International Journal of Computer Mathematics

Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/gcom20

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Tzu-Liang Kung a, Cheng-Kuan Lin b, Tyne Liang b, Jimmy J.M. Tan b & Lih-Hsing Hsu c

a Department of Computer Science and Information Engineering, Asia University, 500 Lioufeng Road, Taichung, 41354, Taiwan, Republic of China
b Department of Computer Science, National Chiao Tung University, 1001 University Road, Hsinchu, 30010, Taiwan, Republic of China
c Department of Computer Science and Information Engineering, Providence University, 200 Chung Chi Road, Taichung, 43301, Taiwan, Republic of China

Published online: 23 Dec 2010.


To link to this article: http://dx.doi.org/10.1080/00207161003786614

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Fault-free mutually independent Hamiltonian cycles of faulty star graphs

Tzu-Liang Kung\textsuperscript{a*}, Cheng-Kuan Lin\textsuperscript{b}, Tyne Liang\textsuperscript{b}, Jimmy J.M. Tan\textsuperscript{b} and Lih-Hsing Hsu\textsuperscript{c}

\textsuperscript{a}Department of Computer Science and Information Engineering, Asia University, 500 Lioufeng Road, Taichung 41354, Taiwan, Republic of China; \textsuperscript{b}Department of Computer Science, National Chiao Tung University, 1001 University Road, Hsinchu 30010, Taiwan, Republic of China; \textsuperscript{c}Department of Computer Science and Information Engineering, Providence University, 200 Chung Chi Road, Taichung 43301, Taiwan, Republic of China

(Received 24 April 2009; revised version received 08 February 2010; accepted 15 March 2010)

The star graph interconnection network has been recognized as an attractive alternative to the hypercube for its nice topological properties. Unlike previous research concerning the issue of embedding exactly one Hamiltonian cycle into an injured star network, this paper addresses the maximum number of fault-free mutually independent Hamiltonian cycles in the faulty star network. To be precise, let $SG_n$ denote an $n$-dimensional star network in which $f \leq n - 3$ edges may fail accidentally. We show that there exist $(n - 2 - f)$-mutually independent Hamiltonian cycles rooted at any vertex in $SG_n$ if $n \in \{3, 4\}$, and there exist $(n - 1 - f)$-mutually independent Hamiltonian cycles rooted at any vertex in $SG_n$ if $n \geq 5$.

**Keywords:** Hamiltonian; interconnection network; star graph; fault tolerance

2000 AMS Subject Classifications: 05C38; 05C45; 05C75; 05C90; 68M10

1. Introduction

The problem of finding Hamiltonian cycles in a graph is well known to be NP-complete and has been discussed in many areas. In 1969, Lovasz [27] asked whether every finite connected vertex-transitive graph has a Hamiltonian path, that is, a simple path that traverses every vertex exactly once.

**Definition 1** [4] A graph is said to be vertex-transitive if for every pair $u$, $v$ of vertices, there exists an automorphism of the graph that maps $u$ into $v$. A graph is said to be edge-transitive if for any two edges $a$ and $b$, there exists an automorphism of the graph that maps $a$ into $b$.

All known vertex-transitive graphs have a Hamiltonian path, but only four vertex-transitive graphs without any Hamiltonian cycle are known to exist. Since none of these four graphs...
is a Cayley graph, there is a folklore conjecture [6] that every Cayley graph with more than two vertices has a Hamiltonian cycle. In the last decades, this problem was extensively studied [2,3,5–7,10,11,17–19,28–30,35]. For those Cayley graphs for which the existence of Hamiltonian cycles has already been proved, more advanced properties, such as edge-Hamiltonicity, Hamiltonian connectivity, and Hamiltonian laceability, etc., are investigated [2,22]. In this paper, we address one of such properties, the concept of mutually independent Hamiltonian cycles [36,37], which is related to the number of Hamiltonian cycles in a given graph. Since its introduction, this topic has gained many researchers’ attention [12,15,16,25,26,33]. In particular, Lin et al. [25] showed that the maximum number of mutually independent Hamiltonian cycles rooted at any vertex can be constructed recursively in the star graph interconnection network (for the detailed definitions, see Sections 2 and 3).

The interconnection network is of great interest in the area of parallel and distributed computer systems. Because it is usually multi-objected and complicated to design an interconnection network, its underlying topology can be modelled as a graph, whose vertices correspond to processors and whose edges correspond to connections/communication links. Hence, we use the terms graphs and networks interchangeably. Among various kinds of network topologies, the star graph is attractive for its high degree of symmetry. However, when some edges are removed at random from the star graph, the symmetry will be broken. Hence, we wonder, in a theoretical point of view, how many mutually independent Hamiltonian cycles can be formed in such an injured network. In this paper, the maximum number of fault-free mutually independent Hamiltonian cycles in the faulty star graph will be studied. To be precise, let $SG_n$ denote an $n$-dimensional star graph with $f \leq n - 3$ faulty edges. Then we aim at proving the following result: $SG_n$ has $(n - 2 - f)$-mutually (respectively, $(n - 1 - f)$-mutually) independent Hamiltonian cycles rooted at any vertex if $n \in \{3, 4\}$ (respectively, $n \geq 5$).

The rest of this paper is organized as follows. In Section 2, graph-theoretic notations and the definition of mutually independent Hamiltonian cycles are introduced. In Section 3, the star graph and its basic properties are presented. Section 4 consists of the proof of our main result. Finally, directions for future research are discussed in Section 5.

2. Preliminaries

Throughout this paper, graphs are simple, loopless, and undirected. For definitions and notations not defined here, see [4]. A graph $G$ is an ordered pair $(V(G), E(G))$, where $V(G)$ is a non-empty set, and $E(G)$ is a subset of $\{\{u, v\}| u, v \text{ is a two-element subsets of } V(G)\}$. The set $V(G)$ is called the vertex set of $G$, and the set $E(G)$ is called the edge set of $G$. Two vertices $u$ and $v$ of $G$ are adjacent if $\{u, v\} \in E(G)$. The degree of a vertex $u$ in $G$ is the number of edges incident to $u$. A graph $G$ is $k$-regular if all its vertices have the same degree $k$. A graph $G$ is bipartite if its vertex set can be partitioned into two disjoint subsets, denoted by $V_0(G)$ and $V_1(G)$, such that every edge joins a vertex of $V_0(G)$ to a vertex of $V_1(G)$.

