An Optimal Algorithm for Sampled-Data Robust Servomechanism Controller Using Exponential Hold

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Abstract—A new structure of sampled-data robust servomechanism controller using exponential hold is developed. An optimal algorithm is also proposed for choosing the controller parameters of two important special classes. The algorithm is derived by minimizing a square-error performance index, and the solution can be solved from a discrete-time algebraic Riccati equation.

I. INTRODUCTION

The problem of robust servomechanism controller design has been widely considered in the literature (see reference). Generally, the purpose for one to construct a robust servomechanism controller is to attain the capability of asymptotic tracking and disturbance rejection with the permission of plant variations. In the literature, a general structure of linear time-invariant robust servomechanism controllers has been characterized [2]-[4], and the well-known “continuous internal model principle” has been given [8], [5], [7]. With this principle, it can be seen that if the steady-state value of the reference input or the disturbance is not constant, then in general, one cannot use sampled data with zero-order hold to construct a ripple-free [7] robust servomechanism controller because ripple errors would occur even if there is no tracking error at the sampling instants.

In this note, a new structure of sampled-data robust servomechanism controllers using exponential hold is developed. Such a structure is convenient for design because it leads to a simple closed-loop form. In particular, controller design for two important special cases classified as the “minimal-order class” and the “one-step prediction class,” respectively, are derived. For the former class, the controller has the simplest structure so that it needs less on-line computations. For the latter class, on-line control values are calculated by one step ahead of the output measurements so that it allows a leisure time to implement the control scheme. An optimal algorithm for choosing the parameters of the two important special classes is also developed. The algorithm is derived by minimizing a square-error performance index, and the solution can be solved from a discrete-time algebraic Riccati equation. A distinctive feature of the algorithm is that the solution does not depend on the weighting matrix of the performance index, but only on the correlation of the initial values of system state, reference input, and disturbance. Hence, the statistical information of the initial conditions becomes very important to this algorithm.

II. PRELIMINARY

A. System Description

Consider the command tracking and disturbance rejection problem of a linear time-invariant system described as follows:

\[ y(t) = Cx(t) + Gd(t) \]  \hspace{1cm} (1.a)
\[ e(t) = y(t) - r(t) \]  \hspace{1cm} (1.b)

where \( x \in \mathbb{R}^n \) is the state, \( u \in \mathbb{R}^m \) is the control, \( y \in \mathbb{R}^n \) is the measurable output, \( d \in \mathbb{R}^p \) is the disturbance, \( r \in \mathbb{R}^n \) is the command or reference input, and \( e \in \mathbb{R}^n \) is the tracking error. The reference input and the disturbance satisfy the following models:

\[ x_1(t) = A_1x_1(t) \]  \hspace{1cm} (2.a)
\[ r(t) = C_1x_1(t) \]  \hspace{1cm} (2.b)
\[ d(t) = C_2u(t) \]  \hspace{1cm} (2.c)

where \( x_1 \in \mathbb{R}^{n_1} \) and \( x_2 \in \mathbb{R}^{n_2} \). The system described above is said to have no transmission zero [2] at the eigenvalues of \( A_1 \) and \( A_2 \), if

\[ \text{rank}\left[ \begin{bmatrix} -sI_n + A & B \\ C & O_m \end{bmatrix} \right] = n + m, \quad \forall s \in \{ \text{eig}(A_1) \cup \text{eig}(A_2) \} \]  \hspace{1cm} (3)

where \( I \) denotes the \( n \)-dimensional identity matrix, \( O_m \) denotes the \( m \)-dimensional zero matrix, \( \text{eig}(\#) \) denotes the set of eigenvalues of a matrix \( \# \), and \( \text{eig}(A_1) \subset \mathbb{C}^+ \), \( \text{eig}(A_2) \subset \mathbb{C}^+ \), where \( \mathbb{C}^+ \) is the right-half complex plane including the imaginary axis.

B. Robust Servomechanism Controller

A controller \( u = f(e, r) \) (with input \( e, r \) and output \( u \)) is called a robust servomechanism controller of system (1), if it can satisfy the following three conditions [4], [6]:

**Condition 1:** The resultant closed-loop system is asymptotically stable. Thus, if \( r(t) \equiv 0 \) and \( d(t) \equiv 0 \), then \( x(t) \rightarrow 0 \) and \( u(t) \rightarrow 0 \) as \( t \rightarrow \infty \).

**Condition 2:** Asymptotic tracking action occurs, i.e., \( e(t) \rightarrow 0 \) as \( t \rightarrow \infty \) for all initial conditions of \( x, z, z_1, \) and the controller state.

