The super-connected property of recursive circulant graphs

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Abstract

In a graph G, a k-container Ck(u,v) is a set of k disjoint paths joining u and v. A k-container Ck(u,v) is k*-container if every vertex of G is passed by some path in Ck(u,v). A graph G is k*-connected if there exists a k*-container between any two vertices. An m-regular graph G is super-connected if G is k*-connected for any k with 1 ≤ k ≤ m. In this paper, we prove that the recursive circulant graphs G(2m, 4), proposed by Park and Chwa [Theoret. Comput. Sci. 244 (2000) 35–62], are super-connected if and only if m ≠ 2.

Keywords: Super-connected; Container; Recursive circulant; Interconnection networks

1. Introduction

For the graph definitions and notations we follow Bondy and Murty [2]. G = (V, E) is a graph if V is a finite set and E is a subset of \{(a, b) | (a, b) is an unordered pair of V\}. We say that V is the vertex set and E is the edge set. Two vertices a and b are adjacent if (a, b) ∈ E. A path of length k from x to y is a finite set of distinct vertices \(\langle v_0, v_1, v_2, \ldots, v_k \rangle\), where \(x = v_0, y = v_k, (v_{i+1}, v_i) \in E\) for all \(1 ≤ i ≤ k\). For convenience, we use the sequence \(\langle v_0, \ldots, v_i, Q, v_j, \ldots, v_k \rangle\), where \(Q = \langle v_i, v_{i+1}, \ldots, v_j \rangle\) to denote the path \(\langle v_0, v_1, v_2, \ldots, v_k \rangle\). Note that we allow Q to be a path of length zero. Let P be the path \(\langle v_0, v_1, \ldots, v_{k-1}, v_k \rangle\). We say that the vertex \(v_i, 0 ≤ i ≤ k\), is passed by the path P. We use \(P^{-1}\) to denote the path \(\langle v_k, v_{k-1}, \ldots, v_1, v_0 \rangle\). In particular, let l(P) denote the length of the path P. The distance between u and v in G, denoted by \(d(u, v)\), is the length of the shortest path joining u and v. A path is a Hamiltonian path if its vertices are distinct and span V. A graph, G, is Hamiltonian connected if there exists a Hamiltonian path joining any two vertices of G. A cycle is a path (except the first vertex is the same as the last vertex) that contains at least three vertices. A Hamiltonian cycle of G is a cycle that traverses every vertex of G exactly once. A graph is Hamiltonian if it has a Hamiltonian cycle.

A circulant graph can be defined as follows. Let n be a positive integer and let \(S = \{k_1, k_2, \ldots, k_r\}\) with \(1 ≤ k_1 < k_2 < \cdots < k_r ≤ n/2\). Then the vertex set of
the circulant graph \((G, S)\) is \([0, 1, \ldots, n - 1]\) and the set of neighbors of the vertex \(u\) is \([u + k_j \mod n | j = 1, \ldots, r]\). The graph we deal with here is the circulant graph \(G(2^m, 4)\) proposed by Park and Chwa [7]. This family belongs to the family of circulant graphs denoted by \(G(N, d)\) with \(N, d \in \mathbb{N}\). The vertex set of \(G(N, d)\) is \([0, 1, \ldots, N - 1]\). Two vertices, \(u\) and \(v\), are adjacent if and only if \(u \pm d^i \equiv v \mod (N)\) for some \(i\) with \(0 \leq i \leq \lceil \log_d N \rceil - 1\). For example, \(G(8, 4)\) and \(G(16, 4)\), as shown in Fig. 1. Several interesting properties of \(G(2^m, 4)\) have been studied in the literature [3, 6–8]. For example, it was proved by Park and Chwa [7] that \(G(2^m, 4)\) is an \(m\)-connected graph. The embedding of meshes and hypercubes are studied in Park and Chwa [7]. The embedding of trees are studied by Lim et al. [3]. The Hamiltonian decomposable property is studied by Micheneau [6].