A graph $H$ is a subgraph of a graph $G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Let $S$ be a non-empty subset of $V(G)$. The subgraph of $G$ induced by $S$ is a subgraph of $G$ with vertex set $S$, whose edge set consists of all the edges joining any two vertices in $S$. We use $G - S$ to denote the subgraph of $G$ induced by $V(G) - S$. Let $F$ be any subset of $E(G)$. Then we use $G - F$ to denote the subgraph of $G$ with vertex set $V(G)$ and edge set $E(G) - F$. For any $S \subseteq V(G)$ and $F \subseteq E(G)$, graph $G - (S \cup F)$ is defined to be the graph $(G - F) - S$.

A walk of length $k \geq 1$ in a graph is a sequence of vertices, $W := v_1v_2\cdots v_{k+1}$, such that $v_i$ and $v_{i+1}$ are adjacent for $i = 1, 2, \ldots, k$. If $v_1 = x$ and $v_{k+1} = y$, we refer to $W$ as an $xy$-walk. The notation $xWy$ is also used simply to signify an $xy$-walk $W$. Moreover, we use $W^{-1}$ to denote the reversed walk $v_{k+1}v_k\cdots v_1$. For any three vertices $x$, $y$, $z$ in a graph, if $xW_1y$ and $yW_2z$ are walks,
A path is a walk in which no vertex is repeated. For convenience, the $i$th vertex of a path $P$ is denoted by $P(i)$. For any two vertices $u$ and $v$ in a graph $G$, the distance between $u$ and $v$, denoted by $d_G(u, v)$, is the length of the shortest path between $u$ and $v$. A cycle is a walk $v_1v_2\cdots v_n$ in which $n \geq 3$, $v_1 = v_n + 1$, and the $n$ vertices $v_1, v_2, \ldots, v_n$ are distinct.

A path (or cycle) in a graph $G$ is a Hamiltonian path (or Hamiltonian cycle) of $G$ if it spans $G$. A graph is Hamiltonian if it has a Hamiltonian cycle. A bipartite graph is Hamiltonian laceable [34] if there exists a Hamiltonian path between any two vertices that are in different partite sets. A Hamiltonian laceable graph $G$ is said to be hyper-Hamiltonian laceable [22] if, for any $i \in \{0, 1\}$ and for any vertex $v \in V_i(G)$, there exists a Hamiltonian path in $G \setminus \{v\}$ between any two vertices of $V_{i−1}(G)$.

Let $G$ be a graph with $N$ vertices. A rooted Hamiltonian cycle $C$ in $G$ can be described as $v_1v_2\cdots v_Nv_1$ to emphasize the order of vertices on $C$. Accordingly, $v_1$ is seen as the root vertex, and $v_j$ is seen as the $j$th vertex on $C$. Two Hamiltonian cycles rooted at a given vertex $s$ of $G$, namely $C_1 := v_1v_2\cdots v_Nv_1$ and $C_2 := u_1u_2\cdots u_Nu_1$ with $v_1 = u_1 = s$, are independent if $v_i \neq u_i$ for $2 \leq i \leq N$. A collection of $m$ Hamiltonian cycles $C_1, \ldots, C_m$ in $G$, rooted at the same vertex, are said to be $m$-mutually independent if $C_i$ and $C_j$ are independent whenever $i \neq j$. Moreover, the mutually independent Hamiltonicity of $G$, denoted by $\mathcal{THC}(G)$, is defined to be the maximum integer $m$ such that for any vertex $v$ of $G$, there exists a set of $m$-mutually independent Hamiltonian cycles rooted at $v$ in $G$. The concept of mutually independent Hamiltonian cycles can be applied in many different areas [12,15,16,25,26,33].

3. The star graph

The hypercube has long been one of the most popular network topologies [21] because of its nice topological properties. The star graph, proposed by Akers and Krishnamurthy [1], is an attractive alternative to the hypercube for interconnecting processors in parallel computers. Since then, star networks have received many researchers’ attention. For example, the diameter and fault diameter were computed in [1,20,32]. Moreover, Fragopoulou and Akl [8,9] studied how to embed directed edge-disjoint spanning trees into the star graph. The Hamiltonian properties of star graphs are addressed in [13,14,23,38]. In particular, because processors or links may fail accidentally to affect network performance, Tseng et al. [38] addressed fault-tolerant ring embedding in an injured star network if no more than $n − 3$ edge faults occur.

The definition of star graphs is described as follows. Let $n$ be any positive integer. For convenience, we use $\mathbb{I}_n$ to denote the set $\{1, 2, \ldots, n\}$. A permutation $u_1u_2\cdots u_n$ on $\mathbb{I}_n$ is a sequence of all elements of $\mathbb{I}_n$. Every permutation can be written as a product of transpositions. An even permutation (respectively, odd permutation) is a permutation that can be written as a product of an even (respectively, odd) number of transpositions. The $n$-dimensional star graph $SG_n$ is a graph whose vertex set is the set of all permutations on $\mathbb{I}_n$. Two vertices, $u_1\cdots u_i \cdots u_n$ and $v_1\cdots v_i \cdots v_n$, are adjacent through an edge of dimension $i$ with $2 \leq i \leq n$ if $u_1 = v_i$, $v_1 = u_i$, and $u_j = v_j$ for $j \in \mathbb{I}_n - \{1, i\}$. Clearly, $SG_n$ is $(n−1)$-regular with $n!$ vertices. Moreover, it is precisely a Cayley graph of the symmetric group with edge set consisting of all the transpositions of form $(1 i)$, where $2 \leq i \leq n$. So it is vertex-transitive and edge-transitive [1]. The star graphs $SG_2$, $SG_3$, and $SG_4$ are illustrated in Figure 1.