**Condition 3:** Condition 2 remains true for any parameter variations in \( A, B, C, F, \) and \( G \) as long as Condition 1 remains true.

C. Deviation Model

It is known [4] that for every linear time-invariant robust servomechanism controller of system (1), there exist matrices \( T_{i1} \in \mathbb{R}^{n \times m_1}, T_{12} \in \mathbb{R}^{n_1 \times m_2}, T_{21} \in \mathbb{R}^{m_1 \times m_2}, \) and \( T_{22} \in \mathbb{R}^{m \times m_2} \), such that as \( t \rightarrow \infty \), then \( x(t) \rightarrow x_{ss}(t) \) and \( u(t) \rightarrow u_{ss}(t) \) for all \( z(0), x_1(0) \) and \( x_2(0) \), where

\[ x_{ss}(t) = T_{11}x_1(t) + T_{12}x_2(t) \]  \hspace{1cm} (4.a)
\[ z_{ss}(t) = T_{21}x_1(t) + T_{22}x_2(t) \]  \hspace{1cm} (4.b)

denote the ultimate steady-state trajectories of \( x \) and \( u \), respectively. Thus, by defining the “deviation variables” as [10]

\[ \delta x(t) = x(t) - x_{ss}(t) \]  \hspace{1cm} (4.a)
\[ \delta u(t) = u(t) - u_{ss}(t) \]  \hspace{1cm} (4.b)

and using the fact that \( \text{eig}(A_1) \subset \mathbb{C}^+ \) and \( \text{eig}(A_2) \subset \mathbb{C}^+ \), it can be easily checked that the deviation variables satisfy the following model:

\[ \dot{\delta x}(t) = A\delta x(t) + \delta Gu(t) \]  \hspace{1cm} (5.a)
\[ e(t) = C\delta x(t) \]  \hspace{1cm} (5.b)

D. An Augmented Model

Let

\[ \lambda(s) = s^2 - \sum_{i=0}^{p-1} \alpha_i s^i \]  \hspace{1cm} (6)
be the lowest order polynomial satisfying
\[
\lambda(A_r) = A_r^p - \sum_{i=0}^{p-1} \alpha_i A_r^i = 0.
\]
Also, let
\[
\Omega = \begin{bmatrix}
O_m & I_m & O_m & \cdots & O_m \\
O_m & O_m & I_m & \cdots & O_m \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
O_m & O_m & O_m & \cdots & I_m \\
\end{bmatrix}
\]
\[
\rho = [I_m \ O_m \ (p-1)m].
\]

Theorem 1: If the matrix
\[
A_r = \begin{bmatrix}
\mathbf{A} & \mathbf{B}_1 \\
\mathbf{O}_{m \times n} & \mathbf{B}_1 \\
\end{bmatrix}
\]
is stable (i.e., all eigenvalues lie inside the unit complex circle), then the following is a robust servomechanism controller of system (1):
\[
\begin{bmatrix}
\hat{\xi}(k+1)T \\
\hat{h}(k+1)T
\end{bmatrix}
= \begin{bmatrix}
\mathbf{C} & \mathbf{O}_{m \times n} \\
\mathbf{O}_{m \times n} & \mathbf{H}_1 \\
\mathbf{B}_1 & \mathbf{H}_1 \\
\end{bmatrix}
\begin{bmatrix}
\hat{\xi}(kT) \\
\hat{h}(kT) \\
\mathbf{e}(kT)
\end{bmatrix}
+ \begin{bmatrix}
\mathbf{L}_2 \\
\mathbf{L}_2 \\
\end{bmatrix}
\mathbf{e}(kT)
\]
(14a)
\[
\mathbf{u}(kT + \theta) = [\mathbf{I} \mathbf{C} \exp(\mathbf{D}T) \mathbf{e}(kT)]
\]
(14b)
where \(k = 0, 1, 2, \ldots\) and \(\theta \in [0, T)\).

