviously, \(\mathcal{H}_l(G) \leq \delta(G) - 2\), where \(\delta(G) = \min\{\deg(v) : v \in V(G)\}\). A regular graph \(G\) is optimal fault-tolerant Hamiltonian if \(\mathcal{H}_l(G) = \delta(G) - 2\). Twisted-cubes, crossed-cubes, and Möbius cubes are proved to be optimal fault-tolerant Hamiltonian [7–9]. All these families of graphs have some good properties in common, including that they can all be recursively constructed. In establishing their fault-tolerant Hamiltonicity, another parameter called fault-tolerant Hamiltonian connectivity is used. The fault-tolerant Hamiltonian connectivity, \(\mathcal{H}_l^c(G)\), is defined to be the maximum integer \(l\) such that \(G - F\) remains Hamiltonian connected for every \(F \subset V(G) \cup E(G)\) with \(|F| \leq l\) if \(G\) is Hamiltonian connected, and undefined if otherwise. Obviously, \(\mathcal{H}_l^c(G) \leq \delta(G) - 3\). A regular graph \(G\) is optimal fault-tolerant Hamiltonian if \(\mathcal{H}_l^c(G) = \delta(G) - 3\). Again, twisted-cubes, crossed-cubes, and Möbius cubes are proved to be optimal fault-tolerant Hamiltonian connected [7–9]. We call those regular graphs super fault-tolerant Hamiltonian if \(\mathcal{H}_l^c(G) = \delta(G) - 2\) and \(\mathcal{H}_l^c(G) = \delta(G) - 3\).
All the proofs of super fault-tolerant Hamiltonicity of the aforementioned families of graphs are done by induction. We observe that there are certain common phenomena behind the recursive structures so that we may construct other super fault-tolerant Hamiltonian graphs. In this paper, we try to investigate these phenomena and establish some construction schemes of super fault-tolerant Hamiltonian graphs.

The rest of this article is organized as follows. In Section 2, some terminologies and notations are introduced. Section 3 describes four lemmas which we shall use in our main results. The main results are proved in Section 4. Finally, the conclusion is given in Section 5.

2. Terminology and notation

Before introducing the terminologies and notations, let us briefly explain our motivation. The hypercube is a popular network because of its attractive properties, including regularity, symmetry, small diameter, strong connectivity, recursive construction, partitionability, and relatively low link complexity [3,11,13]. There are some variations of the hypercube appearing in literature; such as twisted-cubes, crossed-cubes, Möbius cubes, and so on. These variations preserve most of the good topological properties of the hypercube, and even better. For example, the diameter of these variation cubes is around half of that of the hypercube. Recently, twisted-cubes, crossed-cubes, and Möbius cubes are proved to be super fault-tolerant Hamiltonian graphs. We note, however, the hypercube is a bipartite graph, and therefore its fault-tolerant Hamiltonicity is zero. Basically, all those variations of the hypercube, including the hypercube itself, are recursively constructed in a very similar way as follows: An \( n \)-dimension cube \( Q_n = (V,E) \) is constructed from two identical \( (n-1) \)-dimension cubes, \( Q_{n-1}^0 \) and \( Q_{n-1}^1 \). The node set is \( V(Q_n) = V(Q_{n-1}^0) \cup V(Q_{n-1}^1) \), and the edge set is \( E(Q_n) = E(Q_{n-1}^0) \cup E(Q_{n-1}^1) \cup M \) where \( M \) is a set of edges connecting the nodes of \( Q_{n-1}^0 \) and \( Q_{n-1}^1 \) in a one to one fashion.