For the sake of clarity, we use boldface letters to denote vertices of $SG_n$. Moreover, we use $e$ to denote the vertex $12, \ldots, n$. It is known that $SG_n$ is a bipartite graph with one partite set $V_0(SG_n)$ consisting of all the even permutations and the other partite set $V_1(SG_n)$ consisting of all the odd permutations. Let $u = u_1u_2\cdots u_n$ be a vertex in $SG_n$. Then $u_i$ is the $i$th coordinate of $u$, denoted
by \( (u)_i \), for \( 1 \leq i \leq n \). For any \( 2 \leq i \leq n \), the \( i \)-neighbour of vertex \( u \), denoted by \( (u)^i \), is a vertex adjacent to \( u \) through an edge of dimension \( i \). Obviously, \( ((u)^i)^i = u \).

For any \( 1 \leq i \leq n \), let \( SG^{(i)}_n \) denote the subgraph of \( SG_n \) induced by the set of vertices \( \{u \in V(SG_n) \mid (u)_n = i\} \). Then \( SG_n \) can be partitioned into \( n \) vertex-disjoint subgraphs \( SG^{(1)}_n, SG^{(2)}_n, \ldots, SG^{(n)}_n \), and every of them is isomorphic to \( SG_{n-1} \). Let \( I \subseteq \mathbb{I}_n \). We use \( SG^I_n \) to denote the subgraph of \( SG_n \) induced by \( \bigcup_{i \in I} V(SG^{(i)}_n) \). For any pair \( i, j \) of distinct integers in \( \mathbb{I}_n \), we use \( E^{(i,j)} \) to denote the set of edges between \( SG^{(i)}_n \) and \( SG^{(j)}_n \).

In the rest of this section, we introduce some results to be used later.

**Theorem 1** [38] \( \text{Let } F \subseteq E(SG_n) \text{ with } |F| \leq n - 3 \text{ for } n \geq 3. \text{ Then } SG_n - F \text{ is Hamiltonian.} \)

Li et al. [23] introduced the edge-fault-tolerant Hamiltonian laceability of a bipartite graph \( G \), which is the integer \( f \) such that for any \( F \subseteq E(G) \) with \( |F| \leq f \), \( G - F \) is still Hamiltonian laceable and there exists a subset \( F' \) of \( E(G) \) with \( |F'| = f + 1 \) such that \( G - F' \) is not Hamiltonian laceable. Moreover, they also defined the edge-fault-tolerant hyper-Hamiltonian laceability of a graph \( G \) as the integer \( f \) such that for any \( F \subseteq E(G) \) with \( |F| \leq f \), \( G - F \) is hyper-Hamiltonian laceable and there exists a subset \( F' \) of \( E(G) \) with \( |F'| = f + 1 \) such that \( G - F' \) is no longer Hyper-Hamiltonian laceable.

**Theorem 2** [23] \( \text{The star graph } SG_n \text{ is } (n - 3)\text{-edge-fault-tolerant Hamiltonian laceable and } (n - 4)\text{-edge-fault-tolerant hyper-Hamiltonian laceable for } n \geq 4. \)

**Lemma 1** [31] \( \text{Assume that } n \geq 3. \text{ Then } |E^{(i,j)}| = (n - 2)! \text{ for any } 1 \leq i \neq j \leq n. \text{ Moreover, there are } (n - 2)!/2 \text{ pairwise disjoint edges joining vertices of } V_t(SG^{(i)}_n) \text{ to vertices of } V_{1-t}(SG^{(j)}_n) \text{ for any } t \in \{0, 1\}. \)

**Lemma 2** \( \text{For } n \geq 3, \text{ let } u \text{ and } v \text{ be two distinct vertices of } SG_n \text{ with } d_{SG_n}(u, v) \leq 2. \text{ Then } (u)_1 \neq (v)_1. \)

**Lemma 3** \( \text{Let } n \geq 5 \text{ and } F \subseteq E(SG_n) \text{ with } |F| \leq n - 4. \text{ Assume that } I = \{a_1, \ldots, a_r\} \text{ is an } r\text{-element subset of } \mathbb{I}_n \text{ for any } r \in \mathbb{I}_n. \text{ Suppose that } u \in V_t(SG^{(a_1)}_n) \text{ and } v \in V_{1-t}(SG^{(a_{r+1})}_n) \text{ for any} \)
t ∈ {0, 1}. Then there exists a Hamiltonian path $H := x_1 P_1 y_1 x_2 P_2 y_2 \cdots x_r P_r y_r$ in $SG_n^t − F$ such that $x_1 = u, y_r = v$, and $P_i$ is a Hamiltonian path of $SG_n^{a_i} − F$ joining $x_i$ to $y_i$ for every $1 ≤ i ≤ r$.

**Proof** Without loss of generality, we can assume that $t = 0$. Since $SG_n^{a_i}$ is isomorphic to $SG_{n-1}$, this statement holds for $r = 1$ by Theorem 2. Thus, suppose that $r ≥ 2$ and set $x_1 = u$ and $y_r = v$. By Lemma 1, there are $(n - 2)!/2 > n - 4$ pairwise disjoint edges joining vertices of $V_1(SG_n^{a_1})$ to vertices of $V_0(SG_n^{a_1})$ for every $i ∈ I_{r-1}$. Therefore, we can choose $\{y_i, x_{i+1}\} ∈ E_n^{a_i} − F$ with $y_i ∈ V_1(SG_n^{a_1})$ and $x_{i+1} ∈ V_0(SG_n^{a_1})$ for $i ∈ I_{r-1}$. By Theorem 2, $SG_n^{a_i} − F$ has a Hamiltonian path $P_i$ joining $x_i$ to $y_i$ for every $i ∈ I_r$. As a result, the sequence of vertices, $x_1 P_1 y_1 x_2 P_2 y_2 \cdots x_r P_r y_r$, forms a desired Hamiltonian path of $SG_n^t − F$ joining $u$ to $v$. ■

**Lemma 4** Let $n ≥ 5$. Assume that $F ⊂ E(SG_n)$ with $|F| ≤ n - 4$, and $|F ∩ SG_n^{k_1}| ≤ n - 5$ for every $i ∈ I_n$. Moreover, assume that $I = \{a_1, \ldots, a_r\}$ is an $r$-element subset of $I_n$ for any $2 ≤ r ≤ n$. Suppose that $u ∈ V_1(SG_n^{a_1}), w ∈ V_{k-1}(SG_n^{a_1})$, and $v ∈ V_r(SG_n^{a_1})$ for any $t ∈ \{0, 1\}$. Then there exists a Hamiltonian path $H$ of $(SG_n^t − F) − \{w\}$ joining $u$ to $v$.