Proof: By (11) and (12a)-(12c), it is clear that a necessary condition for \(A_r\) to be stable is that the triple
\[
\begin{bmatrix}
\mathbf{C} & \mathbf{O}_{m \times n} \\
\mathbf{O}_{m \times n} & \mathbf{H}_1 \\
\mathbf{B}_1 & \mathbf{H}_1 \\
\end{bmatrix}
\]
is stabilizable and detectable. This in turn implies that \((C, A, B)\) is stabilizable and detectable, and the transmission zero assumption (3) holds (a simple rank test easily checks this fact). Thus, a linear time-invariant robust servomechanism controller of system (1) can be found ([2]-[4]), and the deviation model (5) and the augmented model (10) exist. Therefore, by defining
\[
\hat{\xi}(t) = \mathbf{C} \xi(t)
\]
and subtracting (9) from (14), one obtains
\[
\begin{bmatrix}
\hat{\xi}(t) \\
\mathbf{e}(t)
\end{bmatrix}
= \begin{bmatrix}
\mathbf{A} & \mathbf{B}_1 \\
\mathbf{O}_{m \times n} & \mathbf{B}_1 \\
\end{bmatrix}
\begin{bmatrix}
\xi(t) \\
\mathbf{e}(t)
\end{bmatrix}
+ \begin{bmatrix}
\mathbf{L}_2 \\
\mathbf{L}_2 \\
\end{bmatrix}
\mathbf{e}(t)
\]
(17a)
\[
\mathbf{u}(kT + \theta) = [\mathbf{I} \mathbf{C} \exp(\mathbf{D}T) \mathbf{e}(kT)]
\]
(17b)
By combining (17) and the deviation model (5), one obtains the following closed-loop system:
\[
\begin{bmatrix}
\hat{\xi}(k+1)T \\
\hat{h}(k+1)T
\end{bmatrix}
= \begin{bmatrix}
\mathbf{A} & \mathbf{L}_1 \mathbf{C} & \mathbf{B}_1 \\
\mathbf{B}_1 & \mathbf{H}_1 \\
\mathbf{O}_{m \times n} & \mathbf{H}_1 \\
\end{bmatrix}
\begin{bmatrix}
\hat{\xi}(kT) \\
\mathbf{e}(kT)
\end{bmatrix}
(18)
\]
It is easily checked that
\[
\begin{bmatrix}
\mathbf{I}_{m+n} \\
\mathbf{O}_{m \times n} \\
\end{bmatrix}
= \begin{bmatrix}
\mathbf{A} & \mathbf{L}_1 \mathbf{C} & \mathbf{B}_1 \\
\mathbf{B}_1 & \mathbf{H}_1 \\
\end{bmatrix}
\begin{bmatrix}
\mathbf{A} & \mathbf{L}_1 \mathbf{C} & \mathbf{B}_1 \\
\mathbf{B}_1 & \mathbf{H}_1 \\
\end{bmatrix}^{-1}
\begin{bmatrix}
\mathbf{I}_m \\
\mathbf{O}_{m \times n}
\end{bmatrix}
(19)
so that by giving
\[
\mathbf{h}(kT) = \mathbf{h}(kT) + \begin{bmatrix}
\mathbf{A} \\
\mathbf{L}_1 \mathbf{C} \\
\mathbf{O}_{m \times n}
\end{bmatrix}
\begin{bmatrix}
\hat{\xi}(kT) \\
\mathbf{e}(kT)
\end{bmatrix}
(20)
and substituting it into (18), one obtains
\[
\begin{bmatrix}
\hat{\xi}(k+1)T \\
\hat{h}(k+1)T
\end{bmatrix}
= \begin{bmatrix}
\mathbf{A} & \mathbf{H}_1 \\
\mathbf{B}_1 & \mathbf{H}_1 \\
\end{bmatrix}
\begin{bmatrix}
\hat{\xi}(kT) \\
\mathbf{e}(kT)
\end{bmatrix}
(21)
\]
Since \(A_r\) is stable, all the closed-loop poles lie inside the unit complex circle. Thus, it is true that \(\hat{\xi}(kT) \rightarrow 0\), \(\mathbf{e}(kT) \rightarrow 0\) and \(\mathbf{h}(kT) \rightarrow 0\) as \(k \rightarrow \infty\). By (10b) and (17b), it is also true that \(\mathbf{e}(kT) \rightarrow 0\) and
B. Two Special Cases

Fig. 1. The sampled-data robust servomechanism controller of (14.a) and (23).

δu(t) → 0 as t → ∞. Since the deviation model (5) is an ordinary continuous model, one has δx(t) → 0 and e(t) → 0, as t → ∞. Hence, the asymptotic tracking action occurs as long as A1 is stable. Moreover, from (21), it is clear that the resultant closed-loop system is asymptotically stable if and only if A1 is stable (if r ≡ 0 and A, are stable). Thus, the theorem is proved.

