A solution to the forest leave problem for partial 6-cycle systems

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Abstract

In this paper, we find necessary and sufficient conditions for the existence of a 6-cycle system of $K_n - E(F)$ for every forest $F$ of $K_n$.

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1. Introduction

An $H$-decomposition of the graph $G$ is a partition of $E(G)$ such that each element of the partition induces a subgraph isomorphic to $H$. In the case where $H$ is an $m$-cycle, such a decomposition is referred to as an $m$-cycle system of $G$. An $m$-cycle system of $G$ will be formally described as an ordered pair $(V,B)$, where $V$ is the vertex set of $G$ and $B$ is the set of $m$-cycles.

Results in this area date back to the nineteenth century [7], but have received a lot of attention over the past 40 years. There have been many results found on $H$-decompositions of $G$ for various graphs $H$ and $G$, usually with $G = K_n$. One particularly enticing but difficult problem was to solve the case where $H$ is an $m$-cycle (see [8,9] for surveys of results). This all culminated in two papers. Based on a result

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by Hoffman et al. [6], a paper by Alspach and Gavlas [1] and another by Šajna [10] settled the problem of finding the values of \( n \) for which there exists an \( m \)-cycle system of \( K_n \).

What about the many cases where \( K_n \) does not have an \( H \)-decomposition into \( m \)-cycles? Recently a paper by Alspach and Gavlas [1] and another by Šajna [10] settled the problem of finding the values of \( n \) for which there exists an \( m \)-cycle system of \( K_n - I \), where \( I \) is a 1-factor. This can alternatively be viewed as a partial \( m \)-cycle system in which the set of edges not in any \( m \)-cycle induces a 1-factor. These edges not in any \( m \)-cycle (or the subgraph they induce) are called the leave \( L \).

Continuing with the theme of finding graph decompositions of graphs which are close to complete, one way to extend these results is by finding necessary and sufficient conditions for the existence of a \( m \)-cycle system of \( K_n - E(F) \), where the leave \( L \) is a spanning forest (a forest is a graph that contains no cycle; a connected forest is a tree). Clearly this would generalize the result described above where the leave is a 1-factor. This generalization is so extensive that it becomes very difficult to solve unless further restrictions are made, such as fixing the value of \( m \). Fu and Rodger [4] have obtained such a result by finding necessary and sufficient conditions for the existence of a 4-cycle system of \( K_n - E(F) \), where \( F \) is any forest in \( K_n \). Here we extend these results by finding necessary and sufficient conditions for the existence of a 6-cycle system of \( K_n - E(F) \) for any spanning forest \( F \).

As will be shown, when the leave is a forest, it must be that \( n \) is even. Results obtaining \( m \)-cycle systems of graphs that are close to complete have also been found when \( m \) is odd, in which case each vertex has even degree in the leave (for example [3] when \( m = 3 \), [5] when \( m = 4 \) and [2] when \( m = 6 \)).

Let \( Z_n = \{0, 1, \ldots, n-1\} \). On occasions arithmetic operations will be defined on \( Z_n \), in which case we assume they are performed modulo \( n \).

### 2. The small cases

As stated in the introduction, this section will provide necessary and sufficient conditions for the existence of a 6-cycle system of \( K_n - E(F) \) for any spanning forest \( F \). The approach taken in this paper requires that we know the number of components in \( F \). This can be seen in Lemma 2.1, the result being summarized in Table 1.

#### Lemma 2.1

Let \( n \) be even and let \( F \) be a spanning forest of the complete graph \( K_n \) with \( c(F) \) components. The number of edges in \( K_n - E(F) \) is divisible by 6 if and only if \( n \) and \( c(F) \) are related as in Table 1.

#### Proof

Let \( C_1, C_2, C_3, \ldots, C_{c(F)} \) be the components of \( F \). Then

\[
|E(F)| = \sum_{i=1}^{c(F)} |E(C_i)| = \sum_{i=1}^{c(F)} (V(C_i) - 1) = \sum_{i=1}^{c(F)} V(C_i) - \sum_{i=1}^{c(F)} 1 = n - c(F).
\]

Therefore, \( c(F) = n - |E(F)| \).
Table 1
The number of components required in $F$ in order that 6 divides $|E(K_n - E(F))|$ when $n$ is even

<table>
<thead>
<tr>
<th>$n$</th>
<th>$12k$</th>
<th>$12k + 2$</th>
<th>$12k + 4$</th>
<th>$12k + 6$</th>
<th>$12k + 8$</th>
<th>$12k + 10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c(F)(\text{mod } 6)$</td>
<td>0</td>
<td>1</td>
<td>4</td>
<td>3</td>
<td>4</td>
<td>1</td>
</tr>
</tbody>
</table>

So if 6 divides $|E(K_n - E(F))|$ then $c(F)(\text{mod } 6) \equiv (n - |E(F)|)(\text{mod } 6) \equiv (n - (n^2 - n)/2)(\text{mod } 6)$.

Also, if $c(F)$ and $n$ are related as in Table 1 then $c(F) \equiv n - (n^2 - n)/2 (\text{mod } 6)$, and so $|E(F)| \equiv |E(K_n)| (\text{mod } 6)$. So the result follows. □

Another tool that we will need is from a theorem by Sotteau [11]. Sotteau proved a generalization of the following result. It is stated here for 6-cycles only.

**Lemma 2.2.** There exists a 6-cycle system of $K_{a,b}$ if and only if:

1. $a$ and $b$ are even,
2. 6 divides $a$ or $b$, and
3. $a, b \geq 4$.

The following lemma is a special case of the results in [1], but is easily proved so is included for completeness:

**Lemma 2.3.** Let $n \in \{2, 6, 8\}$, and let $F$ be a 1-factor in $K_n$. There exists a 6-cycle system of $K_n - E(F)$.

**Proof.** In each case, we define the 6-cycle system on the vertex set $\mathbb{Z}_n$.

1. If $n = 2$, let $B = \phi$ be the required 6-cycle system with leave $F = \{\{0, 1\}\}$.
2. If $n = 6$, let $B = \{(0, 2, 1, 4, 3, 5), (0, 4, 2, 5, 1, 3)\}$ be the required 6-cycle system with leave $F = \\{\{0, 1\}, \{2, 3\}, \{4, 5\}\}$.
3. If $n = 8$, let $B = \{(0, 1, 2, 3, 5, 7), (0, 2, 5, 6, 3, 4), (0, 3, 1, 7, 4, 6), (1, 5, 4, 2, 7, 6)\}$ be the required 6-cycle system with leave $F = \\{\{0, 5\}, \{7, 3\}, \{6, 2\}, \{1, 4\}\}$. □

**Lemma 2.4.** If $F$ is a spanning forest of $K_{10}$ in which each vertex has odd degree and for which 6 divides $|E(K_{10} - E(F))|$, then there exists a 6-cycle system of $K_{10} - E(F)$.