**Proof** Without loss of generality, we can assume that $t = 0$. By Lemma 1, there are $(n - 2)!/2 > n - 3$ pairwise disjoint edges joining vertices of $V_1(SG_n^{a_1})$ to vertices of $V_1(SG_n^{a_1})$. Thus, we can choose a vertex $x$ of $V_0(SG_n^{a_1}) − \{u\}$ with $(x)_1 = a_2$ and $(x, (x)^n) ∉ F$. By Theorem 2, there exists a Hamiltonian path $P$ of $(SG_n^{a_1} − F) − \{w\}$ joining $u$ to $x$. By Lemma 3, there exists a Hamiltonian path $Q$ of $SG_n^{k-1} − F$ joining $(x)^n$ to $v$. As a result, the sequence of vertices, $u P x (x)^n Q v$, forms a desired Hamiltonian path. ■

**Lemma 5**[24] Let $w$ and $b$ denote two adjacent vertices of $SG_n$ with $n ≥ 4$. For any vertex $u$ in $V_1(SG_n) − \{w, b\}$, $t ∈ \{0, 1\}$, and for any $i ∈ I_n$, there exists a Hamiltonian path $P$ of $SG_n − \{w, b\}$ joining $u$ to some vertex $v$ in $V_1 - (SG_1) − \{w, b\}$ with $(v)_1 = i$.

**Lemma 6** Let $i ∈ I_n$ and $F ⊂ E(SG_n)$ with $|F| ≤ n - 4$ for $n ≥ 4$. Suppose that $w$ and $b$ are two adjacent vertices of $SG_n$, and $u ∈ V_i(SG_n) − \{w, b\}$ for any $t ∈ \{0, 1\}$. Then there exists a Hamiltonian path of $(SG_n − F) − \{w, b\}$ joining $u$ to some vertex $v$ of $V_1 - (SG_1) − \{w, b\}$ with $(v)_1 = i$.

**Proof** Without loss of generality, we can assume that $t = 0$. Since $SG_n$ is vertex-transitive, we can assume that $w = e$ and $b = (e)^j$ with some $j ∈ I_n − \{1\}$. We set $F_k = F ∩ E(SG_n^{[k]})$ for every $k ∈ I_n$. The proof is done by induction on $n$. The induction basis, that is, the case $n = 4$, follows from Lemma 5. Suppose that this statement holds for $SG_{n-1}$ with $n ≥ 5$. We consider the dimensions of all edges in $F ∪ \{(e, (e)^j)\}$. If there is an edge in $F$ whose dimension, say $j'$, is different from $j$, then $SG_n$ can be partitioned into $n$ vertex-disjoint subgraphs with the $j'$th coordinate of each vertex (that is, the subgraph of $SG_n$ induced by the vertices with the same $j'$th coordinate is $SG_{n-1}$). Otherwise, every edge of $F$ has the same dimension $j$.

Case 1 The dimension $j'$ exists. Without loss of generality, we can assume that $j' = n$. Thus, we have $(e, (e)^j) ∈ E(SG_n^{[n]})$ and $|F_k| ≤ n - 5$ for every $k ∈ I_n$.

Subcase 1.1 Suppose that $u ∈ V_0(SG_n^{[n]})$. Since $|F| ≤ n - 4$, we can choose an integer $r ∈ I_{n-1}$ such that $|F ∩ E^{r,n}| = 0$. By the induction hypothesis, there exists a Hamiltonian path $P$ of $(SG_n^{[n]} − F_u) − \{e, (e)^j\}$ joining $u$ to a vertex $x ∈ V_1(SG_n^{[n]})$ with $(x)_1 = r$. We can choose a vertex $v$ in $V_1(SG_n^{k-1} − [r])$ with $(v)_1 = i$. By Lemma 3, there exists a Hamiltonian path $Q$ of $SG_n^{k-1} − F$ joining $(x)^n$ to $v$. Then the sequence of vertices, $u P x (x)^n Q v$, is a desired path.
Subcase 1.2 Suppose that \( u \in V_0(\text{SG}_{n}^{[k]} \) for some \( k \in \mathbb{I}_{n-1} \). By Lemma 1, there are \((n - 2)!/2 > n - 3 \) pairwise disjoint edges joining vertices of \( V_1(\text{SG}_{n}^{[k]} \) to vertices of \( V_0(\text{SG}_{n}^{[n]} \). We can pick out a vertex \( y \) of \( V_1(\text{SG}_{n}^{[k]} \) such that \( (y)^{\ell} \in V_0(\text{SG}_{n}^{[k]} \) and \( (y, (y)^{\ell}) \notin F \). By Theorem 2, there exists a Hamiltonian path \( H \) of \( \text{SG}_{n}^{[k]} - F_k \) joining \( u \) to \( y \). We can choose an integer \( r \) of \( \mathbb{I}_{n-1} - \{k\} \) such that \( |F \cap E_{r,n}| = 0 \). By the induction hypothesis, there exists a Hamiltonian path \( P \) of \( (\text{SG}_{n}^{[n]} - F_r) \) joining \( (y)^{\ell} \) to a vertex \( x \) of \( V_1(\text{SG}_{n}^{[n]} \) - \( \{x\} \) with \( (x) = r \). Besides, we choose a vertex \( v \) of \( V_1(\text{SG}_{n}^{[n-1,k,r]} \) with \( (v) = i \). By Lemma 3, there exists a Hamiltonian path \( Q \) of \( \text{SG}_{n}^{[n-1,k]} - F \) joining \( (x)^{\ell} \) to \( v \). Then the sequence of vertices, \( u H y (y)^{\ell} P x (x)^{\ell} Q v \), turns out to be a desired path.