For different lower dimensional cube \( Q_{n-1} \) and different matching \( M \), we obtain different higher dimensional cubes \( Q_n \), such as twisted-cubes, crossed-cubes, Möbius cubes, and hypercubes, with variant fault-tolerant Hamiltonicity. This motivates us to study some construction schemes of super fault-tolerant Hamiltonian graphs, and it leads to the following definition. Let \( G_1 \) and \( G_2 \) be two graphs with the same number of nodes. Let \( M \) be an arbitrary perfect matching between the nodes of \( G_1 \) and \( G_2 \); i.e., \( M \) is a set of edges connecting the nodes of \( G_1 \) and \( G_2 \) in a one to one fashion. For convenience, \( G_1 \) and \( G_2 \) are called components. Formally, we define graph \( G(G_1,G_2;M) \), which has node set \( V(G(G_1,G_2;M)) = V(G_1) \cup V(G_2) \), and edge set \( E(G(G_1,G_2;M)) = E(G_1) \cup E(G_2) \cup M \). See Fig. 1. What we have in mind...
is the following: Let $G_1$ and $G_2$ be two $k$-regular super fault-tolerant Hamiltonian graphs with the same number of nodes, and let $M$ be an arbitrary perfect matching. Then $G(G_1, G_2; M)$ is $(k+1)$-regular. The degree of $G(G_1, G_2; M)$, as compared with $G_1$ and $G_2$, is increased by 1. We expect that its fault-tolerant Hamiltonicity $\mathcal{H}_f(G)$ and fault-tolerant Hamiltonian connectivity $H^*_f(G)$ are also increased by 1. This is indeed the case under the condition that $k \geq 5$. Then $G(G_1, G_2; M)$ is also a super fault-tolerant Hamiltonian graph.

For ease of exposition, we make some convention about our notations. Consider the graph $G(G_1, G_2; M)$. For each component $G_i$, we use small letters with subscript $i$ to denote the nodes in $G_i$, e.g., $u_i, v_i$, etc. Thus, $u_1$ is a node in $G_1$, and $u_2$ is a node in $G_2$. A perfect matching $M$ connecting the nodes of $G_1$ and $G_2$ in pairs, such pairs of nodes are called matching nodes, and these edges are called matching edges. We shall use the same letter with different subscripts to denote matching nodes of each other; e.g., $u_1$ and $u_2$ are the matching nodes of each other in components $G_1$ and $G_2$.

We need some more terms. We shall consider graphs with some faults. Our objective is to find a fault free Hamiltonian cycle (Hamiltonian path respectively). In this paper, each fault can be a faulty node or a faulty edge. If a node $v$ is not faulty, we say $v$ is a healthy node. We call an edge $e$ (respectively a matching edge $e$) healthy if both edge $e$ and its two endpoints are not faulty. We use $F_i$ to denote the set of faults in $G_i$ for $i = 1, 2$. Let $f_i = |F_i|$ for $i = 1, 2$. Given two distinct healthy nodes $x$ and $y$, we use $x, y$-Hamiltonian path to call a fault free Hamiltonian path joining $x$ and $y$, $HP_i$ to denote a fault free Hamiltonian path in $G_i - F_i$ for $i = 1, 2$. A fault free $x, y$-Hamiltonian path in $G_i - F_i$ can be written as $\langle x, HP_i, y \rangle$ for $i = 1, 2$. In addition, path $\langle x, HP_i, y \rangle$ is a cycle if $x = y$.

3. Preliminaries

Consider an interconnection network $G$, and suppose that there are faults in it. Let $F_G$ be the set of faults in $G$, and $f_G$ be the number of faults in $G$. Suppose that $G$ is $k$-Hamiltonian ($k$-Hamiltonian connected respectively) and $f_G \leq k$. Let
Lemma 1. Let \( G \) be a \( k \)-Hamiltonian graph, \( F_G \) be a set of faults in \( G \) with \( |F_G| \leq k \), and \( u \) be a healthy node in \( G \). Then there are at least \( k - f_G + 2 \) edges incident to node \( u \), such that each one of them is on some Hamiltonian cycle in \( G - F_G \).

Proof. We know that \( G \) is \( k \)-Hamiltonian, and there are \( f_G \) faults in \( G \). Hence, \( G - F_G \) is still Hamiltonian even if we add \( k - f_G \) more faults to \( G - F_G \). Suppose \( f_G < k \). Let \( HC \) be a Hamiltonian cycle in \( G - F_G \), and let \( e \) be an edge on \( HC \) and incident to node \( u \). Deleting edge \( e \), \( G - F_G \) - \{\( e \)\} still contains a Hamiltonian cycle. Repeating this process \( k - f_G \) times, we find \( k - f_G + 2 \) edges incident to node \( u \) and each one of them is on some Hamiltonian cycle in \( G - F_G \). \( \square \)

Lemma 2. Let \( G \) be a \( k \)-Hamiltonian connected graph, \( F_G \) be a set of faults in \( G \) with \( |F_G| \leq k \), and \( \{x, y, u\} \) be three distinct healthy nodes in \( G \). Then there are at least \( k - f_G + 2 \) edges incident to node \( u \), such that each one of them is on some \( x, y \)-Hamiltonian path in \( G - F_G \).