**Proof.** The maximum number of components possible in a subgraph of $K_{10}$ in which each vertex has odd degree is 5 (i.e. a 1-factor). So by Lemma 2.1 and Table 1, since we are given that 6 divides $|E(K_{10} - E(F))|$ it follows that $c(F) = 1$; so $F$ is a tree.

There are 7 possibilities for the leave $F$. For $1 \leq i \leq 7$, a 6-cycle system $(\mathbb{Z}_{10}, B_i)$ of $K_{10} - E(F_i)$ is given below, where $F_i$ is the tree induced by the edges occurring in no 6-cycle in $B_i$. 

---

<table>
<thead>
<tr>
<th>$i$</th>
<th>$B_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>${0, 1, 2, 3, 5, 7}$</td>
</tr>
<tr>
<td>2</td>
<td>${0, 2, 5, 6, 3, 4}$</td>
</tr>
<tr>
<td>3</td>
<td>${0, 3, 1, 7, 4, 6}$</td>
</tr>
<tr>
<td>4</td>
<td>${1, 5, 4, 2, 7, 6}$</td>
</tr>
<tr>
<td>5</td>
<td>${0, 5}$</td>
</tr>
<tr>
<td>6</td>
<td>${7, 3}$</td>
</tr>
<tr>
<td>7</td>
<td>${6, 2}$</td>
</tr>
<tr>
<td>8</td>
<td>${1, 4}$</td>
</tr>
</tbody>
</table>
Lemma 3.1. Let $B_1 = \{(1,4,5,8,2,9), (2,5,6,9,3,7), (3,6,4,7,1,8), (1,2,4,8,7,5), (2,3,5,9,8,6), (3,1,6,7,9,4)\}$, so $E(F_1) = \{\{0,1\}, \{0,2\}, \{0,3\}, \{0,4\}, \{0,5\}, \{0,6\}, \{0,7\}, \{0,8\}, \{0,9\}\}$. 

$B_2 = \{(1,4,5,8,2,9), (2,5,6,9,3,7), (3,6,4,0,1,8), (1,2,4,8,7,5), (2,3,5,9,8,6), (3,1,6,7,9,4)\}$, so $E(F_2) = \{\{0,2\}, \{0,3\}, \{0,5\}, \{0,6\}, \{0,7\}, \{0,8\}, \{0,9\}, \{7,4\}, \{7,1\}\}$. 

$B_3 = \{(1,4,5,8,2,9), (2,5,6,9,3,7), (3,6,4,0,1,8), (1,2,4,8,7,5), (2,3,5,9,8,6), (3,0,6,7,9,4)\}$, so $E(F_3) = \{\{0,2\}, \{0,5\}, \{0,7\}, \{0,8\}, \{0,9\}, \{7,4\}, \{7,1\}, \{1,3\}, \{1,6\}\}$. 

$B_4 = \{(1,4,5,8,2,9), (2,5,0,9,3,7), (3,6,4,0,1,8), (1,2,4,8,7,5), (2,3,5,9,8,6), (3,0,6,7,9,4)\}$, so $E(F_4) = \{\{0,2\}, \{0,7\}, \{0,8\}, \{7,4\}, \{7,1\}, \{1,3\}, \{1,6\}, \{6,5\}, \{6,9\}\}$. 

$B_5 = \{(1,4,5,8,2,9), (2,5,0,9,3,7), (3,6,4,0,1,8), (1,2,4,8,7,5), (2,3,5,9,8,6), (3,1,6,7,9,4)\}$, so $E(F_5) = \{\{0,2\}, \{0,3\}, \{0,6\}, \{0,7\}, \{0,8\}, \{7,4\}, \{7,1\}, \{6,5\}, \{6,9\}\}$. 

$B_6 = \{(1,4,5,8,2,9), (2,5,6,9,3,7), (3,6,4,0,1,8), (1,2,4,8,0,5), (2,3,5,9,8,6), (3,1,6,7,9,4)\}$, so $E(F_6) = \{\{0,2\}, \{0,3\}, \{0,6\}, \{0,7\}, \{0,9\}, \{7,4\}, \{7,1\}, \{7,5\}, \{7,8\}\}$. 

$B_7 = \{(1,4,5,8,2,9), (2,5,6,9,3,7), (3,6,4,0,1,8), (1,2,0,8,7,5), (2,3,5,9,8,6), (3,0,6,7,9,4)\}$, so $E(F_7) = \{\{0,5\}, \{0,7\}, \{0,9\}, \{7,4\}, \{7,1\}, \{1,3\}, \{1,6\}, \{4,2\}, \{4,8\}\}$. \hfill $\Box$

3. Some building blocks

In this section, we provide some 6-cycle systems of small graphs which will be used to build 6-cycle systems of $K_n - E(F)$ in Section 4. For the next six lemmas, let $G_{\gamma,\beta} = G_{\gamma,\beta}(V)$ be a graph labeled with the vertices in the sequence $V$ (the order in which the vertices are listed is important) and let $B$ be a 6-cycle system of $G_{\gamma,\beta}(V)$ ($\gamma$ and $\beta$ are indices that represent the $\beta$th graph defined in the $\gamma$th lemma of this section (i.e. in Lemma 3,7)). Let $G^c$ denote the complement of a graph $G$. Also let $G \vee H$ denote the join of two vertex disjoint graphs $G$ and $H$ (so $E(G \vee H) = E(G) \cup E(H) \cup \{\{u, v\}: u \in V(G), v \in V(H)\}$).

Lemma 3.1. Let $G_{1,1}(0,1,2,\ldots,11)$ be the graph $K_{10}^c \circ K_4 + \{0,2\}, \{1,2\} - \{0,11\}, \{1,10\}, \{2,9\}, \{2,8\}$ with $V(K_{10}^c) = \mathbb{Z}_8$ and $V(K_4) = \mathbb{Z}_{12} \setminus \mathbb{Z}_8$. There exists a 6-cycle system of $G_{1,1}$.