Case 2 Every edge in \( F \) has the same dimension \( j \). Without loss of generality, we may assume that \( j = n \). Thus, we have \(|F_i| = 0\) for every \( i \in \mathbb{I}_n \).

Subcase 2.1 Suppose that \( u \in V_0(\text{SG}_{n}^{[k]} \) for some \( k \in \mathbb{I}_{n-1} - \{1\} \). By Lemma 1, there are \((n - 2)!/2 > n - 4 \) pairwise disjoint edges joining vertices of \( V_1(\text{SG}_{n}^{[k]} \) to vertices of \( V_0(\text{SG}_{n}^{[n]} \). Thus, we can choose a vertex \( x \) of \( V_1(\text{SG}_{n}^{[k]} \) with \( (x) = 1 \) and \( (x, (x)^{\ell}) \notin F \). By Theorem 2, there exits a Hamiltonian path \( H \) of \( \text{SG}_{n}^{[k]} \) joining \( u \) to \( x \). Similarly, we can choose a vertex \( y \) of \( V_0(\text{SG}_{n}^{[n]} \) with \( (y) = n \), \( (y, (y)^{\ell}) \notin F \), and \( y \neq (x)^{\ell} \). This can be done because there are \((n - 2)!/2 \geq n - 2 \) pairwise disjoint edges between the sets \( V_0(\text{SG}_{n}^{[k]} \) and \( V_1(\text{SG}_{n}^{[k]} \). By Theorem 2, \( (x)^{\ell} \) has a Hamiltonian path \( P \) joining \( (x)^{\ell} \) to \( y \). Let \( v \) be a vertex in \( V_1(\text{SG}_{n}^{[n-1,k]} \) with \( (v) = i \). By Lemma 4, there exists a Hamiltonian path \( Q \) of \( (\text{SG}_{n}^{[n-1,k]} - F) - \{e\} \) joining \( (x)^{\ell} \) to \( v \). Then the sequence of vertices, \( u H x (x)^{\ell} P y (y)^{\ell} Q v \), turns out to be a desired path.

Subcase 2.2 Suppose that \( u \in V_0(\text{SG}_{n}^{[1]} \). By Lemma 1, there are \((n - 2)!/2 > n - 4 \) pairwise disjoint edges joining vertices of \( V_0(\text{SG}_{n}^{[n]} \) to vertices of \( V_1(\text{SG}_{n}^{[1]} \). Thus, we can choose a vertex \( x \) of \( V_0(\text{SG}_{n}^{[1]} \) - \( \{u\} \) with \( (x) = n \) and \( (x, (x)^{\ell}) \notin F \). By Theorem 2, there exists a Hamiltonian path \( H \) of \( \text{SG}_{n}^{[1]} \) - \( \{(x)^{\ell}\} \) joining \( u \) to \( x \). Furthermore, we choose a vertex \( v \) of \( V_1(\text{SG}_{n}^{[n-1,1]} \) with \( (v) = i \). By Lemma 4, there exists a Hamiltonian path \( Q \) of \( (\text{SG}_{n}^{[n-1,1]} - F) - \{e\} \) joining \( (x)^{\ell} \) to \( v \). Then the sequence of vertices, \( u H x (x)^{\ell} Q v \), forms a desired path.

Subcase 2.3 Suppose that \( u \in V_0(\text{SG}_{n}^{[n]} \). Since \(|F| \leq n - 4 \), we can choose two integers \( k_1 \) and \( k_2 \) in \( \mathbb{I}_{n-1} - \{1\} \) such that \( \{(e)^{k_i}, ((e)^{k_i})^{n}\} \notin F \) and \( \{(e)^{k_1}, ((e)^{k_2})^{n}\} \notin F \). Let \( X = \{(e), (e)\} \in \mathbb{I}_{n-1} - \{1, k_1, k_2\} \). Obviously, \(|X| = n - 4 \). Moreover, we can choose a vertex \( x \in V_1(\text{SG}_{n}^{[n]} \) such that \( (x) \in \mathbb{I}_{n-1} - \{1, k_1, k_2\} \) and \( (x, (x)^{\ell}) \notin F \). Since \( (x) \neq k_1 \) and \( (x) \neq k_2 \), we have \( x \neq (e)^{k_1} \) and \( x \neq (e)^{k_2} \). By Theorem 2, there exists a Hamiltonian path \( H \) of \( \text{SG}_{n}^{[n]} - X \) joining \( u \) to \( x \). Because vertex \( e \) has precisely two neighbours, that is, \( (e)^{k_1} \) and \( (e)^{k_2} \), in \( \text{SG}_{n}^{[n]} - X \), the edges \( \{e, (e)^{k_1}\} \) and \( \{e, (e)^{k_2}\} \) must be consecutive on \( H \). Thus, with no loss of generality, we can write \( H = u H_1(e)^{k_1} e (e)^{k_2} H_2 x \). Let \( y = (e)^{k_2} \). Since \( (y) = (x) \), we have \( i \neq (x) \) or \( i \neq (y) \).