Remark 1: If (A, B) is controllable, then for any matrices A1 and H1, there exist infinitely many choices of ϕ1(θ) and ϕ2(θ) to satisfy (12.b) and (12.c), respectively. In particular, if n ≤ mp and rank [B1] = n [no loss of generality by increasing the number of modes of λ(s)], then a simple possible choice may be the exponential hold as follows:

\[
ϕ1(θ) = Γ \exp(φ θ) L_1, \quad ϕ2(θ) = Γ \exp(φ θ) H_1
\]

(22.b)

where

\[
L_1 = B_1(B_1 B_1^T)^{-1}_1, \quad H_1 = B_2(B_2 B_2^T)^{-1}_1.
\]

(22.c)

With this choice, the control (14.b) can be simplified as (see Fig. 1):

\[
u(t) = Γ \exp(φ θ) [ξ(t) + H_1 h(kT) + L_1 e(kT)].
\]

(23)

B. Two Special Cases

By letting ϕ2(θ) = 0 (i.e., H1 = 0) and H2 = 0 or ϕ1(θ) = 0 (i.e., L1 = 0) and L2 = 0, respectively, then Theorem 1 leads to the following two corollaries.

Corollary 1: (Minimal-order class) If the matrix

\[
A_1 = \left[ \begin{array}{cc}
\bar{A} & B_1 \\
o_{mp \times n} & 0
\end{array} \right] + \left[ \begin{array}{c}
L_1 \\
L_2
\end{array} \right] \left[ \begin{array}{c}
C \\
o_{mp \times mp}
\end{array} \right]
\]

is stable, then the following is a robust servomechanism controller of system (1):

\[
\dot{ξ}(k+1)T = \bar{A} ξ(kT) + L_1 e(kT)
\]

\[
u(kT + θ) = Γ \exp(θ) [ξ(kT) + H_1 h(kT) + L_1 e(kT)].
\]

(25.a)

(25.b)

Corollary 2: (One-step prediction class) If the matrix

\[
A_1 = \left[ \begin{array}{cc}
\bar{A} & B_1 \\
o_{mp \times n} & 0
\end{array} \right] + \left[ \begin{array}{c}
H_1 \\
H_2
\end{array} \right] C_σ
\]

is stable, then the following is a robust servomechanism controller of system (1):

\[
\dot{ξ}(k+1)T = \bar{A} ξ(kT) + L_1 e(kT)
\]

\[
u(kT + θ) = Γ \exp(θ) [ξ(kT) + H_1 h(kT)].
\]

(26)

Remark 2: From (21) and (20), one has \(\bar{h}(kT) = 0\) and \(h(kT) = C_σ [\bar{A} ξ(kT) ξ(kT)]\) for all \(k \geq 1\). Since \(h(kT)\) is stable, and \(\bar{ξ}(kT)\) can be calculated from (27.a) as long as \(h(kT)\) is measurable, therefore, from (27.b), the values of \(w(kT + θ)\) in-between the sampling instants \(kT\) and \((k + 1)T\) can be calculated. The class name of (27) reflects this prediction property.

IV. AN OPTIMAL APPROACH

In the rest of this note, one assumes that (C, A, B) is controllable and observable, and both the minimal order class (25) and the one-step prediction class (27) are not empty (i.e., there exist \(L_1, L_2\) and \(H_1, H_2\) such that both the matrices (24) and (26) are stable). Besides, we select a quadratic performance index as follows:

\[
J = \sum_{k=0}^{∞} \left[ \bar{δ} ξ(kT) + H_1 h(kT) + L_1 e(kT) \right] \left[ \bar{δ} ξ(kT) + H_1 h(kT) + L_1 e(kT) \right]^	op
\]

(28)

where \(Q \in R^{(n+mp)×(n+mp)}\) is positive-definite, and \(\bar{ξ}(kT) = ξ(kT) - ξ(kT)\). Notice that the index serves as a measure of the deviation errors from the ultimate steady-state trajectories. Now, it is desired to find the optimal gains \(L_1, L_2, H_1, H_2\), such that the index \(J\) subject to either class of (23) or (27) is minimized.

A. Minimal-Order Class

Since the minimal-order class (25) is a special case of the general class (14) with \(ϕ2(θ) = 0\) (i.e., \(H_1 = 0, H_2 = 0, H_3 = 0\), thus the closed-loop system (21) can be simplified as

\[
\bar{δ} ξ(k + 1)T = \bar{A} ξ(kT) + L_1 e(kT)
\]

\[
u(kT + θ) = Γ \exp(θ) [ξ(kT) + H_1 h(kT) + L_1 e(kT)].
\]

(29)

Since the closed-loop system is asymptotically stable (a necessary condition of the robust servomechanism controller), the index \(J\) subject to (29) equals (13):

\[
J = Tr(Ψ V)
\]