Proof. There exists a 6-cycle system $(\mathbb{Z}_{12}, B)$ of $G_{1,1}$ defined by $B = \{(0,2,1,8,9,10), (2,10,5,8,4,11), (0,8,6,10,11,9), (1,9,4,10,8,11), (3,10,7,9,6,11), (3,8,7,11,5,9)\}$. \hfill $\Box$

Lemma 3.2. Let $G_{2,1}(0,1,2,\ldots,7)$ be the graph $K_4^c \circ K_4 - \{0, i\} | 4 \leq i \leq 7\}$, and let $G_{2,2}(0,1,2,\ldots,7)$ be the graph $K_4^c \circ K_4 - \{i, 7 - 2i\}, \{i, 6 - 2i\} | i \in \mathbb{Z}_2\}$ with $V(K_{10}^c) = \mathbb{Z}_4$ and $V(K_4) = \mathbb{Z}_8 \setminus \mathbb{Z}_4$ in each case. There exists a 6-cycle system $B$ of $G_{2,\beta}$ for $1 \leq \beta \leq 2$. 

Proof. For $1 \leq \beta \leq 2$, there exists a 6-cycle system $(\mathbb{Z}_8, B_\beta)$ defined by $B_1 = \{(1, 7, 2, 4, 6, 5), (1, 6, 3, 7, 5, 4), (2, 6, 7, 4, 3, 5)\}$ and $B_2 = \{(0, 4, 6, 7, 3, 5), (1, 7, 4, 5, 2, 6), (2, 4, 3, 6, 5, 7)\}$. \qed

Lemma 3.3. Let $G_{3,1}(0, 1, 2, \ldots, 15)$ be the graph $K_6^c \oplus K_6 = \{(0, 3), \{1, 4\}, \{2, 4\}\} - \{(i, 15-i) \mid i \in \mathbb{Z}_5\}$, and let $G_{3,2}(0, 1, 2, \ldots, 15)$ be the graph $K_6^c \oplus K_6 = \{(0, 3), \{1, 3\}, \{2, 3\}\} - \{(i, 15-i) \mid i \in \mathbb{Z}_4\}$ with $V(K_6^c) = \mathbb{Z}_6$ and $V(K_6) = \mathbb{Z}_6$. There exists a 6-cycle system $B$ of $G_{3, \beta}$ for $1 \leq \beta \leq 2$. \qed

Proof. For $1 \leq \beta \leq 2$, there exists a 6-cycle system $(\mathbb{Z}_{16}, B_\beta)$ of $G_{3, \beta}$ defined by

$B_1 = \{(2, 4, 1, 12, 11, 10), (10, 3, 11, 15, 14, 0), (3, 0, 11, 7, 12, 13), (4, 13, 1, 15, 10, 12), (5, 10, 9, 14, 2, 11), (5, 15, 6, 11, 13, 14), (6, 13, 7, 15, 8, 12), (3, 15, 13, 10, 7, 14), (4, 15, 12, 9, 11, 14), (5, 13, 9, 15, 2, 12), (6, 14, 8, 11, 1, 10), (0, 13, 8, 10, 14, 12)\}$ and

$B_2 = \{(2, 3, 1, 12, 11, 10), (10, 4, 11, 15, 14, 0), (3, 0, 11, 7, 12, 13), (4, 13, 1, 15, 10, 12), (5, 10, 9, 14, 2, 11), (5, 15, 6, 11, 13, 14), (6, 13, 7, 15, 8, 12), (3, 15, 13, 10, 7, 14), (4, 15, 12, 9, 11, 14), (5, 13, 9, 15, 2, 12), (6, 14, 8, 11, 1, 10), (0, 13, 8, 10, 14, 12)\}$. \qed

Lemma 3.4. Let $G_{4,1}(0, 1, 2, \ldots, 15)$ be the graph $K_5^c \oplus K_8 = \{(0, i) \mid 8 \leq i \leq 15\}$, let $G_{4,2}(0, 1, 2, \ldots, 15) = K_5^c \oplus K_8 - \{(i, 15-i), (i, 14-4i), (i, 13-4i), (i, 12-4i) \mid i \in \mathbb{Z}_2\}$, let $G_{4,3}(0, 1, 2, \ldots, 15) = K_5^c \oplus K_8 - \{(i, 15-i), (i, 14-4i), (i, 13-4i), (i, 12-4i) \mid i \in \mathbb{Z}_2\}$, let $G_{4,4}(0, 1, 2, \ldots, 15) = K_5^c \oplus K_8 - (0, i), (1, j), (2, k) \mid 12 \leq i \leq 15, 10 \leq j \leq 11, 8 \leq k \leq 9\}$, and let $G_{4,5}(0, 1, 2, \ldots, 15) = K_5^c \oplus K_8 - \{(i, 15-2i), (i, 14-2i) \mid i \in \mathbb{Z}_4\}$, with $V(K_5^c) = \mathbb{Z}_8$ and $V(K_8) = \mathbb{Z}_8 \setminus \mathbb{Z}_9$ for each case. There exists a 6-cycle system $(\mathbb{Z}_{16}, B_\beta)$ of $G_{4, \beta}$ for $1 \leq \beta \leq 5$. \qed

Proof. Let $(\mathbb{Z}_{16}/\mathbb{Z}_7, B)$ be a 6-cycle system of $K_9$ (this is easy to do; or see [1], or see [8] for a survey). For $1 \leq \beta \leq 5$, there exists a 6-cycle system $(\mathbb{Z}_{16}, B_\beta)$ of $G_{4, \beta}$ defined by