Subcase 2.3.1 Suppose that \( i \neq (x) \). Let \( k_3 = (x) \). We choose a vertex \( v \) of \( V_1(\text{SG}_{n}^{[k_3]} \) with \( (v) = i \). By Lemma 1, there are \((n - 2)!/2 > n - 4 \) pairwise disjoint edges joining vertices of \( V_1(\text{SG}_{n}^{[k_3]} \) to vertices of \( V_0(\text{SG}_{n}^{[1]} \). Thus, we can choose a vertex \( z \) of \( V_1(\text{SG}_{n}^{[k_3]} \) with \( (z) = 1 \) and \( (z, (z)^{\ell}) \notin F \). By Theorem 2, there exists a Hamiltonian path \( T \) of \( \text{SG}_{n}^{[k_3]} \) joining \( (x)^{\ell} \) to \( v \). Similarly, there exists a Hamiltonian path \( P \) of \( \text{SG}_{n}^{[k_3]} \) joining \( ((e)^{k_1})^{n} \) to \( z \). By Lemma 4, there exists a Hamiltonian path \( Q \) of \( (\text{SG}_{n}^{[n-1,1,k_3]} - F) - \{(e)^{k_1}\} \) joining \( (z)^{n} \) to \( v \). Then the sequence of vertices, \( u H_1 (e)^{k_1} ((e)^{k_1})^{n} P z (z)^{n} Q (y)^{n} y H_2 x (x)^{n} T v \), is a desired one.
Subcase 2.3.2 Suppose that \( i \neq (y)_1 \). Let \( k_3 = (y)_1 \). Then the proof of this case happens to be similar to that of Subcase 2.3.1. Thus, we omit the details.

**Lemma 7** Let \( \{a, b\} \subset \mathbb{I}_n \) with \( a < b \), and let \( F \subset E(SG_n) \) with \( |F| \leq n - 4 \) for \( n \geq 4 \). Suppose that \( x \in V_0(SG_n) \), and \( x_1 \) and \( x_2 \) are two distinct neighbours of \( x \). Then there exists a Hamiltonian path of \( (SG_n - F) - \{x, x_1, x_2\} \) between two vertices \( u \) and \( v \) in \( V_0(SG_n) - \{x\} \) such that \( (u)_1 = a \) and \( (v)_1 = b \).

**Proof** Since \( SG_n \) is vertex-transitive, we can assume that \( x = e, x_1 = (e)^{i_1}, \) and \( x_2 = (e)^{i_2} \) with some \( \{i_1, i_2\} \subset \{2, 3, \ldots, n\} \). Then this lemma is proved by induction on \( n \).

Suppose that \( n = 4 \). Thus, we have \( |F| = 0 \). Since \( SG_4 \) is edge-transitive, we can assume that \( x_1 = (e)^2 = 2134 \) and \( x_2 = (e)^3 = 3214 \). The required paths of \( SG_4 - \{1234, 2134, 3214\} \) are listed in Table 1.

Suppose that the statement holds for \( SG_{n-1} \) with \( n \geq 5 \). Let \( F_k = F \cap E(SG_n^{[k]}) \) for every \( k \in \mathbb{I}_n \). Without loss of generality, suppose that \( F \) contains at least one edge of dimension \( n \). Thus, we have \( |F_k| \leq n - 5 \) for every \( k \in \mathbb{I}_n \). Because \( a < b \), we have \( a \neq n \) and \( b \neq 1 \). Since \( |F| \leq n - 4 \), we can choose an integer \( c \in \mathbb{I}_{n-1} - \{1, a\} \) such that \( |F \cap E^{c,n}| = 0 \). Moreover, we can choose a vertex \( v \) of \( V_0(SG_n^{[l]}) \) with \( (v)_1 = b \).

**Case 1** Suppose that \( i_1 \neq n \) and \( i_2 \neq n \). By the induction hypothesis, there exists a Hamiltonian path \( H \) of \( (SG_n^{[n]} - F_n) - \{(e), (e)^{i_1}, (e)^{i_2}\} \) joining a vertex \( u \) of \( V_0(SG_n^{[n]}) \) with \( (u)_1 = a \) to a vertex \( y \) of \( V_0(SG_n^{[n]}) \) with \( (y)_1 = c \). By Lemma 3, there exists a Hamiltonian path \( R \) of \( SG_n^{[i_1]} - F \) joining \( (y)^n \) to \( v \). As a result, the sequence of vertices, \( u H y (y)^n R v \), forms a desired path in \( (SG_n - F) - \{(e), (e)^{i_1}, (e)^{i_2}\} \).

**Case 2** Either \( i_1 = n \) or \( i_2 = n \). Without loss of generality, we can assume that \( i_2 = n \). We choose a vertex \( u \in V_0(SG_n^{[n]}) \) with \( (u)_1 = a \). By Lemma 6, there exists a Hamiltonian path \( H \) of \( (SG_n^{[n]} - F_n) - \{(e), (e)^{i_1}\} \) joining a vertex \( u \) to some vertex \( y \) of \( V_1(SG_n^{[n]}) \) with \( (y)_1 = c \). By Lemma 4, there exists a Hamiltonian path \( Q \) of \( (SG_n^{[i_1]} - F) - \{(e)^n\} \) joining \( (y)^n \) to \( v \). As a result, the sequence of vertices, \( u H y (y)^n Q v \), is a desired path.

4. Mutually independent Hamiltonian cycles in faulty star graphs

Lin et al. [25] showed the next theorem.

**Theorem 3** [25] \( \mathcal{IHC}(SG_3) = 1, \mathcal{IHC}(SG_4) = 2, \) and \( \mathcal{IHC}(SG_n) = n - 1 \) if \( n \geq 5 \).

For the sake of clarity, our main result, Theorem 4, will be divided into three lemmas (Lemma 8-10).

**Lemma 8** Let \( f \in E(SG_4) \). Then \( \mathcal{IHC}(SG_4 - \{f\}) = 1 \).

**Proof** Since \( SG_4 \) is edge-transitive, we can assume that \( f = \{1234, 4231\} \). By Theorem 1, there exists a Hamiltonian cycle in \( SG_4 - \{f\} \). Thus, we have \( \mathcal{IHC}(SG_4 - \{f\}) \geq 1 \). To show that \( \mathcal{IHC}(SG_4 - \{f\}) \leq 1 \), it suffices to point out that there will be no two-mutually independent Hamiltonian cycles rooted at vertex 1234. In Table 2, we list all Hamiltonian cycles of \( SG_4 - \{f\} \) rooted at 1234. By brute force, we can check that there do not exist two-mutually independent Hamiltonian cycles. Hence, the proof is completed.
Table 1. The required Hamiltonian paths in SG \(_4\) – \{1234, 2134, 3214\}.