A \(k\)-container \(C_k(u, v)\) is a set of \(k\) disjoint paths joining \(u\) and \(v\). The connectivity of \(G, k(G)\), is the minimum number of vertices whose removal leaves the remaining graph disconnected or trivial. When \(G\) is a graph with \(k(G) \geq k\), it follows from Menger’s theorem [5] that there is a \(k\)-container between any two different vertices of \(G\). In this paper, we are interested in another type of container. A \(k\)-container \(C_k(u, v)\) is a \(k^*\)-container if every vertex of \(G\) is passed by some path in \(C_k(u, v)\). A graph \(G\) is \(k^*\)-connected if there exists a \(k^*\)-container between any two vertices. In particular, \(G\) is \(1^*\)-connected if and only if \(G\) is Hamiltonian connected, and \(G\) is \(2^*\)-connected if and only if \(G\) is Hamiltonian. Obviously, all \(1^*\)-connected graphs, except \(K_1\) and \(K_2\), are \(2^*\)-connected. The study of \(k^*\)-connected graphs is motivated by the globally \(3^*\)-connected graphs proposed by Albert, Aldred and Holton [1]. We say a \(k\)-regular graph is super-connected if it is \(i^*\)-connected for all \(1 \leq i \leq k\).

Lin et al. [4] prove that the pancake graph \(P_n\) is super-connected if and only if \(n > 3\). In this paper, we prove that \(G(2^m, 4)\) is super-connected if and only if \(m \neq 2\).

Hypercubes are one of the most popular interconnection networks being used. A hypercube \(Q_m\) is a graph with \(2^m\) vertices. Two vertices in \(Q_m\) are joined by an edge if and only if their binary representations differ in exactly one bit position. The number of vertices of \(G(2^m, 4)\) is \(2^m\), which is equal to that of \(Q_m\). The connectivity of \(G(2^m, 4)\) is \(m\), which is the best possible. The diameter of \(G(2^m, 4)\) is less than that of \(Q_m\). \(G(2^m, 4)\) has good fault-tolerant Hamiltonian properties [8]. The super-connected property of \(G(2^m, 4)\) is important in such a sense that it can be considered as a measure of the reliability of \(G(2^m, 4)\).

In Section 2, we give some basic properties of \(G(2^m, 4)\). Then in Section 3, we discuss the super-connected property of \(G(2^m, 4)\). Finally, conclusions are given in Section 4.

2. Basic properties

For \(0 \leq i < 2^m\), let \(f_i\) be the function from \(V(G(2^m, 4))\) into itself defined by \(f_i(x) \equiv (x + i) \mod (2^m)\). It is easy to see that \(f_i\) is an automorphism of \(G(2^m, 4)\). Similarly, let \(g\) be the function from \(V(G(2^m, 4))\) into itself defined by \(g(x) \equiv -x \mod (2^m)\). Again, \(g\) is an automorphism of \(G(2^m, 4)\). Let \(h_i\) be the function from \(V(G(2^m, 4))\) into itself defined by \(h_i(x) \equiv (x - i) \mod (2^m)\) for all \(0 \leq i < 2^m\). Similarly, \(h_i\) is an automorphism of \(G(2^m, 4)\). Thus, \(G(2^m, 4)\) is vertex transitive. Micheneau [6] also pointed out that \(G(2^m, 4)\) has the following recursive property: For \(0 \leq j \leq 3\), let \(G_j\) be the subgraph of \(G(2^m, 4)\) induced by vertices \(\{v | v \equiv j \mod (4^j)\}\). The edge set \(R\) in \(E(G(2^m, 4))\), but not in \(E(G_0) \cup E(G_1) \cup E(G_2) \cup E(G_3)\), is \(\{(i, i + 1) \mod (2^m) | 0 \leq i < 2^m - 1\}\). Thus, \(R\) forms a Hamiltonian cycles of \(G(2^m, 4)\). Moreover, each \(G_j\) is isomorphic to \(G(2^{m-2}, 4)\). We have the following theorems.

Theorem 1 [7]. The diameter of \(G(2^m, 4)\) is \([(3m - 1)/4]\).
Theorem 2. Assume that $F$ is a subset of $V(G(2^m,4)) \cup E(G(2^m,4))$. Then $G(2^m,4) - F$ is Hamiltonian if $|F| \leq m - 2$ and $G(2^m,4) - F$ is Hamiltonian connected if $|F| \leq m - 3$, where $m \geq 3$.

Therefore, we have the following corollary.

Corollary 1. $G(2^m,4)$ are both 1*-connected and 2*-connected if $m \geq 3$.

Corollary 2. Let $(x, y)$ be any edge of $G(2^m,4)$ with $m \geq 3$. Then there are two Hamiltonian cycles $C_1$ and $C_2$ of $G(2^m,4)$ such that $(x, y) \in E(C_1)$ and $(x, y) \notin E(C_2)$.