$B_1 = B \cup \{(9, 1, 8, 6, 15, 4), (11, 1, 10, 5, 14, 4), (9, 2, 8, 5, 11, 6), (13, 3, 12, 6, 10, 4), (13, 6, 14, 3, 10, 2), (13, 5, 15, 2, 12, 1), (11, 3, 15, 1, 14, 2), (8, 4, 12, 5, 9, 3)\}$, $B_2 = B \cup \{(9, 0, 8, 6, 15, 4), (11, 1, 10, 5, 14, 4), (9, 2, 8, 5, 11, 6), (13, 3, 12, 6, 10, 4), (13, 6, 14, 3, 10, 2), (13, 5, 15, 2, 12, 1), (11, 3, 15, 1, 14, 2), (8, 4, 12, 5, 9, 3)\}$, $B_3 = B \cup \{(9, 0, 8, 6, 15, 4), (11, 0, 10, 5, 14, 4), (9, 2, 8, 5, 11, 6), (13, 3, 12, 6, 10, 4), (13, 6, 14, 3, 10, 2), (13, 5, 15, 2, 12, 1), (11, 3, 15, 1, 14, 2), (8, 4, 12, 5, 9, 3)\}$, $B_4 = B \cup \{(9, 0, 8, 6, 15, 4), (11, 0, 10, 5, 14, 4), (9, 1, 8, 5, 11, 6), (13, 3, 12, 6, 10, 4), (13, 6, 14, 3, 10, 2), (13, 5, 15, 2, 12, 1), (11, 3, 15, 1, 14, 2), (8, 4, 12, 5, 9, 3)\}$, and $B_5 = B \cup \{(9, 0, 8, 6, 15, 4), (11, 0, 10, 5, 14, 4), (9, 1, 8, 5, 11, 6), (13, 3, 15, 1, 14, 2), (8, 4, 12, 5, 9, 2)\}$. \qed

Lemma 3.5. Let $G_{5,1}(0, 1, 2, \ldots, 15)$ be the graph $K_6^c \oplus K_5 = \{(0, 15), \{0, 14\}, \{12, 14\}, \{13, 14\}, \{1, 11\}, \{1, 10\}, \{2, 9\}, \{2, 8\}\}$ and let $G_{5,2}(0, 1, 2, \ldots, 15)$ be the graph $K_6^c \oplus K_5 = \{(0, 15), \{0, 14\}, \{12, 14\}, \{13, 14\}, \{1, 11 - i\} \mid i \in \mathbb{Z}_4\}$ with $V(K_6^c) = \mathbb{Z}_8$ and $V(K_5) = \mathbb{Z}_8 \setminus \mathbb{Z}_9$. There exists a 6-cycle system $B$ of $G$ for $1 \leq \beta \leq 2$. \qed
exists a 6-cycle system \((\overline{Z_{16}}, B)\) of \(G_{5, \beta}\) defined by
\[B_1 = \{(8, 1, 9, 0, 11, 5), (1, 13, 6, 14, 5, 15), (3, 10, 2, 15, 6, 11), (3, 14, 7, 13, 8, 9), (4, 8, 0, 12, 7, 11), (4, 9, 5, 12, 15, 10), (4, 15, 13, 11, 8, 12), (4, 14, 9, 11, 10, 13), (13, 12, 11, 14, 15, 9), (1, 12, 10, 6, 8, 14), (3, 12, 6, 9, 7, 8), (3, 13, 5, 10, 7, 15), (2, 14, 10, 8, 15, 11), (2, 12, 9, 10, 0, 13)\} and
\[B_2 = \{(8, 2, 9, 0, 11, 5), (1, 13, 6, 14, 5, 15), (3, 10, 2, 15, 6, 11), (3, 14, 7, 13, 8, 9), (4, 8, 0, 12, 7, 11), (4, 9, 5, 12, 15, 10), (4, 15, 13, 11, 8, 12), (4, 14, 9, 11, 10, 13), (13, 12, 11, 14, 15, 9), (1, 12, 10, 6, 8, 14), (3, 12, 6, 9, 7, 8), (3, 13, 5, 10, 7, 15), (2, 14, 10, 8, 15, 11), (2, 12, 9, 10, 0, 13)\}.

**Lemma 3.6.** Let \(G_6(0, 1, 2, \ldots, 15)\) be the graph \(K_6 \setminus K_8\) be a spanning forest of \(G_6\). Therefore, there exists a 6-cycle system \((\overline{Z_{16}}, B)\) of \(G_6\) defined by \(B = \{(2, 14, 11, 10, 6, 12), (2, 15, 4, 10, 7, 13), (3, 15, 9, 6, 8, 12), (3, 14, 1, 8, 9, 13), (0, 11, 5, 9, 7, 12), (4, 9, 11, 8, 5, 13), (5, 10, 12, 13, 11, 15), (1, 9, 14, 15, 8, 13), (2, 10, 0, 9, 3, 11), (2, 9, 12, 5, 14, 8), (3, 8, 7, 15, 13, 10), (0, 8, 4, 11, 6, 13), (4, 14, 6, 15, 1, 12), (7, 11, 12, 15, 10, 14)\}.

**4. Main result**

Finally, we are ready to prove the main result of this paper. Let \(G[W]\) denote the subgraph of \(G\) induced by \(W\).

**Theorem 4.1.** Let \(F\) be a forest in the complete graph \(K_n\) with \(|E(F)| \geq 1\). There exists a 6-cycle system of \(G = K_n - E(F)\) if and only if

1. all vertices in \(F\) have odd degree,
2. \(|E(K_n - E(F))|\) is divisible by 6 and
3. \(n\) is even.

**Remark.** Note that condition (1) requires \(F\) to be a spanning forest of \(K_n\).

**Proof.** Suppose that there exists a 6-cycle system \((V, B)\) of \(G = K_n - E(F)\). Then for each \(v \in V\), the 6-cycles in \(B\) partition the edges incident with \(v\) into pairs, so \(d_G(v)\) is even. Since \(|E(F)| \geq 1\) and \(F\) is a forest, \(F\) contains at least one vertex, say \(w\), with \(d_F(w) = 1\), so \(d_G(w) = n - 2\). Therefore, \(n\) is even. Also, for each \(v \in V\), \(d_F(v) = (n - 1) - d_G(v)\) so \(d_F(v)\) is odd. Then clearly \(F\) spans \(K_n\). Since the 6-cycles in \(B\) partition the edges of \(G\), it follows that 6 must divide \(|E(K_n - F)|\).

To prove the sufficiency, first note that the necessary conditions prevent the possibility that \(n = 4\).

Next observe that if \(n \in \{2, 6, 8\}\) then conditions (1)–(3) require that \(F\) be a 1-factor. Lemma 2.3 provides such a 6-cycle system of \(K_n - E(F)\) for each \(n \in \{2, 6, 8\}\). If \(n = 10\), then conditions (1)–(3) require that \(F\) be a spanning tree, thus giving seven
possibilities for $F$, and each is considered in Lemma 2.4. Therefore we can now assume that $n \geq 12$.