(30)

where \(Ψ\) is a correlation matrix given by

\[
Ψ \equiv \text{cor} \left( \begin{array}{c}
\bar{δ} ξ(0) \\
ξ(0)
\end{array} \right) = E \left( \begin{array}{c}
\bar{δ} ξ(0) \\
ξ(0)
\end{array} \right) \left( \begin{array}{c}
\bar{δ} ξ(0) \\
ξ(0)
\end{array} \right)^	op
\]

(31)

and \(V \in R^{(n+mp)×(n+mp)}\) is a positive-definite matrix solved from the following Lyapunov equation:

\[
\left( \begin{array}{cc}
\bar{A} & B_1 \\
O_{mp \times n} & 0
\end{array} \right) + \left[ \begin{array}{c}
L_1 \\
L_2
\end{array} \right] \left[ \begin{array}{c}
C \\
o_{mp \times mp}
\end{array} \right] V \left( \begin{array}{cc}
\bar{A} & B_1 \\
O_{mp \times n} & 0
\end{array} \right)^	op + \left[ \begin{array}{c}
L_1 \\
L_2
\end{array} \right] \left[ \begin{array}{c}
C \\
o_{mp \times mp}
\end{array} \right] = V + Q = 0.
\]

(32)

By approximating \(Ψ\) by \(Ψ + εe_{mp×mp}\), where \(ε\) is a small positive number, and, without loss of generality, by assuming \(Ψ\) to be positive-definite, one obtains the following result.

Theorem 2: Assume \(Ψ\) is positive-definite, then the optimal gains \(L_1, L_2\) of the sampled-data robust servomechanism controller (25) to minimize the performance index (28) is given by

\[
L_1 = \left[ \begin{array}{c}
A \\
O_{mp \times n}
\end{array} \right] P \left[ \begin{array}{c}
C \\
o_{mp \times mp}
\end{array} \right] \left( \begin{array}{c}
C \\
o_{mp \times mp}
\end{array} \right)\]
where $P \in \mathbb{R}^{(n+mp) \times (n+mp)}$ is a positive-definite matrix solved from the following algebraic Riccati equation:

$$
\begin{bmatrix}
\bar{A} & \bar{B}_1 \\
\bar{O}_{mp \times n} & \bar{\phi}
\end{bmatrix} P + P \begin{bmatrix}
\bar{A} & \bar{B}_1 \\
\bar{O}_{mp \times n} & \bar{\phi}
\end{bmatrix}^\top - C^\top \bar{O}_{mp \times np} P \begin{bmatrix}
\bar{A} & \bar{B}_1 \\
\bar{O}_{mp \times n} & \bar{\phi}
\end{bmatrix}^\top - Q = 0.
$$

(33.b)

Proof: To minimize (30) subject to (32), we introduce the following augmented cost [12]:

$$
J_c = \text{Tr} \left( V \Psi + P \left( \begin{bmatrix}
\bar{A} & \bar{B}_1 \\
\bar{O}_{mp \times n} & \bar{\phi}
\end{bmatrix} + \begin{bmatrix}
\bar{L}_1 \\
\bar{L}_2
\end{bmatrix} \begin{bmatrix}
C & \bar{O}_{mp \times np}
\end{bmatrix} \right)^\top \right)
$$

where $P$ is the associated Lagrange multiplier. Letting $\frac{dJ_c}{dV} = 0$, one obtains

$$
\begin{bmatrix}
\bar{A} & \bar{B}_1 \\
\bar{O}_{mp \times n} & \bar{\phi}
\end{bmatrix} P + P \begin{bmatrix}
\bar{A} & \bar{B}_1 \\
\bar{O}_{mp \times n} & \bar{\phi}
\end{bmatrix}^\top - \begin{bmatrix}
\bar{L}_1 \\
\bar{L}_2
\end{bmatrix} \begin{bmatrix}
C & \bar{O}_{mp \times np}
\end{bmatrix} - P + \Psi = 0.
$$

(35.a)

On the other hand, by letting $L = \begin{bmatrix}
\bar{L}_1 \\
\bar{L}_2
\end{bmatrix}^\top$ and $\frac{dJ_c}{dL} = 0$, one obtains

$$
\begin{bmatrix}
\bar{A} & \bar{B}_1 \\
\bar{O}_{mp \times n} & \bar{\phi}
\end{bmatrix} P + P \begin{bmatrix}
\bar{A} & \bar{B}_1 \\
\bar{O}_{mp \times n} & \bar{\phi}
\end{bmatrix}^\top - \begin{bmatrix}
\bar{L}_1 \\
\bar{L}_2
\end{bmatrix} \begin{bmatrix}
C & \bar{O}_{mp \times np}
\end{bmatrix} - P + \Psi = 0.
$$