Lemma 1. Assume that $x$ and $y$ are any two different vertices of $G(2^m,4)$ with $m \geq 3$. Then there exists a 3*-container $C_3(x, y) = \{P_1, P_2, P_3\}$ joining $x$ and $y$ such that $P_1$ is a shortest path between $x$ and $y$. Hence, $G(2^m,4)$ is 3*-connected if $m \geq 3$.

Proof. Since $G(2^m,4)$ is vertex transitive, we only need to find a desired 3*-container between vertex 0 and any vertex $v$ of $G(2^m,4)$ with $v \neq 0$. Let $P_1$ be a shortest path joining 0 and $v$. By Theorem 1, $l(P_1) \leq \lceil(3m - 1)/4\rceil$. We may write $P_1$ as $(0, x_1, x_2, \ldots, x_k, x)$, where $\lceil(3m - 1)/4\rceil \leq m - 1$ for $m \geq 3$, $k \leq m - 2$. Therefore by Theorem 2, there exists a Hamiltonian cycle $C$ of $G(2^m,4)$ - $\{x_i \mid 1 \leq i \leq k\}$. Clearly, $C$ can be written as $(0, P_2, x, (P_3)^{-1}, 0)$. Accordingly, $P_1$, $P_2$, and $P_3$ form a 3*-container joining 0 and $v$. Therefore, $G(2^m,4)$ is 3*-connected.

Lemma 2. Let $x$ and $y$ be any two different vertices of $G(16,4)$. Then there exists a 4*-container $C_4(x, y) = \{P_1, P_2, P_3, P_4\}$ joining $x$ and $y$. In particular, $P_1 = \langle x, y \rangle$ if $x$ and $y$ are adjacent.

Proof. Since $G(2^m,4)$ is vertex transitive, we only need to find a desired 4*-container between vertex 0 and any vertex $x$ of $G(2^m,4)$ with $x \neq 0$. We list this 4*-container in Table 1.

The lemma is proved completely.

3. Super-connected property

Lemma 3. Let $x$ and $y$ be two adjacent vertices in $G(2^m,4)$ with $m \geq 3$ and $k$ be an integer with $2 \leq k \leq m$. Then there exists a $k^*$-container $C_k(x, y) = \{P_1, P_2, \ldots, P_k\}$ of $G(2^m,4)$ such that $P_1 = \langle x, y \rangle$.

Proof. We prove this lemma by induction on $m$. With Corollary 2, the lemma is true for any $m \geq 3$ and $k = 2$. With Lemma 1, the lemma is true for any $m \geq 3$ and $k = 3$. With Lemma 2, the lemma is true for $m = 4$ and $k = 4$. Assume that the lemma holds for any $G(2^t,4)$ with $t < m$. We only need to consider the case $m \geq 5$ and $4 \leq k \leq m$. Since $G(2^m,4)$ is vertex transitive, we only need to find a desired $k^*$-container of $G(2^m,4)$ between vertex 0 and any neighbor $x$ for $4 \leq k \leq m$. Since the function $g$ is an automorphism of $G(2^m,4)$, we have the following cases: (1) $x = 1$ and $(2) x = 4^t \pmod{2^m}$ for all $1 \leq i \leq \lceil m/2 \rceil - 1$.

Case 1: $x = 1$. By induction, there is a $(k - 2)^*$-container $\{Q_1, Q_2, \ldots, Q_{k-2}\}$ of $G_0$ between 0 and 0 such that $Q_1 = (0, 4)$. Obviously, $\ell(Q_1) \geq 2$ for $2 \leq i \leq k - 2$. Thus, we can write $Q_i$ as $(0, R_i, b_i, 4)$ with $b_i \notin \{0, 4\}$ for $2 \leq i \leq k - 2$. Let $\{f_1(Q_1), f_1(Q_2), \ldots, f_1(Q_{k-2})\}$ be the image of $\{Q_1, Q_2, \ldots, Q_{k-2}\}$ under the function $f_1$. Thus, $\{f_1(Q_1), f_1(Q_2), \ldots, f_1(Q_{k-2})\}$ forms a $(k-2)^*$-container of $G_1$ between 1 and 5. Since there are $2^m - 2$ vertices in $G_2$ and $m \geq 4$, $|V(G_2)| \geq 4$. Then there is a vertex $y$ in $G_2$ such that $y \neq 2$ and $y \neq 2^m - 2$. By Theorem 2, there exists a Hamiltonian path $S_2$ of $G_2$ joining $y$ to 2, and there exists a Hamiltonian path $S_3$ of $G_3$ joining $2^m - 1$ to $y + 1$. We set $P_1 = \{\langle 0, 1 \rangle, \langle 0, R_i, b_i + 1, (f_1(R_i))^{-1}, 1 \rangle\}$ for $2 \leq i \leq k - 2$.