Proof. It is known that \( G \) is \( k \)-Hamiltonian connected, and there are \( f_G \) faults in \( G \). Thus, \( G - F_G \) is still Hamiltonian connected even if we add \( k - f_G \) more faults to \( G - F_G \). Suppose \( f_G < k \). Let \( HP \) be an \( x, y \)-Hamiltonian path in \( G - F_G \), and let \( e \) be an edge on \( HP \) and incident to node \( u \). Deleting edge \( e \), \( G - F_G \) - \{\( e \)\} still contains an \( x, y \)-Hamiltonian path. Repeating this process \( k - f_G \) times, we find \( k - f_G + 2 \) edges incident to node \( u \) and each one of them is on some \( x, y \)-Hamiltonian path in \( G - F_G \). \( \square \)

Lemma 3. Let \( G_1 \) and \( G_2 \) be two \( k \)-regular graphs with the same number of nodes. If the total number of faults in \( G(G_1, G_2; M) \) is no greater than \( k \), there exists at least one healthy matching edge between \( G_1 \) and \( G_2 \).

Lemma 4. Let \( G_1 \) and \( G_2 \) be two \( k \)-regular graphs with the same number of nodes, and let \( x \) and \( y \) be two healthy nodes in \( G(G_1, G_2; M) \). If the total number of faults in \( G(G_1, G_2; M) \) is no greater than \( k - 2 \), there exists at least one healthy matching edge between \( G_1 \) and \( G_2 \) whose endpoints are neither \( x \) nor \( y \).
The above two lemmas result immediately from the fact that $|V(G_1)| = |V(G_2)| \geq k + 1$.

4. Super fault-tolerant Hamiltonian graphs

Let $G_1$ and $G_2$ be two $k$-regular super fault-tolerant Hamiltonian graphs. The following two theorems state that the fault-tolerant Hamiltonicity $H_f(G)$ and fault-tolerant Hamiltonian connectivity $H^*_f(G)$ of the graph $G(G_1, G_2; M)$, as compared with $G_1$ and $G_2$, are increased by 1. Hence, $G(G_1, G_2; M)$ is a super fault-tolerant Hamiltonian graph. We make one simple observation first.

Observation 1. To prove that a graph $G$ is $l$-Hamiltonian (respectively $l$-Hamiltonian connected), it suffices to show that $G - F_G$ is Hamiltonian (respectively Hamiltonian connected) for any faulty set $F_G \subseteq V(G) \cup E(G)$ with $|F_G| = l$. If the total number of faults $|F_G|$ is strictly less than $l$, we may arbitrarily designate $l - |F_G|$ healthy edges as faulty to make exactly $l$ faults.

Theorem 1. Assume $k \geq 4$. Let $G_1$ and $G_2$ be two $k$-regular super fault-tolerant Hamiltonian graphs and $|V(G_1)| = |V(G_2)|$. Then graph $G(G_1, G_2; M)$ is $(k - 1)$-Hamiltonian.

Proof. $G(G_1, G_2; M)$ is $(k + 1)$-regular. To prove that $G(G_1, G_2; M)$ is $(k - 1)$-Hamiltonian, it suffices to show that $G(G_1, G_2; M) - F_{\{1, \ldots, 2\}}$ is Hamiltonian for any faulty set $F_{\{1, \ldots, 2\}} \subseteq V(G) \cup E(G)$ with $|F_{\{1, \ldots, 2\}}| = k - 1$.

Case 1. All $k - 1$ faults are in the same component.

We may assume without loss of generality that all faults are in $G_1$. Since $G_1$ is $(k - 2)$-Hamiltonian and $f_1 = k - 1$, $G_1 - F_1$ has a Hamiltonian path $(u_1, HP, v_1)$. Let $(u_1, u_2)$ and $(v_1, v_2)$ be two matching edges between $G_1$ and $G_2$. In $G_2$, there exists a $u_2, v_2$-Hamiltonian path $(u_2, HP, v_2)$ since $f_2 = 0$ and $G_2$ is $(k - 3)$-Hamiltonian connected. Therefore, $(u_1, HP, v_1, v_2, HP, u_2, u_1)$ forms a fault free Hamiltonian cycle in this case. See Fig. 2.