(35.b)

Since $Q$ is positive-definite, the solution $V$ of the Lyapunov equation (32) is positive-definite, hence (35.b) can be reduced to (35.a). Furthermore, by substituting (35.a) into (35.b), one obtains (35.b). Hence, the necessity of the theorem is proved. Besides, by (30), (32) and (35.a), one has

$$
J = \text{Tr}(V \Psi)
$$

(36)

It is known [1], [9] that the algebraic Riccati equation (33.b) and (33.a) has a unique stable solution which minimizes the index (36), so that the theorem is proved. $\square$

B. One-Step Prediction Class

Since the one-step prediction class (27) is a special case of the general class (14) with $\gamma_1(\theta) = 0$ (i.e., $L_1 = 0$, $L_2 = 0$ and $L_3 = 0$), the closed-loop system (21) can be simplified as

$$
\begin{bmatrix}
\bar{A} & \bar{B}_1 \\
\bar{O}_{mp \times n} & \bar{\phi}
\end{bmatrix} \bar{X}(k+1|T) + \bar{H}_1 \bar{C}_0 \bar{X}(k|T) = \left[ \bar{A} + \bar{B}_1 \bar{C}_0 \right] \bar{X}(k|T).
$$

(37)

for all $k \geq 1$. Assume $h(0) = 0$, then from (18), one has

$$
\begin{bmatrix}
\bar{X}(T) \\
\xi(T)
\end{bmatrix} = \begin{bmatrix}
\bar{A} & \bar{B}_1 \\
\bar{O}_{mp \times n} & \bar{\phi}
\end{bmatrix} \bar{X}(0|T) + \left[ \bar{A} + \bar{B}_1 \bar{C}_0 \right] \xi(0)
$$

(38)

Hence, the correlation of the state $\bar{X}(T)\xi(T)$ equals

$$
\Psi_{X} = \text{cov} \begin{bmatrix}
\bar{X}(T) \\
\xi(T)
\end{bmatrix} = \begin{bmatrix}
\bar{A} & \bar{B}_1 \\
\bar{O}_{mp \times n} & \bar{\phi}
\end{bmatrix} \Psi \begin{bmatrix}
\bar{A} & \bar{B}_1 \\
\bar{O}_{mp \times n} & \bar{\phi}
\end{bmatrix}^\top.
$$

(39)

Notice that $\Psi_{X}$ is independent of $H_1$ and $H_2$, so that the minimization of the index $J$ is equivalent to the minimization of the following index

$$
J_1 = E \sum_{k=1}^{\infty} \left[ \bar{X}(k|T)^\top Q \bar{X}(k|T) \right].
$$

(40)

Theorem 3: Assume $\Psi_{X}$ is positive-definite, then the optimal gains $H_1$ and $H_2$ of the sampled-data robust servomechanism controller (27) to minimize the performance index (28) with initial condition $h(0) = 0$ is given by

$$
\begin{bmatrix}
H_1 \\
H_2
\end{bmatrix} = - \begin{bmatrix}
\bar{A} & \bar{B}_1 \\
\bar{O}_{mp \times n} & \bar{\phi}
\end{bmatrix} P \begin{bmatrix}
\bar{A} & \bar{B}_1 \\
\bar{O}_{mp \times n} & \bar{\phi}
\end{bmatrix}^\top - P + \Psi = 0.
$$

(41.a)

where $P \in \mathbb{R}^{(n+mp) \times (n+mp)}$ is a positive-definite matrix solved from the following algebraic Riccati equation:

$$
\begin{bmatrix}
\bar{A} & \bar{B}_1 \\
\bar{O}_{mp \times n} & \bar{\phi}
\end{bmatrix} P + P \begin{bmatrix}
\bar{A} & \bar{B}_1 \\
\bar{O}_{mp \times n} & \bar{\phi}
\end{bmatrix}^\top - \begin{bmatrix}
\bar{L}_1 \\
\bar{L}_2
\end{bmatrix} \begin{bmatrix}
C & \bar{O}_{mp \times np}
\end{bmatrix} - P + \Psi = 0.
$$

(41.b)

Proof: Replacing $\begin{bmatrix}
C & \bar{O}_{mp \times np}
\end{bmatrix}$ by $C$ and $\Psi$ by $\Psi_{X}$ in Theorem 2, the result follows directly.