Thus, $\{P_1, P_2, \ldots, P_k\}$ forms a desired $k^*$-container of $G(2^m,4)$ between 0 and $x$.

Case 2: $x = 4^t \pmod{2^m}$ for all $1 \leq i \leq \lceil m/2 \rceil - 1$. Thus $x \in V(G_0)$. By induction, there is a $(k - 2)^*$-container $\{P_1, P_2, \ldots, P_{k-2}\}$ of $G_0$ between 0 and 0 such that $P_1 = (0, x)$. Since $x \neq 0, x + 1 \neq 1$ and $x - 1 \neq 2^m - 1 \pmod{2^m}$. Since $G_1$ is isomorphic to $G(2^{m-2}, 4)$ for all $0 \leq i \leq 3$, by Theorem 2, there exists a Hamiltonian path $Q_1$ of $G_1$, joining 1 to $x + 1$; and there exists a Hamiltonian path $Q_2$ of $G_3$, joining $2^m - 1$ to $x - 1$. We rewrite $Q_2$ as $\langle 2^m - 1, 0, i, x - 1 \rangle$. Therefore, $t - 1$ and $x - 2$ are two
distinct vertices in $G_2$. By Theorem 2, there exists a Hamiltonian path $Q_3$ of $G_2$, joining $t - 1$ to $x - 2$. Consequently, we set $P_{k-1}$ as $(0, 1, Q_1, x + 1, x)$ and $P_k$ as $(0, 2^m - 1, S, t, t - 1, Q_3, x - 2, x - 1, x)$. Thus, $\{P_1, P_2, \ldots, P_k\}$ forms a $k^*$-container of $G(2^m, 4)$ between 0 and $x$. □

**Theorem 3.** $G(2^m, 4)$ is super-connected if and only if $m \neq 2$.

**Proof.** It is easy to see that $G(2^m, 4)$ is isomorphic to $K_2$ if $m = 1$ and $G(2^m, 4)$ is isomorphic to $C_4$ if $m = 2$. Clearly, $G(2^1, 4)$ is super-connected. However, $C_4$ is not Hamiltonian connected. Hence, $G(2^2, 4)$ is not super-connected. Now, by induction we prove that $G(2^m, 4)$ is super-connected for $m \geq 3$. With Corollary 1 and Lemma 1, $G(2^3, 4)$ is super-connected. With Corollary 1, Lemma 1, and Lemma 2, $G(2^4, 4)$ is super-connected. Assume that $G(2^m, 4)$ is super-connected for any $n$ with $3 \leq n < m$ with $m \geq 5$. By Corollary 1 and Lemma 1, $G(2^m, 4)$ is $k^*$-connected with $k = 1, 2$, and 3. Assume that $4 \leq k \leq m$. By Lemma 3, if $x$ and $y$ are adjacent then there exists a $k^*$-container $C_k(x, y) = \{P_1, P_2, \ldots, P_k\}$ of $G(2^m, 4)$. Consequently, we need to find a $k^*$-container between any two nonadjacent vertices of $G(2^m, 4)$ for $4 \leq k \leq m$.

Since $G(2^m, 4)$ is vertex transitive, we only need to find a $k^*$-container between 0 and $x$ with $x \neq 0$, $x$ is not adjacent to 0, and $4 \leq k \leq m$. We have the following five cases: (1) $x \equiv 0 \pmod 4$ and $x \neq \pm 4^l \pmod {2^m}$ for all $1 \leq l \leq \lceil m/2 \rceil$, (2) $x \equiv \pm 1 \pmod 4$, $x \neq 1$, and $x \neq 2^m - 1$, (3) $x = 2$ or $x = 2^m - 2$, (4) $x \equiv \pm 2^4 \pmod {2^m}$ and $x \neq 2^m - 2$ for all $1 \leq l \leq \lceil m/2 \rceil - 1$, and (5) $x \equiv 2 \pmod 4$ and $x \neq \pm 2^4 \pmod {2^m}$ for all $1 \leq l \leq \lceil m/2 \rceil$.