Case 2. Not all $k - 1$ faults are in the same component.

Without loss of generality, we may assume that $f_2 \leq f_1 \leq k - 2$. In this case, we claim that $G_2 - F_2$ is Hamiltonian connected if $k \geq 4$. Suppose not, then

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**Fig. 2.** Case 1: All $k - 1$ faults are in the same component.
Let \( f_2 \geq k - 2, \) \( k - 2 \leq f_2 \leq f_1 \leq k - 2, \) so \( k - 2 = f_2 = f_1 = k - 2 \) and \( (k - 2) + (k - 2) \leq k - 1. \) Therefore, \( k \leq 3, \) which is a contradiction.

By Lemma 3, there exists a healthy matching edge between \( G_1 \) and \( G_2, \) say \((u_1, u_2).\) Claim that we can find a node \( v_1 \) incident to \( u_1 \) such that \((u_1, v_1)\) is on a Hamiltonian cycle in \( G_1 - F_1, \) and the matching edge \((v_1, v_2)\) incident to \( v_1 \) is healthy. Then, the case is proved since \( G_2 - F_2 \) is Hamiltonian connected.

Now, in \( G_1 - F_1, \) by Lemma 1, of all the healthy nodes incident to \( u_1, \) there are at least \((k - 2) - f_1 + 2 = k - f_1 \) edges which are on some Hamiltonian cycle in \( G_1 - F_1. \) Of all these \( k - f_1 \) edges, there is at least one edge, say \((u_1, v_1),\) such that \( v_1, v_2, \) and \((v_1, v_2)\) are healthy. Were it not true, \( G(G_1, G_2; M) \) would contain at least \( f_1 + (k - f_1) = k \) faults, contradicting the fact that the total number of faults is \( k - 1. \) Therefore, we have a fault free Hamiltonian cycle \((u_1, HP_1, v_1, v_2, HP_2, u_2, u_1)\) in this case. See Fig. 3. This completes the proof of this theorem. \( \square \)

The fault-tolerant Hamiltonian connectivity \( \mathcal{H}^k_f(G) \) of \( G(G_1, G_2; M) \) is also increased by 1, as stated in the following theorem.

**Theorem 2.** Assume \( k \geq 5. \) Let \( G_1 \) and \( G_2 \) be two \( k \)-regular super fault-tolerant Hamiltonian graphs and \( |V(G_1)| = |V(G_2)|. \) Then graph \( G(G_1, G_2; M) \) is \((k - 2)\)-Hamiltonian connected.

**Proof.** Let \( F_{(1, 2)} \) be a set of faults, \( F_{(1, 2)} \subset V(G) \cup E(G) \) and \( |F_{(1, 2)}| = k - 2. \) Let \( x \) and \( y \) be two healthy nodes in \( G(G_1, G_2; M), \) we shall find a fault free Hamiltonian path joining \( x \) and \( y. \) The proof is classified into two cases.

**Case 1.** \( x \) and \( y \) are not in the same component.

Without loss of generality, we may assume \( x \) is in \( G_1, \) and \( y \) is in \( G_2. \) This case can be further divided into two subcases.

**Subcase 1.** All \( k - 2 \) faults are in the same component.

Without loss of generality, we may assume that all \( k - 2 \) faults are in \( G_1. \) So there is a fault free Hamiltonian cycle in \( G_1 - F_1. \) On this fault free cycle, there are two nodes incident to \( x. \) One of these two nodes is not matched with \( y, \) say \( u_1. \) Now, we delete edge \((x, u_1)\) and add matching edge \((u_1, u_2)\) in \( G_2. \) There is a \( u_2, y \)-Hamiltonian path \( \langle u_2, HP_2, y \rangle \) because \( G_2 \) is Hamiltonian connected.

![Fig. 3. Case 2: Not all \( k - 1 \) faults are in the same component.](image-url)
Thus, \((x, HP_1, u_1, u_2, HP_2, y)\) forms a fault free \(x, y\)-Hamiltonian path in this subcase. See Fig. 4.

Subcase 1.2. Not all \(k - 2\) faults are in the same component.