The remaining cases are proved by induction. So now suppose that for each positive integer $x$ with $2 \leq x < n$ and for any forest $F'$ in $K_x$ that together satisfy:

- (1') all vertices in $F'$ have odd degree (so $F'$ is spanning),
- (2') $|E(K_n - E(F'))|$ is divisible by 6 and
- (3') $x$ is even,

there exists a 6-cycle system of $K_x - E(F')$. We can assume that $V(K_n) = \mathbb{Z}_n$. We obtain a 6-cycle system $(\mathbb{Z}_n, B)$ of $K_n - E(F)$ by considering several cases in turn: $F$ has at least three components which are isomorphic to $K_2$; $n = 6t$; $n = 6t+2$; $n = 12k+4$; and $n = 12k+10$ (3 subcases). We regularly make use of Table 1, since it is easier to find the number of components $c(F')$ in $F'$ than it is to check that condition (2') is satisfied.

**Case 1:** Suppose $F$ has three components isomorphic to $K_2$.

Let the vertex sets of these three components be $\{n - i, n - i - 1\}$, where $i \in \{1, 3, 5\}$. Let $F' = F[\mathbb{Z}_{n-6}]$ and let $x = n - 6$.

We must check to see that $F'$ and $x = n - 6$ satisfy conditions (1')--(3'). Since $F'$ is formed by removing the three components of $F$ isomorphic to $K_2$, $d_{F'}(v) = d_F(v)$ for each $v \in \mathbb{Z}_{n-6}$. So all vertices in $F'$ have odd degree, and thus (1') is satisfied. Also $c(F') = c(F) - 3$ and $|V(F')| = n - 6$. Since we assumed that 6 divides $|K_n - E(F)|$, by Table 1, we have that 6 divides $|E(K_{n-6} - E(F'))|$, so (2') is satisfied. Clearly, $n - 6$ is even since $n$ is even, so (3') is satisfied. We apply induction to obtain a 6-cycle system $(\mathbb{Z}_{n-6}, B_1)$ of $K_{n-6} - E(F')$.

Since $n \geq 12$, $n - 6 \geq 6$ and $|\mathbb{Z}_n \setminus \mathbb{Z}_{n-6}| = 6$, by Lemma 2.2 there exists a 6-cycle system $(\mathbb{Z}_n, B_2)$ of $K_{n-6,6}$ with bipartition $\{\mathbb{Z}_{n-6}, \mathbb{Z}_n \setminus \mathbb{Z}_{n-6}\}$ of the vertex set.

By Lemma 2.3, there exists a 6-cycle system $B_3$ of $K_6 - \{\{n - i, n - i - 1\}|i \in \{1, 3, 5\}\}$ defined on the vertex set $\mathbb{Z}_n \setminus \mathbb{Z}_{n-6}$. Then $(\mathbb{Z}_n, B_1 \cup B_2 \cup B_3)$ is a 6-cycle system of $K_n - E(F)$.

In view of Case 1, we can assume that in the remaining cases, $F$ contains at most 2 components that are isomorphic to $K_2$. The remaining cases depend on the congruence of $n$ (mod 12).

**Case 2:** Suppose $n = 6t$, so $n \geq 12$.

By Table 1, we know that $c(F) \equiv 0$ or 3 (mod 6) so in particular $c(F) \geq 3$. Let $C_0$, $C_1$, and $C_2$ be three components in $F$. In view of Case 1, we can assume that one of the components, say $C_2$, is not $K_2$.

For $0 \leq i \leq 2$, let the second vertex in a maximum length path $P_i \in C_i$ be named $i$; then vertex $i$ is adjacent to a vertex of degree 1 in $F$, namely the first vertex in $P_i$, which we call $n - i - 1$. Since we know that $C_2$ is not $K_2$, and since we know that each vertex in $F$ has odd degree, vertex $i = 2$ is incident with at least two additional edges in $F$. At least one of these edges is incident to a leaf (a leaf is a vertex of degree one) or else the path $P_2$ would not be maximal. Name this leaf $n - 4$ (see Fig. 1).
Now that we have selected 4 special vertices, we proceed as follows. Let $F'$ be formed from $F[\mathbb{Z}_{n-4}]$ and adding edges $\{0,2\}$ and $\{1,2\}$, and let $n = n - 4$. We check to see that conditions (1′–3′) are satisfied.

Since $d_{F'}(i) = d_F(i) - 1 + 1$ for $0 \leq i \leq 1$ and $d_{F'}(2) = d_F(2) - 2 + 2$, clearly $d_{F'}(i) = d_F(i)$ for $1 \leq i \leq n - 4$, so all vertices in $F'$ have odd degree, so (1′) is satisfied. Furthermore, the edges $\{0,2\}$ and $\{1,2\}$ connect the three components $C_0, C_1,$ and $C_2$ in $F$ to create a single component in $F'$, so we know that $c(F') = c(F) - 2$. Since $|V(F')| = n - 4$ and we are assuming 6 divides $|E(K_n - E(F))|$, by Table 1, 6 divides $|E(K_{n-4} - E(F'))|$. It follows that (2′) is satisfied. Clearly since $n$ is even, $n - 4$ is even, so (3′) is satisfied. So, we can apply induction to obtain a 6-cycle system $(\mathbb{Z}_{n-4}, B_1)$ of $K_{n-4} - E(F')$.

For convenience, if $n = 12$ then $B_2 = \emptyset$. If $n \geq 18$ then $n - 12 \geq 6 > 3$. Clearly $n - 12 = 6(t - 2)$ which is divisible by 6 (in particular we are interested in $t > 2$). So by Lemma 2.2, there exists a 6-cycle system $(\mathbb{Z}_n \setminus \mathbb{Z}_8, B_2)$ of $K_{n-12,4}$ with bipartition $\{\mathbb{Z}_{n-4} \setminus \mathbb{Z}_8, \mathbb{Z}_n \setminus \mathbb{Z}_{n-4}\}$ of the vertex set.

By Lemma 3.1, there exists a 6-cycle system $B_3$ of $G_1 = G_{1}(0,1,2,3,4,5,6,7,n-4,n-3,n-2,n-1)$. Recall from the definition of $G$ in Lemma 3.1 that $\{\{0,2\}, \{1,2\}\} \subseteq E(G)$. These two edges are in $F'$, so they do not occur in $F$ nor in any 6-cycle in $B_1$, so they are placed in a 6-cycle in $B_3$. Also, $\{\{0,n-1\}, \{1,n-2\}, \{2,n-3\}, \{2,n-4\}\} \cap E(B_3) = \emptyset$, which means those edges are not in a 6-cycle in $B_3$. This is good because these edges occur in $F$. Therefore $(\mathbb{Z}_n, B_1 \cup B_2 \cup B_3)$ is a 6-cycle system of $K_n - E(F)$.