**Case 1:** $x \equiv 0 \pmod 4$ and $x \neq \pm 4^l \pmod {2^m}$ for all $1 \leq l \leq \lceil m/2 \rceil$. Thus $x \in V(G_0)$. By induction, there is a $(k-2)^*$-container $\{P_1, P_2, \ldots, P_{k-2}\}$ of $G_0$ between 0 and $x$. Since $x \neq 0$, $x + 1 \neq 1$ and $x - 1 \neq 2^m - 1 \pmod {2^m}$. Note that $G_t$ is isomorphic to $G(2^{m-2}, 4)$ for all $0 \leq i \leq 3$. By Theorem 2, there exists a Hamiltonian path $Q_1$ of $G_1$ joining 1 to $x + 1$ and there exists a Hamiltonian path $Q_2$ of $G_3$ joining $2^m - 1$ to $x - 1$. We write $Q_2$ as $(2^m - 1, S, t, x - 1)$. Therefore, $t - 1$ and $x - 2$ are two distinct vertices in $G_2$. By Theorem 2, there exists a Hamiltonian path $Q_3$ of $G_2$ joining $t - 1$ to $x - 2$. We set $P_{k-1}$ as $(0, 1, Q_1, x + 1, x)$ and $P_k$ as $(0, 2^m - 1, S, t, t - 1, Q_3, x - 2, x - 1, x)$. Thus, $\{P_1, P_2, \ldots, P_k\}$ forms a $k^*$-container of $G(2^m, 4)$ between 0 and $x$.

**Case 2:** $x \equiv \pm 1 \pmod 4$, $x \neq 1$, and $x \neq 2^m - 1$. Thus, $x \in V(G_1)$ or $x \in V(G_3)$. Since the function $g$ is an automorphism of $G(2^m, 4)$, we may assume that $x \in V(G_1)$. Thus, $x - 1 \neq 0$. By induction, there exists a $(k-2)^*$-container $\{P_1, P_2, \ldots, P_{k-2}\}$ of $G_0$ between 0 and $x - 1$. Without loss of generality, we assume that $l(P_1) \leq l(P_i)$ for all $2 \leq i \leq k - 2$. Hence, $l(P_1) \geq 2$ for $2 \leq i \leq k - 2$. Thus, we can write $P_1$ as $(0, R_1, b_1, x - 1)$ for $1 \leq i \leq k - 2$. Note that $l(R_1) = 0$ if $b_1 = 0$.