Since the total number of faults \(f_{(1, 2)}\) is \(k - 2\) and not all faults are in the same component, we may assume that \(f_2 \leq f_1 < k - 3\). By Lemma 4, we can find a healthy matching edge \((u_1, u_2)\) between \(G_1\) and \(G_2\), where \(u_1 \neq x\) and \(u_2 \neq y\). Since \(f_2 \leq f_1 < k - 3\), \(G_1 - F_1\) and \(G_2 - F_2\) are Hamiltonian connected, there is one \(x, u_1\)-Hamiltonian path \((x, HP_1, u_1)\) in \(G_1 - F_1\) and one \(u_2, y\)-Hamiltonian path \((u_2, HP_2, y)\) in \(G_2 - F_2\). So \((x, HP_1, u_1, u_2, HP_2, y)\) is a fault free \(x, y\)-Hamiltonian path in this subcase. See Fig. 5.

Case 2. \(x\) and \(y\) are in the same component.

Without loss of generality, we may assume \(x\) and \(y\) are in \(G_1\). We shall divide this case into three subcases.

Subcase 2.1. All \(k - 2\) faults are in \(G_1\).

\(G_1\) is \((k - 3)\)-Hamiltonian connected and \(f_1 = k - 2\). Let \(g\) be a faulty edge or a faulty node. In \(G_1 - (F_1 - \{g\})\), there is a Hamiltonian path \((x, P(x, y), y)\) joining \(x\) and \(y\). Removing the fault \(g\), this Hamiltonian path is separated into two subpaths, say \((x, P(x, u_1), u_1)\) and \((u_1, P(v_1, y), y)\), which cover all the nodes of \(G_1 - F_1\). Then we add two matching edges \((u_1, u_2)\) and \((v_1, v_2)\). In \(G_2\), there exists a \(u_2, v_2\)-Hamiltonian path \((u_2, HP_2, v_2)\) since \(f_2 = 0\). Thus, we have a fault free \(x, y\)-Hamiltonian path \((x, P(x, u_1), u_1, u_2, HP_2, v_2, v_1, P(v_1, y), y)\) in this subcase. See Fig. 6.

Subcase 2.2. All \(k - 2\) faults are in \(G_2\).

Let \(x_2\) be the matching node of \(x\) in \(G_2\), and \(y_2\) be the matching node of \(y\) in \(G_2\). This subcase can be further divided into two subcases:

Subcase 2.2.1. At least one of \(x_2\) and \(y_2\) is healthy.

Without loss of generality, we may assume \(y_2\) is healthy. We add the matching edge \((y, y_2)\). In \(G_2 - F_2\), there exists a Hamiltonian cycle since
\[ f_2 = k - 2. \] On this fault free cycle, there are two nodes incident to node \( y_2 \). At least one of these two nodes is not adjacent to \( x \), say \( u_2 \). We then add the matching edge \( (u_1, u_2) \), and delete edge \( (u_2, y_2) \). In \( G_1 - \{ y \} \), we claim that there exists a fault free \( x, u_1 \)-Hamiltonian path \( \langle x, P(x, u_1), u_1 \rangle \). Suppose not, then \( k - 3 < 1 \), so \( k < 4 \). It is a contradiction. Therefore, we have a fault free \( x, y \)-Hamiltonian path \( \langle x, P(x, u_1), u_1, u_2, HP_2, y_2, y \rangle \) in this subcase. See Fig. 7.

**Subcase 2.2.2.** Both \( x_2 \) and \( y_2 \) are faulty.

In \( G_1 \), the number of healthy edges incident to \( y \) is \( k \) and \( f_2 = k - 2 \). We can find a healthy node \( u_1 \) incident to \( y \) such that \( u_1 \neq x \) and \( u_2 \) is healthy, where \( u_2 \) is the matching node of \( u_1 \) in \( G_2 \). In \( G_2 - F_2 \), there exists a Hamiltonian cycle since \( f_2 = k - 2 \). Let \( v_2 \) be a node on this cycle incident to \( u_2 \). Node \( v_2 \) is not matched with \( x \) and \( y \) since \( x_2 \) and \( y_2 \) are faulty in this subcase. Then we add the matching edge \( (v_1, v_2) \) and delete edge \( (u_2, v_2) \). In this subcase, we claim that \( G_1 - \{ u_1, y_2 \} \) has a fault free \( x, v_1 \)-Hamiltonian path \( \langle x, P(x, v_1), v_1 \rangle \) for \( k \geq 5 \). Suppose not, \( k - 3 < 2 \), and \( k < 5 \). It is a contradiction. Thus, \( \langle x, P(x, v_1), v_1, v_2, HP_2, u_2, u_1, y \rangle \) forms a fault free \( x, y \)-Hamiltonian path in this subcase. See Fig. 8.