**Case 3:** Suppose $n = 6t + 2$, so $n \geq 14$.

This proof requires carefully selecting four vertices (named $n - 4, n - 3, n - 2,$ and $n - 1$) in $F$, which we do for several cases. By Table 1, we know that $c(F) \equiv 1$ or 4 (mod 6).

Suppose $c(F) = 1$. If $F$ is a star centered at vertex say 0, then it has at least 13 leaves, so choose any 4 and call them $n - 1, n - 2, n - 3,$ and $n - 4$. If $F$ is not a star, let $P$ be a maximum length path in $F$. Since $n \geq 14$, and since $F$ is not a star, $P$ has length at least 3. Label the second vertex in $P$ with 0 and the second to last vertex in $P$ with 1. Since $P$ is maximal and each vertex in $F$ has odd degree, we know that vertex 0 is adjacent to at least two vertices of degree 1, call them $n - 1$ and $n - 2$,
and similarly vertex 1 is adjacent to at least two vertices of degree 1, call them \( n - 3 \) and \( n - 4 \).

Finally, suppose \( c(F) > 1 \). Then \( F \) has at least 4 components. Since at most 2 components are isomorphic to \( K_2 \), there are at least two components not isomorphic to \( K_2 \), call them \( C_0 \) and \( C_1 \). For \( 0 \leq i \leq 1 \), let vertex \( i \) be the second vertex in a maximum length path \( P_i \subseteq C_i \). Since \( P_i \) is a maximum length path, vertex \( i \) is adjacent to at least 2 vertices of degree 1 call them \( n - 2i - 1 \) and \( n - 2i - 2 \).

Now that we have selected four special vertices, we proceed as follows. Let \( F' = F[Z_{n-4}] \). Therefore, \( F' \) spans \( Z_{n-4} \). Since either

(i) \( d_{F'}(i) = d_F(i) - 4 \) for \( i = 0 \) and \( d_{F'}(i) = d_F(i) \) for \( 1 \leq i \leq n - 5 \) or
(ii) \( d_{F'}(i) = d_F(i) - 2 \) for \( 0 \leq i \leq 1 \) and \( d_{F'}(i) = d_F(i) \) for \( 2 \leq i \leq n - 5 \),

all vertices in \( F' \) have odd degree and thus (1') is satisfied. Since \( F' \) is formed from \( F \) by deleting vertices of degree 1, \( c(F') = c(F) \). Therefore, since \( |V(F')| = n - 4 \) and we are assuming 6 divides \( |E(K_n - E(F))| \), by Table 1, 6 divides \( |E(K_n - E(F'))| \) so (2') is satisfied. Clearly, since \( n \) is even, so is \( n - 4 \), so (3') is satisfied. We can apply induction to obtain a 6-cycle system \((Z_{n-4}, B_1)\) of the vertex set \( K_{n-4} - E(F') \).

Since \( n \geq 14, n - 8 \geq 6 \), clearly \( n - 8 = 6(t - 1) \) which is divisible by 6 (in particular, we are interested when \( t > 1 \)). By Lemma 2.2, there exists a 6-cycle system \((Z_n \setminus Z_4, B_2)\) of \( K_{n-8,4} \) with bipartition \( \{Z_{n-4} \setminus Z_4, Z_n \setminus Z_{n-4}\} \) of the vertex set.

By Lemma 3.2, there exists a 6-cycle system \( B_3 \) of \( G_{2,i} = G_{2,i}(0,1,2,3,n-4,n-3,n-2,n-1) \), where \( i = 1 \) when \( F \) is a star and \( i = 2 \) otherwise.

It follows from the definition of \( G_{2,1} \) that \( \{0,n-1\}, \{0,n-2\}, \{0,n-3\}, \{0,n-4\} \} \cap E(B_3) = \emptyset \) which is good because these edges occur in \( F \) when \( F \) is a star. Similarly, the definition of \( G_{2,2} \) ensures that \( \{0,n-1\}, \{0,n-2\}, \{1,n-3\}, \{1,n-4\} \} \cap E(B_3) = \emptyset \) which is good because these edges occur in \( F \) when \( F \) is not a star. Therefore \((Z_n,B_1 \cup B_2 \cup B_3)\) is a 6-cycle system of \( K_n - E(F) \).

Case 4: Suppose \( n = 12k + 4 \), so \( n \geq 16 \).

By Table 1, we know that \( c(F) \equiv 4 \pmod{6} \), so in particular \( c(F) \geq 4 \). Let \( C_0, C_1, C_2, \) and \( C_3 \) be components in \( F \). We also know that one of the components, say \( C_3 \), is not a \( K_2 \). For \( 0 \leq i \leq 3 \), let \( P_i \) be a maximum path in \( C_i \), and let \( n - i - 1 \) be the first vertex in \( P_i \) and let \( i \) be the second vertex in \( P_i \). If \( C_3 \) is a star then let \( n - 5 \) and \( n - 6 \) be two additional vertices of degree one adjacent to vertex 3. If \( C_3 \) is not a star then let vertex 3 be the second to last vertex on \( P_3 \). Since \( P_3 \) is maximal, vertex 4 is adjacent to at least 2 vertices of degree 1, call them \( n - 5 \) and \( n - 6 \) (see Fig. 2).

Now that we have selected 6 special vertices, namely \( n - 6, n - 5, n - 4, n - 3, n - 2, n - 1 \), we proceed as follows. Let \( F' \) be formed from \( F[Z_{n-6}] \) by adding edges

(i) \( \{0,3\}, \{1,4\} \) and \( \{2,4\} \) if \( C_3 \) is not a star and
(ii) \( \{0,3\}, \{1,3\} \) and \( \{2,3\} \) if \( C_3 \) is a star.