### Table 1

<table>
<thead>
<tr>
<th>$x$</th>
<th>$4^*$-container $C_4(0, x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$(0, 1), (0, 4, 3, 2, 1), (0, 15, 14, 13, 1), (0, 12, 11, 10, 9, 8, 7, 6, 5, 1)$</td>
</tr>
<tr>
<td>2</td>
<td>$(0, 1, 2), (0, 15, 3, 2), (0, 4, 5, 6, 2), (0, 12, 8, 7, 11, 10, 9, 13, 14, 2)$</td>
</tr>
<tr>
<td>3</td>
<td>$(0, 4, 3), (0, 15, 3), (0, 1, 13, 14, 2, 3), (0, 12, 11, 10, 6, 5, 9, 8, 7, 3)$</td>
</tr>
<tr>
<td>4</td>
<td>$(0, 4), (0, 1, 5, 4), (0, 15, 3, 4), (0, 12, 11, 10, 9, 13, 14, 2, 6, 7, 8, 4)$</td>
</tr>
<tr>
<td>5</td>
<td>$(0, 4, 5), (0, 1, 5, 5), (0, 15, 3, 2, 6, 5), (0, 12, 13, 14, 10, 11, 7, 8, 9, 5)$</td>
</tr>
<tr>
<td>6</td>
<td>$(0, 1, 5, 6), (0, 4, 3, 2, 6), (0, 12, 11, 10, 6), (0, 15, 14, 13, 9, 8, 7, 6)$</td>
</tr>
<tr>
<td>7</td>
<td>$(0, 15, 11, 7), (0, 12, 8, 7), (0, 1, 2, 3, 7), (0, 4, 5, 9, 13, 14, 10, 6, 7)$</td>
</tr>
<tr>
<td>8</td>
<td>$(0, 4, 8), (0, 12, 8), (0, 15, 11, 10, 6, 5, 9, 8), (0, 1, 13, 14, 2, 3, 7, 8)$</td>
</tr>
<tr>
<td>9</td>
<td>$(0, 1, 13, 9), (0, 12, 8, 9), (0, 4, 3, 7, 11, 10, 9), (0, 15, 14, 2, 6, 5, 9)$</td>
</tr>
<tr>
<td>10</td>
<td>$(0, 15, 11, 10), (0, 4, 5, 6, 10), (0, 12, 13, 14, 10), (0, 1, 2, 3, 7, 8, 9, 10)$</td>
</tr>
<tr>
<td>11</td>
<td>$(0, 12, 11), (0, 15, 11), (0, 1, 13, 14, 10, 11), (0, 4, 3, 2, 6, 5, 9, 8, 7, 11)$</td>
</tr>
<tr>
<td>12</td>
<td>$(0, 12), (0, 4, 8, 12), (0, 15, 14, 13, 12), (0, 1, 2, 3, 7, 6, 5, 9, 10, 11, 12)$</td>
</tr>
<tr>
<td>13</td>
<td>$(0, 12, 13), (0, 1, 13), (0, 15, 11, 10, 14, 13), (0, 4, 5, 6, 2, 3, 7, 8, 9, 13)$</td>
</tr>
<tr>
<td>14</td>
<td>$(0, 15, 14), (0, 12, 11, 10, 14), (0, 4, 3, 2, 14), (0, 1, 5, 6, 7, 8, 9, 13, 14)$</td>
</tr>
<tr>
<td>15</td>
<td>$(0, 15), (0, 1, 2, 3, 15), (0, 12, 13, 14, 15), (0, 4, 5, 6, 7, 8, 9, 10, 11, 15)$</td>
</tr>
</tbody>
</table>
Obviously, \( b_j + 1 \) is a neighborhood of \( x \) for \( 1 \leq i \leq k - 2 \). Let \( B = \{ (x, x \pm 4^i \pmod{2^m}) \mid 1 \leq i \leq [m/2] - 1 \} \) and \( x \pm 4^i \neq b_j + 1 \pmod{2^m} \) for all \( 1 \leq j \leq k - 2 \). We set \( F_1 \) to be the union of \( B \) and the set \( \{ a_i \mid 3 \leq i \leq k - 2 \} \) and \( a_1 = b_1 + 1 \). Clearly, \( |F_1| = m - 4 \) and the only neighbors of \( x \) in \( G_1 - F_1 \) are \( a_1 \) and \( a_2 \). By Theorem 2, there exists a Hamiltonian cycle \( C \) of \( G_1 - F_1 \). We can write \( C \) as \( (x, a_1, S_1, 1, S_2, a_2, x) \). Without loss of generality, we may assume that \( l(S_1) \leq l(S_2) \). Since the number of vertices in \( G_1 - F_1 \) are \( 2^{m-2} - k + 4 \) with \( k \leq m \), \( l(C) \geq 7 \). Thus, \( l(S_2) \geq 3 \). We can rewrite \( S_2 \) as \( (1, v, T, u, a_2) \) with \( l(T) \geq 0 \).

Clearly, \( u + 1 \) and \( v + 1 \) are two distinct vertices in \( G_2 \). By Theorem 2, there exists a Hamiltonian path \( S \) of \( G_2 \) joining \( u + 1 \) and \( v + 1 \). We write \( S \) as \( (u + 1, S_3, x, t + 1, t, S_4, v + 1) \). Thus, one of vertices \( w \) and \( t \) is not \( 2^{m-2} \). Without loss of generality, we assume that \( t \neq 2^{m-2} \). Again, we can write \( S \) as \( (u + 1, S_3, x + 1, t, S_4, v + 1) \). Since \( G_3 \) is Hamiltonian connected, there exists a Hamiltonian path \( S_6 \) of \( G_3 \) joining \( 2^{m-2} - 1 \) and \( v + 1 \). We set

\[
Q_1 = \begin{cases} 
(0, P_1, x - 1, x) & \text{for } i = 1, \\
(0, R_i, b_i, b_i + 1, x) & \text{for } 2 \leq i \leq k - 2, \\
(0, 1, S_1, a_1, x) & \text{for } i = k - 1, \\
(0, 2^{m-1} - 1, S_6, t + 1, t, S_4, v + 1, v, T, u, \\
u + 1, S_5, x + 1, x) & \text{for } i = k. 
\end{cases}
\]

Apparently, \( \{Q_1, Q_2, \ldots, Q_k\} \) forms a \( k \)-star-container of \( G(2^m, 4) \) between vertices \( 0 \) and \( x \), as shown by Fig. 2.