**Subcase 2.3.** Neither all \( k - 2 \) faults are in \( G_1 \) nor all \( k - 2 \) faults are in \( G_2 \).

Since \( f(1, 2) = k - 2 \) and not all faults are in one component, we have \( f_1 \leq k - 3 \) and \( f_2 \leq k - 3 \). Consequently, both \( G_1 - F_1 \) and \( G_2 - F_2 \) are Hamiltonian.
Corollary 1. Assume that $G_1$ and $G_2$ are $k$-regular super fault-tolerant Hamiltonian where $k \geq 5$ and $|V(G_1)| = |V(G_2)|$. Then $G(G_1,G_2;M)$ is $(k+1)$-regular super fault-tolerant Hamiltonian.

In the following, we briefly introduce the definitions of the twisted-cubes, the crossed-cubes, and the Möbius cubes. It is straightforward to see that these cubes are all special cases of the construction scheme proposed in the previous section.

In [1], the twisted $n$-cube $TQ_n$ is defined for odd values of $n$. The vertex set of the twisted $n$-cube $TQ_n$ is the set of all binary strings of length $n$. Let $u = u_{n-1}u_{n-2}\ldots u_1u_0$ be any vertex in $TQ_n$. For $0 \leq i \leq n-1$, let the $i$th parity function be $P_i(u) = u_i \oplus u_{i+1} \oplus \cdots \oplus u_0$, where $\oplus$ is the exclusive or operation. We can recursively define $TQ_n$ as follows: A twisted 1-cube, $TQ_1$, is a complete graph with two vertices 0 and 1. Suppose that $n \geq 3$. We can decompose the vertices of $TQ_n$ into four sets, $TQ^0_{n-2}$, $TQ^0_{n-2}$, $TQ^0_{n-2}$, and $TQ^0_{n-2}$ where $TQ^0_{n-2}$ consists of those vertices $u$ with $u_{n-1} = i$ and $u_{n-2} = j$. For each $(i, j) \in \{(0,0), (0,1), (1,0), (1,1)\}$, the induced subgraph of $TQ^0_{n-2}$ in $TQ_n$ is isomorphic to $TQ^0_{n-2}$. The edges that connect these four subtwisted cubes can be described as follows: Any vertex $u_{n-1}u_{n-2}\ldots u_1u_0$ with $P_{n-3}(u) = 0$ is connected to $u_{n-1}u_{n-2}\ldots u_1u_0$ and $\bar{u}_{n-1}u_{n-2}\ldots u_1u_0$; and to $u_{n-1}u_{n-2}\ldots u_1u_0$ and $\bar{u}_{n-1}u_{n-2}\ldots u_1u_0$ if $P_{n-3}(u) = 1$. 

Fig. 9. Subcase 2.3: Neither all $k - 2$ faults are in $G_1$ nor all $k - 2$ faults are in $G_2$. 
From the definition, both the subgraph induced by $TQ^{0,0}_{n-2} \cup TQ^{1,0}_{n-2}$ and the subgraph induced by $TQ^{0,1}_{n-2} \cup TQ^{1,1}_{n-2}$ are isomorphic to $TQ_{n-2} \times K_2$, where $K_2$ is the complete graph with two vertices. Moreover, the edges joining $TQ^{0,0}_{n-2} \cup TQ^{0,0}_{n-2}$ and $TQ^{0,1}_{n-2} \cup TQ^{1,1}_{n-2}$ form a perfect matching of $TQ_n$. Recursively applying Theorems 1 and 2, we can easily prove that $TQ_n$ is super fault-tolerant Hamiltonian for $n > 5$. As for $n \leq 5$, it can be checked by a computer program that it is super fault-tolerant Hamiltonian.