Clearly \( F' \) spans \( K_{n-6} \). Since either

(i) \( d_{F'}(i) = d_F(i) + 1 \) for \( 0 \leq i \leq 3 \) and \( d_{F'}(i) = d_F(i) + 2 \) for \( i = 4 \) or
(ii) \( d_{F'}(i) = d_F(i) + 1 \) for \( 0 \leq i \leq 2 \) and \( d_{F'}(i) = d_F(i) + 3 \) for \( i = 3 \),
all of the vertices in $F'$ have odd degree so (1') is satisfied. Furthermore, by connecting the four components $C_0$, $C_1$, $C_2$, and $C_3$ in $F$ to create a single component in $F'$, we know that $c(F') = c(F) - 3$. Therefore, since $c(F) \equiv 4 \mod 6$, we have $c(F') \equiv 1 \mod 6$. Since $|V(F')| = n - 6$ and we are assuming 6 divides $|E(K_n - E(F))|$ by Table 1, 6 divides $|E(K_{n-6} - E(F'))|$ so (2') is satisfied. Clearly, since $n$ is even, so is $n - 6$, so (3') is satisfied. Therefore, we can apply induction to obtain a 6-cycle system $(Z_{n-6}, B_1)$ of $K_{n-6} - E(F')$.

For convenience, if $n = 16$ then $B_2 = \emptyset$. If $n \geq 28$, then clearly $n - 16 = 6(2k - 2)$ which is divisible by 6. So, by Lemma 2.2, there exists a 6-cycle system $(Z_n \setminus Z_{10}, B_2)$ of $K_{n-16,6}$ with bipartition $\{Z_{n-6} \setminus Z_{10}, Z_n \setminus Z_{n-6}\}$ of the vertex set.

Let $\beta = 1$ if $C_3$ is not a star and $\beta = 2$ if $C_3$ is a star. By Lemma 3.3, there exists a 6-cycle system $B_3$ of $G_{3,\beta} = G_{3,\beta}(0,1,2,3,4,5,6,7,8,9,n - 6,n - 5,n - 4,n - 3,n - 2,n - 1)$. Notice that the edges of $G_{3,\beta}$ joining vertices in $Z_{n-6}$ are precisely the edges in $E(F') \setminus E(F)$, and do not occur in a 6-cycle in $B_1$ but do occur in a 6-cycle in $B_3$. Also, the edges joining a vertex in $Z_{n-6}$ to a vertex in $Z_n \setminus Z_{n-6}$ that occur in no 6-cycle in $B_3$ are precisely the edges in $E(F) \setminus E(F')$. Therefore, $(Z_n, B_1 \cup B_2 \cup B_3)$ is a 6-cycle system of $K_n - E(F)$.

**Case 5:** Suppose $n = 12k + 10$, so $n \geq 22$.

By Table 1, we know that $c(F) \equiv 1 \mod 6$. This proof requires the term leaf pair to be defined. A leaf pair is a set $Y$ of two vertices each of degree 1 in $F$ that have a common neighbor, $N(Y)$. We call $N(Y)$ the center of $Y$. Additionally, 2, 3, or 4 leaf pairs that have a common center are called a double, triple, or quadruple leaf pair respectively. This proof requires carefully selecting 8 vertices (named: $n - 8,n - 7,n - 6,n - 5,n - 4,n - 3,n - 2,$ and $n - 1$) in $F$ which we do for 3 subcases.

**Subcase 1:** Suppose $F$ has at least four disjoint leaf pairs, call them $\{n - (2i + 1),n - (2i + 2)\}$ for $0 \leq i \leq 3$.

Name each of the centers of the four leaf pairs with a vertex in $\{0,1,2,3\}$ in such a way that for $0 \leq i < j \leq 3$, vertex $i$ is the center of at least as many leaf pairs as vertex $j$. Naming the vertices in this manner gives rise to five possibilities for the set $L$ consisting of these leaf pairs (see Fig. 3):

1. $L$ is 1 quadruple leaf pair, in which case let $\beta = 1$,
2. $L$ is 1 leaf pair and a triple leaf pair, in which case let $\beta = 2$, 

Fig. 3. Naming the vertices when $n = 12k + 10$ (subcase 1).
(3) \( L \) is 2 double leaf pairs, in which case let \( \beta = 3 \),
(4) \( L \) is 2 leaf pairs and a double leaf pair, in which case let \( \beta = 4 \) and
(5) \( L \) is 4 leaf pairs, in which case let \( \beta = 5 \).

Now that we have selected 8 special vertices, we proceed as follows. Let \( F' = F'[Z_{n-8}] \). In each case, all \( n - 8 \) vertices in \( F' \) have odd degree so (1') is satisfied. Since \( F' \) is formed from \( F \) by deleting vertices of degree 1 in \( F \), \( c(F') = c(F) \). Since \( |V(F')|=n-8 \) and we are assuming 6 divides \( |E(K_n - E(F))| \), by Table 1, we see that 6 divides \( |E(K_{n-8} - E(F'))| \) so (2') is satisfied. Clearly, since \( n \) is even \( n-8 \) is even, so (3') is satisfied. So we can apply induction to obtain a 6-cycle system \((Z_{n-8}, B_1)\) of \( K_{n-8} - E(F') \).

Since \( n \geq 22 \), \( n - 16 \geq 6 \). Clearly \( n - 16 = 6(2k - 1) \) which is divisible by 6, so by Lemma 2.2 there exists a 6-cycle system \((Z_{n-8}, B_2)\) of \( K_{n-16,8} \) with bipartition \( \{Z_{n-8} \setminus Z_8, Z_n \setminus Z_{n-8}\} \) of \( Z_{n-8} \).

Finally, by Lemma 3.4, there exists a 6-cycle system \( B_3 \) of \( G_{4,b} \{0, 1, 2, 3, 4, 5, 6, n - 8, n - 7, n - 6, n - 5, n - 4, n - 3, n - 2, n - 1\} \). Notice the edges joining vertices in \( Z_{n-8} \) to vertices in \( Z_n \setminus Z_{n-8} \) that occur in no 6-cycle in \( B_3 \) are precisely the edges in \( E(F') \setminus E(F) \). Thus \( (Z_n, B_1 \cup B_2 \cup B_3) \) is a 6-cycle system in \( K_n - E(F) \).

Subcase 2: Notice that in Case 5, if \( c(F) > 1 \) then by Table 1 it is clear that \( c(F) \geq 7 \). Since \( F \) has at most two components that are isomorphic to \( K_2 \), if \( c(F) > 1 \) then there are at least 4 components each of which contains a leaf pair. So in view of subcase 1, we can now assume that \( F \) is a tree.