<table>
<thead>
<tr>
<th>(a = 1) and (b = 2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1324</td>
</tr>
<tr>
<td>(a = 1) and (b = 3)</td>
</tr>
<tr>
<td>1423</td>
</tr>
<tr>
<td>(a = 1) and (b = 4)</td>
</tr>
<tr>
<td>1324</td>
</tr>
<tr>
<td>(a = 2) and (b = 3)</td>
</tr>
<tr>
<td>2314</td>
</tr>
<tr>
<td>(a = 2) and (b = 4)</td>
</tr>
<tr>
<td>2314</td>
</tr>
<tr>
<td>(a = 3) and (b = 4)</td>
</tr>
<tr>
<td>3124</td>
</tr>
</tbody>
</table>

Table 2. All Hamiltonian cycles rooted at 1234 in SG \(_4\) – \{1234, 4231\}.

| 1234 | 2134 | 3124 | 1324 | 2314 | 4312 | 1432 | 3412 | 4312 | 2314 | 1324 | 3124 | 4132 | 1342 | 2341 | 4321 | 1243 | 4123 | 1234 | 3142 | 4312 | 1312 |
| 1234 | 2134 | 3124 | 1324 | 2314 | 4312 | 1432 | 3412 | 4312 | 2314 | 1324 | 3124 | 4132 | 1342 | 2341 | 4321 | 1243 | 4123 | 1234 | 3142 | 4312 | 1312 |
| 1234 | 2134 | 3124 | 1324 | 2314 | 4312 | 1432 | 3412 | 4312 | 2314 | 1324 | 3124 | 4132 | 1342 | 2341 | 4321 | 1243 | 4123 | 1234 | 3142 | 4312 | 1312 |
| 1234 | 2134 | 3124 | 1324 | 2314 | 4312 | 1432 | 3412 | 4312 | 2314 | 1324 | 3124 | 4132 | 1342 | 2341 | 4321 | 1243 | 4123 | 1234 | 3142 | 4312 | 1312 |
| 1234 | 2134 | 3124 | 1324 | 2314 | 4312 | 1432 | 3412 | 4312 | 2314 | 1324 | 3124 | 4132 | 1342 | 2341 | 4321 | 1243 | 4123 | 1234 | 3142 | 4312 | 1312 |
| 1234 | 2134 | 3124 | 1324 | 2314 | 4312 | 1432 | 3412 | 4312 | 2314 | 1324 | 3124 | 4132 | 1342 | 2341 | 4321 | 1243 | 4123 | 1234 | 3142 | 4312 | 1312 |
| 1234 | 2134 | 3124 | 1324 | 2314 | 4312 | 1432 | 3412 | 4312 | 2314 | 1324 | 3124 | 4132 | 1342 | 2341 | 4321 | 1243 | 4123 | 1234 | 3142 | 4312 | 1312 |
| 1234 | 2134 | 3124 | 1324 | 2314 | 4312 | 1432 | 3412 | 4312 | 2314 | 1324 | 3124 | 4132 | 1342 | 2341 | 4321 | 1243 | 4123 | 1234 | 3142 | 4312 | 1312 |
| 1234 | 2134 | 3124 | 1324 | 2314 | 4312 | 1432 | 3412 | 4312 | 2314 | 1324 | 3124 | 4132 | 1342 | 2341 | 4321 | 1243 | 4123 | 1234 | 3142 | 4312 | 1312 |
| 1234 | 2134 | 3124 | 1324 | 2314 | 4312 | 1432 | 3412 | 4312 | 2314 | 1324 | 3124 | 4132 | 1342 | 2341 | 4321 | 1243 | 4123 | 1234 | 3142 | 4312 | 1312 |
| 1234 | 2134 | 3124 | 1324 | 2314 | 4312 | 1432 | 3412 | 4312 | 2314 | 1324 | 3124 | 4132 | 1342 | 2341 | 4321 | 1243 | 4123 | 1234 | 3142 | 4312 | 1312 |
Lemma 9  Suppose that $n \geq 5$ and $F \subset E(SG_n)$ with $|F| = n - 3$. Let $u \in V(SG_n)$. Then there exist two mutually independent Hamiltonian cycles rooted at $u$ in $SG_n - F$.

Proof  Because $SG_n$ is edge-transitive, there exists an automorphism $\phi_1$ of $SG_n$ mapping any edge in $F$ into an edge of dimension $n$. For convenience, let $w = \phi_1(u)$. Moreover, let $\phi_2 : V(SG_n) \to V(SG_n)$ be a function defined as follows: $\phi_2(v) = h((v_1)h((v_2)), \ldots, h((v_n))$ for any $v \in V(SG_n)$, where $h : \mathbb{I}_n \to \mathbb{I}_n$ is a function such that $h((w_j)) = j$ for each $j \in \mathbb{I}_n$. Clearly, $\phi_2$ is also an automorphism of $SG_n$. It is easy to check that the composition of $\phi_1$ and $\phi_2$, namely $\phi_2 \circ \phi_1$, is an automorphism of $SG_n$ such that $\phi_2 \circ \phi_1(u) = e$. For this reason, we can assume that $u = e$, and $F$ contains at least one edge of dimension $n$. Let $F_k = F \cap E(SG_n^{(k)})$ for every $k \in \mathbb{I}_n$.

As a result, we have $|F_k| \leq n - 4$ for every $k \in \mathbb{I}_n$.