Now, we introduce the definition of the crossed-cubes. Two two-digit binary strings $x = x_1x_0$ and $y = y_1y_0$ are pair related, denoted by $x \sim y$, if and only if $(x, y) \in \{(00, 00), (10, 10), (01, 11), (11, 01)\}$. An $n$-dimension crossed-cube $CQ_n$ [6] is a graph $CQ_n = (V, E)$ that is recursively constructed as follows: $CQ_1$ is a complete graph with two vertices labeled by 0 and 1. $CQ_n$ consists of two identical $(n-1)$-dimension crossed-cubes, $CQ^{0}_{n-1}$ and $CQ^{1}_{n-1}$. The vertex $u = 0u_{n-2}\ldots u_0 \in V(CQ^0_{n-1})$ and vertex $v = 1v_{n-2}\ldots v_0 \in V(CQ^1_{n-1})$ are adjacent in $CQ_n$ if and only if (1) $u_{n-2} = v_{n-2}$ if $n$ is even; and (2) for $0 \leq i < \lfloor n - 1/2 \rfloor$, $u_{2i+1}u_{2i} \sim v_{2i+1}v_{2i}$.

According to the definition, $CQ_n$ can be viewed as $G(CQ^0_{n-1}, CQ^1_{n-1}; M)$ for some perfect matching $M$. Again, recursively applying Theorems 1 and 2, we can easily prove that $CQ_n$ is super fault-tolerant Hamiltonian for $n > 5$. As for $n \leq 5$, it can be checked by a computer program that it is super fault-tolerant Hamiltonian.

The Möbius cube [5], $MQ_n = (V, E)$, of dimension $n$ has $2^n$ vertices. Each vertex is labeled by a unique $n$-bit binary string as its address and has connections to $n$ other distinct vertices. The vertex with address $X = x_{n-1}x_{n-2}\ldots x_0$ connects to $n$ other vertices $Y_i$, $0 \leq i \leq n - 1$, where the address of $Y_i$ satisfies (1) $Y_i = (x_{n-1}\ldots x_{i+1}0\ldots x_0)$ if $x_{i+1} = 0$; or (2) $Y_i = (x_{n-1}\ldots x_{i+1}1\ldots x_0)$ if $x_{i+1} = 1$.

From the above definition, $X$ connects to $Y_i$ by complementing the bit $x_{i+1}$ if $x_{i+1} = 0$, or by complementing all bits of $x_{i+1}\ldots x_0$ if $x_{i+1} = 1$. For the connection between $X$ and $Y_{n-1}$, we can assume that the unspecified $x_n$ is either 0 or 1, which gives slightly different topologies. If $x_n$ is 0, we call the network generated the “0-Möbius cube”, denoted by $0-MQ_n$; and if $x_n$ is 1, we call the network generated the “1-Möbius cube”, denoted by $1-MQ_n$.

According to the above definition, $0-MQ_{n+1}$ and $1-MQ_{n+1}$ can be recursively constructed from a $0-MQ_n$ and a $1-MQ_n$ by adding a perfect matching. Recursively applying Theorems 1 and 2, we can easily prove that $MQ_n$ is super fault-tolerant Hamiltonian for $n > 5$. As for $n \leq 5$, it can be checked by a computer program that it is super fault-tolerant Hamiltonian.

5. Conclusion

The fault-tolerant Hamiltonicity and the fault-tolerant Hamiltonian connectivity are essential parameters of an interconnection network. In this paper,
we propose a family of $k$-regular, $(k - 2)$-Hamiltonian, and $(k - 3)$-Hamiltonian connected graphs. These graphs are maximally fault-tolerant, and we call them super fault-tolerant Hamiltonian graphs.

One of the contributions of this paper is the following. We propose a construction scheme to construct, with flexibility, many $k$-regular super fault-tolerant Hamiltonian graphs for $k \geq 6$. As for small values of $k$, $k \leq 5$, there are some examples in literature, such as twisted-cubes, crossed-cubes, and Möbius cubes, etc.

There are many popular interconnection networks which are $k$-regular graphs. Some of them, e.g., twisted-cubes, crossed-cubes, and Möbius cubes, can be recursively constructed using our construction schemes. And therefore, they are in fact a subclass of our proposed family of graphs. Then, we know that they are super fault-tolerant Hamiltonian as long as the case is true for initial cases $k \leq 5$. For small values of $k$, $k \leq 5$, we may use a computer program to check that it is $(k - 2)$-Hamiltonian and $(k - 3)$-Hamiltonian connected.

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References