Suppose \( F \) has exactly three leaf pairs. Let \( P \) be a maximum length path in \( F \). We know the second and second to last vertices are the centers of leaf pairs. Since \( P \) has only one other leaf pair, all other internal vertices in \( P \), except possibly one, have degree 3 in \( F \) and are adjacent to a leaf. Since \( n \geq 22 \), \( P \) must have length at least 8. Therefore, by changing the direction of \( P \) if necessary, we can assure ourselves that the third leaf pair is at least as close to the end of \( P \) as it is to the beginning. Now let \( n - 2 \) be the second vertex of \( P \), and let the two vertices of degree 1 adjacent to it in \( P \) be \( n - 3 \) and \( n - 4 \). Now let 0 be the third vertex of \( P \) and call the leaf adjacent to it \( n - 1 \). Let 1 be the second to last vertex of \( P \), and let the two vertices of degree 1 adjacent to it in \( P \) be \( n - 5 \) and \( n - 6 \). Finally, call the last leaf pair \( n - 7 \) and \( n - 8 \) and name its center 2 if it is not vertex 1 (see Fig. 4). If vertex 1 is a single or double leaf pair let \( \beta = 1 \) or 2, respectively.

Now that we have selected at least 10 special vertices, we proceed as follows. Let \( F' = F[Z_{n-8}] \). Therefore, \( F' \) spans \( Z_{n-8} \). Since, when \( \beta = 1 \), \( d_{F'}(i) = d_F(i) - 2 \) for \( i \in Z_3 \) and \( d_{F'}(i) = d_F(i) \) for \( 3 \leq i \leq n - 9 \) or if \( \beta = 2 \), \( d_{F'}(0) = d_F(0) - 2 \), \( d_{F'}(1) = d_F(1) - 4 \) and \( d_{F'}(i) = d_F(i) \) for \( 2 \leq i \leq n - 9 \), then all vertices in \( F' \) have odd degree so (1') is satisfied. Since \( F' \) is formed from \( F \) by deleting vertex \( n - 2 \) and its leaf pair (each of degree 1 in \( F \)), \( c(F') = c(F) \). Since \( |V(F')|=n-8 \) and we are assuming 6 divides \( |E(K_{n-8} - E(F'))| \), by Table 1, 6 divides \( |E(K_{n-8} - E(F'))| \), so (2') is satisfied. Clearly since \( n \) is even we know \( n - 8 \) is even, so (3') is satisfied. We can apply induction to obtain a 6-cycle system \((Z_{n-8}, B_1)\) of \( K_{n-8} - E(F') \).

Since \( n \geq 22 \), \( n - 16 \geq 6 \). Clearly \( n - 16 = 6(2k - 1) \) which is divisible by 6, so by Lemma 2.2, there exists a 6-cycle system \((Z_{n-8}, B_2)\) of \( K_{n-16,8} \) with bipartition \( \{Z_n \setminus Z_{n-8}, Z_{n-8} \setminus Z_8\} \) of the vertex set.
By Lemma 3.5, there exists a 6-cycle system $B_3$ of $G_{5,7}(0,1,2,3,4,5,6,7,n-8,n-7,n-6,n-5,n-4,n-3,n-2,n-1)$. Notice the edges joining a vertex in $Z_{n-8}$ to a vertex in $Z_n \setminus Z_{n-8}$ that occur in no 6-cycle in $B_3$ are precisely the edges in $E(F) \setminus E(F')$. Then $(Z_n,B_1 \cup B_2 \cup B_3)$ is a 6-cycle system of $K_n - E(F)$.

Subcase 3: Suppose $F$ has exactly two leaf pairs. Let $P$ be a maximum length path in $F$. We know the second and second to last vertices are the centers of the two leaf pairs. Now let $n-2$ be the second vertex of $P$ and call its adjacent leaves $n-3$ and $n-4$, call the third vertex 0, and call the leaf adjacent to it $n-1$. Similarly, let $n-6$ be the second to last vertex of $P$ and call its adjacent leaves $n-7$ and $n-8$, call the third to last vertex 1 and call the leaf adjacent to it $n-5$, these vertices are all distinct since $n \geq 22$ (See Fig. 5).

Now that we have selected 8 special vertices, we proceed as follows. Let $F' = F[Z_{n-8}]$. Therefore, $F'$ spans $Z_n \setminus Z_{n-8}$. Since $d_{F'}(i) = d_F(i) - 2$ for $0 \leq i \leq 1$ and $d_{F'}(i) = d_F(i)$ for $2 \leq i \leq n-9$, then all vertices in $F'$ have odd degree so (1') is satisfied. Since $F'$ is formed from deleting vertex $n-6$ and vertex $n-2$ whose leaf pairs have degree 1 and deleting $n-1$ and $n-2$ in $F$, $c(F') = c(F)$. Since $|V(F')| = n-8$ and we are assuming 6 divides $|E(K_{n-8} - E(F'))|$, so by Table 1, 6 divides $|E(K_{n-8} - E(F'))|$ so (2') is satisfied. Clearly since $n$ is even $n-8$ is even so (3') is satisfied. We can apply induction to obtain a 6-cycle system $(Z_{n-8},B_1)$ of $K_{n-8} - E(F')$.

Since $n \geq 22, n-16 \geq 6$. Clearly $n-16 = 6(2k-1)$ which is divisible by 6, so by Lemma 2.2, there exists a 6-cycle system $(Z_{n-8},B_2)$ of $K_{n-16,8}$ with bipartition $\{Z_n \setminus Z_{n-8}, Z_{n-8} \setminus Z_8\}$ of the $K_{n-16,18}$ vertex set.

By Lemma 3.6, there exists a 6-cycle system $B_3$ of $G_6 = G_6(0,1,2,3,4,5,6,7,n-8,n-7,n-6,n-5,n-4,n-3,n-2,n-1)$. Notice the edges joining a vertex in
Fig. 5. Naming the vertices when $n = 12k + 10$ (subcase 3).

$Z_{n-8}$ to a vertex in $Z_n \setminus Z_{n-8}$ that occur in no 6-cycle in $B_3$ are precisely the edges in $E(F) \setminus E(F')$. Then $(Z_n, B_1 \cup B_2 \cup B_3)$ is a 6-cycle system of $K_n - E(F)$.

This completes the proof. \hfill \Box

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