C. Computation of the Correlation Matrix

A convenient method to calculate the correlation matrix $\Psi_{X}$ (or $\Psi_{X}$) can be done by way of the augmented model (10). To do so, subtracting (10) from (1) and using (4.b), one obtains

$$
\begin{bmatrix}
\bar{X}(t) \\
\xi(t)
\end{bmatrix} = \begin{bmatrix}
\bar{A} & \bar{B}_1 \\
\bar{O}_{mp \times n} & \bar{\phi}
\end{bmatrix} \begin{bmatrix}
\bar{X}(t) \\
\xi(t)
\end{bmatrix} + \begin{bmatrix}
\bar{F}_C & \bar{F}_{GD}
\end{bmatrix} \begin{bmatrix}
\bar{X}(t) \\
\xi(t)
\end{bmatrix}.
$$

(42.a)

Now, define

$$
\begin{bmatrix}
\bar{X}_{s}(t) \\
\xi_{s}(t)
\end{bmatrix} = \begin{bmatrix}
\bar{A} & \bar{B}_1 \\
\bar{O}_{mp \times n} & \bar{\phi}
\end{bmatrix} \begin{bmatrix}
\bar{X}_{s}(t) \\
\xi_{s}(t)
\end{bmatrix} + \begin{bmatrix}
\bar{F}_C & \bar{F}_{GD}
\end{bmatrix} \begin{bmatrix}
\bar{X}_{s}(t) \\
\xi_{s}(t)
\end{bmatrix}.
$$

(42.b)

and differentiating (43) continuously, one obtains

$$
\frac{d}{dt} \begin{bmatrix}
\bar{X}_{s}(t) \\
\xi_{s}(t)
\end{bmatrix} = N_1 \begin{bmatrix}
\bar{X}_{s}(t) \\
\xi_{s}(t)
\end{bmatrix} + N_2 \begin{bmatrix}
\bar{X}_{s}(t) \\
\xi_{s}(t)
\end{bmatrix}.
$$

(44)

where $N_1 = [N_{10} \cdots N_{1p}]^\top \in \mathbb{R}^{mp \times (m+p)}$, $N_2 = [N_{20} \cdots N_{2p}]^\top \in \mathbb{R}^{mp \times (m+p)}$, $N_3 = [N_{30} \cdots N_{3p}]^\top \in \mathbb{R}^{mp \times (m+p)}$.
for \( q = 0, 1, 2, \ldots, g - 1 \) (except \( N_{20} = O_{m \times (m_r + m_d)} \)). Since \((C, A)\) is observable, the augmented model (10) is observable, hence rank \( N_3 = n + pm \) can be guaranteed by a sufficiently large positive integer \( g \), so that one obtains

\[
\mathbf{N}[\mathbf{x}_t(t)] = N[\mathbf{x}_t(t)] \quad (46.1)
\]

where

\[
N = (N_2^2 N_2)^{-1} N_2 (N_1 - N_3) \in \mathbb{R}^{(n+mp)\times (n+mp)}. \quad (46.2)
\]

Now, from (4.1b), (16) and (46.1), one has

\[
\begin{bmatrix}
\dot{\mathbf{z}}(t) \\
\dot{\mathbf{z}}(0)
\end{bmatrix} = \begin{bmatrix}
\mathbf{z}(t) \\
\mathbf{z}(0)
\end{bmatrix} - \begin{bmatrix}
\mathbf{O}_{n \times 1} \\
\mathbf{O}_{n \times 1}
\end{bmatrix} + \begin{bmatrix}
\mathbf{S}\mathbf{z}(t)
\end{bmatrix}
\]

where

\[
\mathbf{S} = \begin{bmatrix}
I_n \\
\mathbf{O}_{mp \times n}
\end{bmatrix} - N \in \mathbb{R}^{(n+mp)\times (n+mp)}. \quad (48)
\]

Hence, the correlation matrix \( \mathbf{Q} \) equals

\[
\begin{bmatrix}
\mathbf{Q}_{xz}(t) \\
\mathbf{Q}_{xz}(0)
\end{bmatrix} = \begin{bmatrix}
\mathbf{Q}_{xz}(0) \\
\mathbf{Q}_{xz}(0)
\end{bmatrix} = \begin{bmatrix}
\mathbf{Q}_{xz}(0) \\
\mathbf{Q}_{xz}(0)
\end{bmatrix} + \mathbf{S}^{'}
\]

where

\[
\mathbf{Q}_{xz}(0) = \mathbf{O}_{n \times 1} \mathbf{O}_{n \times 1} + \mathbf{S}^{'}
\]

and approximating \( \mathbf{P} \) (calculated from (50) and (39)) by \( \mathbf{P} + 10^{-3} \mathbf{d} \), then from theorem (3), a one-step prediction optimal sampled-data robust servomechanism controller is

\[
\begin{bmatrix}
\dot{\mathbf{z}}(t) \\
\mathbf{z}(t)
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix} \mathbf{z}(t) + \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}
\]