**Case 3:** \( x = 2 \) or \( x = 2^{m-2} \). Since \( g \) is an automorphism of \( G(2^m, 4) \), we consider only the case \( x = 2 \). Note that \( 0 \) and \( 4 \) are adjacent in \( G_0 \). By Lemma 3, there exists a \((k - 2)\)-star-container \( \{P_1, P_2, \ldots, P_{k-2}\} \) of \( G_0 \) between 0 and \( x = 2^{m-2} \) such that \( P_i = (0, x, -2) \). Hence \( l(P_i) \geq 2 \) for \( 2 \leq i \leq k - 2 \). We can write \( P_i \) as \( (0, a_i, R_i, b_i, x) \) for \( 2 \leq i \leq k - 2 \). Since \( x = 2^{m-2} \), \( x + 1 \) and \( 2^{m-2} - 1 \) are two distinct vertices of \( G_3 \). By Theorem 2,

\[
Q_1 = \begin{cases} 
(0, 4, 3, 2) & \text{for } i = 1, \\
(0, a_i, R_i, b_i, b_i - 1, (h_1(R_i))^{-1}, a_i - 1, \\
(0, a_i - 2, h_2(R_i), b_i - 2, 2) & \text{for } 2 \leq i \leq k - 3, \\
(0, a_{k-2}, R_{k-2}, y, z, z - 1, y - 1, \\
(h_1(R_{k-2}))^{-1}, a_{k-2} - 1, a_{k-2} - 2, \\
h_2(R_{k-2}), y - 2, y - 3, z - 2, 2) & \text{for } i = k - 2, \\
(0, 1, 2) & \text{for } i = k - 1, \\
(0, 2^{m-1} - 1, 2^{m-2} - 2) & \text{for } i = k. 
\end{cases}
\]

Clearly, \( (y - 3, z - 3) \) is in \( \{(9, 5), (29, 25, 13, 17), (25, 9, 29, 13, 17)\} \). We can find a Hamiltonian path \( S \) of \( G_1 - \{1\} \) joining \( y - 3 \) and \( z - 3 \) in Table 2.

Now, we set

\[
Q_1 = \begin{cases} 
(0, 4, 3, 2) & \text{for } i = 1, \\
(0, a_i, R_i, b_i, b_i - 1, (h_1(R_i))^{-1}, a_i - 1, \\
(0, a_i - 2, h_2(R_i), b_i - 2, 2) & \text{for } 2 \leq i \leq k - 3, \\
(0, a_{k-2}, R_{k-2}, y, z, z - 1, y - 1, \\
(h_1(R_{k-2}))^{-1}, a_{k-2} - 1, a_{k-2} - 2, \\
h_2(R_{k-2}), y - 2, y - 3, z - 2, 2) & \text{for } i = k - 2, \\
(0, 1, 2) & \text{for } i = k - 1, \\
(0, 2^{m-1} - 1, 2^{m-2} - 2) & \text{for } i = k. 
\end{cases}
\]

Accordingly, \( \{Q_1, Q_2, \ldots, Q_k\} \) forms a \( k \)-star-container of \( G(2^m, 4) \) between 0 and \( x \).

**Case 4:** \( x = 2 \pm 4^i \pmod{2^m} \) and \( x \neq 2^{m-2} - 2 \) for all \( 1 \leq i \leq [m/2] - 1 \). Clearly, \( x \) is in \( G_2 \). Therefore, \( x - 2 \) is adjacent to \( 0 \) in \( G_0 \). By Lemma 3, there exists a \((k - 2)\)-star-container \( \{P_1, P_2, \ldots, P_{k-2}\} \) of \( G_0 \) between \( 0 \) and \( x - 2 \) such that \( P_i = (0, x, -2) \). Hence \( l(P_i) \geq 2 \) for \( 2 \leq i \leq k - 2 \). We can write \( P_i \) as \( (0, a_i, R_i, b_i, x) \) for \( 2 \leq i \leq k - 2 \). Since \( x = 2^{m-2} \), \( x + 1 \) and \( 2^{m-2} - 2 \) are two distinct vertices of \( G_3 \). By Theorem 2,
there exists a Hamiltonian path $T$ of $G_3$ joining $x + 1$ and $2^m - 1$. We set