Assume the correlation of the initial states is

\[
E\left( \begin{bmatrix}
\mathbf{z}(0) \\
\mathbf{z}(0)
\end{bmatrix} \begin{bmatrix}
\mathbf{z}(0) \\
\mathbf{z}(0)
\end{bmatrix}^{'} \right) = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} \quad (52)
\]

where \( \mathbf{z}(0) = \mathbf{d} \) and \( \gamma \) is a positive number. By replacing \( \mathbf{u}(t-1) \) by \( \mathbf{u}(t) \) to remove the time-delay, selecting \( T = 1 \), choosing

\[
\phi = \begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}, \quad \Gamma = \begin{bmatrix}
1 & 0
\end{bmatrix}
\]

In particular, if \( \dot{\mathbf{z}}(0) = 0 \), then \( \mathbf{B} \) is simplified to

\[
\mathbf{B} = \mathbf{S}^{'}
\]

where

\[
n(\alpha^{(K+1)}T) = \begin{bmatrix}
1 & 1 \\
0 & 1
\end{bmatrix} - 3.5209 \begin{bmatrix}
\xi(t) \\
h(t)
\end{bmatrix} + \begin{bmatrix}
0 \\
0
\end{bmatrix} + \begin{bmatrix}
0 \\
1
\end{bmatrix} \quad (55)
\]

Example 1: Consider a linear time-delay process described as follows (e.g., a tank temperature control [7], or a paper machine [11]):

\[
\dot{x}(t) = -x(t) + u(t-1) + d \\
y(t) = x(t) \\
e(t) = y(t) - r(t)
\]

where \( u(\theta) = 0 \) for \( \theta \in (-1, 0), d \) is a constant disturbance, and
Example 2: Consider the following system (Rosenbrock problem [4]):

\[
\begin{align*}
\dot{z}(t) &= \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} z(t) + \begin{bmatrix} -1/6 & 0 \\ 2/3 & 1 \\ 0 & 1/2 \end{bmatrix} u(t), \\
z(0) &= 0 \\
y(t) &= \begin{bmatrix} 3 & -3/4 & -1/2 \\ 2 & -1 & 0 \end{bmatrix} z(t).
\end{align*}
\] (56. a)

This system is to track a sinusoidal signal described as:

\[
\begin{align*}
r(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} z(t), \\
r(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} z(t).
\end{align*}
\] (56. b)

where \(z(0)\) is a random vector with correlation

\[
E(z(0)z'(0)) = \begin{bmatrix} 0 & 0 \\ 0 & \gamma \end{bmatrix}
\] (58)

where \(\gamma\) is a positive real number. Selecting \(\gamma = 0.5\), choosing \(\gamma_0 = 0\), and approximating \(\Psi\) (calculated from (50)) by \(\Psi + 10^{-5} J\), then from Theorem 2, a minimal-order optimal sampled-data robust servomechanism controller is

\[
\begin{align*}
\xi((k+1)T) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \xi(k+1)T \\
+ \begin{bmatrix} 2.9014 & -12.3172 \\ 1.6216 & -4.6230 \\ -1.5642 & 3.5657 \end{bmatrix} e(kT) \\
u(kT + \theta) &= \begin{bmatrix} \cos(\pi \theta) & \sin(\pi \theta) & 0 & 0 \\ 0 & 0 & \cos(\pi \theta) & \sin(\pi \theta) \end{bmatrix} \\
\begin{bmatrix} -6.5989 & 6.6012 \\ 3.5575 & -3.5529 \\ -7.3429 & 7.3395 \\ 8.2651 & -8.2661 \end{bmatrix} e(kT)
\end{align*}
\] (60. a)

where \(\xi(0) = 0\) is assumed. The responses of the system with this controller is shown in Fig. 3.

VI. CONCLUSION

In this note, a new structure of sampled-data robust servomechanism controller using exponential hold is presented. The proposed structure is simple for design and can be easily implemented by digital computers. An optimal algorithm is also derived for choosing the parameters of the two important special classes. The solution of the algorithm can be solved from a discrete-time algebraic Riccati equation.

It is of interest that the solution of the derived algorithm does not depend on the weighting matrix of the performance index, but only on the correlations of the initial values of the system state, reference input and the disturbance. In a general robust servomechanism controller problem, the uncertain signals to be tracked or the unknown disturbance can be treated as the random vector of the initial values, so that from a statistical viewpoint, the derived algorithm can reflect the capability of treating such uncertainty.

REFERENCES