$$Q_i = \begin{cases} 
(0, x - 2, x - 1, x) & \text{for } i = 1, \\
(0, a_l, R_l, b_l, f_l(b_l), (f_l(R_l))^{-1}, f_l(a_l), \\
f_2(a_l), f_2(R_l), f_2(b_l), x) & \text{for } 2 \leq i \leq k - 2, \\
(0, 1, 2, x) & \text{for } i = k - 1, \\
(0, 2^m - 1, T, x + 1, x) & \text{for } i = k.
\end{cases}$$

Thus, $\{Q_1, Q_2, \ldots, Q_k\}$ forms a $k^*$-container of $G(2^m, 4)$ between 0 and $x$.

**Case 5:** $x \equiv 2 \pmod{4}$ and $x \not\equiv 2 \pm 4^l \pmod{2^m}$ for all $1 \leq l \leq [m/2]$. By induction, there is a $(k - 2)^*$-container $\{P_1, P_2, \ldots, P_{k-2}\}$ of $G_0$ between 0 and $x - 2$. Since $x - 2 \not\equiv 4^l \pmod{2^m}$, $l(P_i) \geq 2$ for all $1 \leq i \leq k - 2$. We can write $P_i$ as $(0, a_l, R_l, b_l, x - 2)$ for $1 \leq i \leq k - 2$. We recursively define a sequence of vertices in $G_3$ as follows: Set $z_1 = 3$ and $z_i = z_i - 1 + 4$ for $2 \leq i \leq 2^{m-2}$. Clearly, $(3 = z_1, z_2, \ldots, z_{2^{m-2}} = 2^m - 1, 3 = z_1)$ forms a Hamiltonian cycle $C$ of $G_3$. Since $x - 2 \not\equiv 4^l \pmod{2^m}$, $x - 3, x + 1, 2^m - 1$, and 3 are four distinct vertices of $G_3$. We may write $C$ as $(3, S, x - 3, x + 1, T, 2^m - 1, 3)$. Now, we set

$$Q_i = \begin{cases} 
(0, a_l, R_l, b_l, f_l(b_l), (f_l(R_l))^{-1}, f_l(a_l), \\
f_2(a_l), f_2(R_l), f_2(b_l), x) & \text{for } 1 \leq i \leq k - 2, \\
(0, 1, 2, 3, S, x - 3, x - 2, x - 1, x) & \text{for } i = k - 1, \\
(0, 2^m - 1, T, x + 1, x) & \text{for } i = k.
\end{cases}$$

Thus, $\{Q_1, Q_2, \ldots, Q_k\}$ forms a $k^*$-container of $G(2^m, 4)$ between 0 and $x$.

### 4. Conclusions

Recursive circulant graphs $G(2^m, 4)$ are the major concern in this paper. $G(2^m, 4)$ has the connectivity $m$ and the diameter $\lceil (3m - 1)/4 \rceil$; which is less than $m$, the diameter of the hypercube $Q_m$. The main result of this paper is proving that the recursive circulant graphs $G(2^m, 4)$ have super-connected property if and only if $m \not\equiv 2$. A $k$-container $C_k(u, v)$ between two distinct vertex $u$ and $v$ in $G$ is a set of $k$ disjoint paths between $u$ and $v$. The length of a $C_k(u, v)$, written as $l(C_k(u, v))$, is the length of the longest path in $C_k(u, v)$. The $k$-wide distance between $u$ and $v$ is $d_k(u, v)$, which is the minimum length among all $k$-containers between $u$ and $v$. Let $k$ be the connectivity of $G$. The wide diameter of $G$, denoted by $D_k(G)$, is the maximum of $k$-wide distances among all pairs of vertices $u, v$ in $G$, $u \neq v$. Assume that $G$ is $k^*$-connected. We may define the $k^*$-wide distance between any two vertices $u$ and $v$, denoted by $d^*_k(u, v)$, which is the minimum length among all $k^*$-containers between $u$ and $v$. Let $D^*_k(G) = \max\{|d^*_k(u, v)| \mid u \neq v \}$ for two different vertices of $G$. We say that $D^*_k(G)$ is the $k^*$-diameter of $G$. In our future work, we are interested to find $D_k^*(G(2^m, 4))$ for $2 \leq k \leq m$.

### References