Case 1  Suppose that $\{e, (e)^n\} \notin F$. Let $B = (b_{i,j})$ be a $2 \times n$ matrix with

$$b_{i,j} = \begin{cases} j & \text{if } i = 1, \\ n & \text{if } i = 2 \text{ and } j = 1, \\ j + 1 & \text{if } i = 2 \text{ and } 2 \leq j \leq n - 2, \\ 2 & \text{if } i = 2 \text{ and } j = n - 1, \\ 1 & \text{if } i = 2 \text{ and } j = n. \end{cases}$$

By Lemma 3, there exists a Hamiltonian path $P$ of $SG_n^{(b_{i,j})} - F$ joining $(e)^n$ to $e$. Similarly, there exists a Hamiltonian path $H$ of $SG_n^{(b_{i,j})} - F$ joining $e$ to $(e)^n$. Then we set $C_1 := e (e)^n P e$ and $C_2 := e H (e)^n e$. Obviously, $\{C_1, C_2\}$ forms a set of two mutually independent Hamiltonian cycles rooted at $e$ in $SG_n - F$ (see Figure 2(a) for illustration).

Case 2  Suppose that $\{e, (e)^n\} \in F$ and $|F_n| = n - 4$. Obviously, we have $|F_k| = 0$ for every $k \in \mathbb{I}_{n-1}$. By Theorem 1, there exists a Hamiltonian cycle $H = e R q p e$ of

![Figure 2](https://example.com/figure2.png)

Figure 2. The two mutually independent Hamiltonian cycles in $SG_5 - F$ for Lemma 9.
SG_{n}\) - \(F_n\). Accordingly, we have that \(\{p, (p^n)\} \notin F\) and \(\{q, (q^n)\} \notin F\). By Lemma 2, \((p_1) \neq (q)\). We set \((p_1) = i_{n-1}\) and \((q_1) = i_1\). Let \(i_2i_3 \cdots i_{n-2}\) be an arbitrary permutation of \(\{i_1, i_{n-1}\}\).

For \(1 \leq k \leq n - 2\), let \(x_k\) be a vertex of \(V_0(SG_{n}^{[i_1]}))\) such that \((x_k)_1 = i_{k+1}\) and \((x_k, (x_k^n)) \notin F\). By Theorem 2, there exists a Hamiltonian path \(P_0\) of \(SG_{n}^{[i_1]}\) joining \((q^n)\) to \(x_1\). Similarly, there is a Hamiltonian path \(P_{n-1}\) of \(SG_{n}^{[i_{n-1}]}\) joining \((x_{n-2})\) to \((p^n)\). Then we set \(C_1 := e R q (q^n) P_1 x_1 (x_1^n) P_2 x_2 (x_2^n) \cdots x_{n-2} (x_{n-2}^n) P_{n-1} (p^n) p e\).

We can pick out a vertex \(y_{n-1}\) of \(V_1(SG_{n}^{[i-1]}))\) such that \((y_{n-1})_1 = i_2\) and \((y_{n-1}, (y_{n-1}^n)) \notin F\). For \(2 \leq k \leq n - 3\), we have \(|\{u \in V_1(SG_{n}^{[i_1]}))|(u_1) = i_{k+1}\) and \(d_{SG_{n}}(u, (x_{k}^n)) = 2\)\} = \(n - 3 < (n - 2)!/2\) if \(n \geq 5\). Thus, we can choose a vertex \(y_k\) of \(V_1(SG_{n}^{[i_1]}))\) such that \(d_{SG_{n}}(y_k, (x_{k}^n)) > 2, (y_k)_1 = i_{k+1}\) and \((y_k, (y_k^n)) \notin F\) for \(2 \leq k \leq n - 3\). Since \(|\{u \in V_1(SG_{n}^{[i-2]}))|(u_1) = i_1\) and \(d_{SG_{n}}(u, (x_{k}^n)) = 2\} = \(n - 3 < (n - 2)!/2\) if \(n \geq 5\), we can choose a vertex \(y_{n-2}\) of \(V_1(SG_{n}^{[i-2]}))\) such that \(d_{SG_{n}}(y_{n-2}, (x_{n-3}^n)) > 2, (y_{n-2})_1 = i_1\) and \((y_{n-2}, (y_{n-2}^n)) \notin F\). By Theorem 2, there exists a Hamiltonian path \(Q_1\) of \(SG_{n}^{[i]}\) joining \((y_{n-2}^n)\) to \((q^n)\). Again, there exists a Hamiltonian path \(Q_2\) of \(SG_{n}^{[i]}\) joining \((y_{n}^n)\) to \((y_{n}^n)\). Then we set \(C_2 := e p (p^n) Q_{n-1} y_{n-1} (y_{n-1}^n) Q_2 y_2 (y_2^n) Q_3 y_3 (y_3^n) \cdots y_{n-2}^n Q_1 (q^n) q R^{-1} e\).

In summary, \(C_1, C_2\) forms a set of 2-mutually independent Hamiltonian cycles rooted at \(e\) in \(SG_{n} - F\). Figure 2(b) illustrates \(C_1\) and \(C_2\) in \(S_3\).

Case 3 Suppose that \(e, (e^n) \in F\) and \(|F_n| \leq n - 5\). Since \(|F| = n - 3\), there exists an integer of \(\{i_{n-1} - \{i_{n-1}\}\}\) to vertices of \(V_0(SG_{n}^{[i_1]}))\). Thus, we can choose a vertex \(w \in V_0(SG_{n}^{[i_1]})) - \{e\}\) such that \((w_1) = i_1\) and \((w, (w^n)) \notin F\). By Theorem 2, there exists a Hamiltonian path \(R\) of \(SG_{n}^{[i]} - F_{i_1}\) - \((e^{i-1})\) joining \(e\) to \(w\). For each \(1 \leq k \leq n - 2\), we have \(i_{k+1}\) and \((x_k, (x_k^n)) \notin F\). By Theorem 2, there exists a Hamiltonian path \(P_{k}\) of \(SG_{n}^{[i_1]} - F_{i_1}\) joining \((x_{k}^n)\) to \((x_{k-1}^n)\) for each \(2 \leq k \leq n - 2\). Moreover, let \(i_{3}i_{4} \cdots i_{n-2}\) be an arbitrary permutation of \(\{i_{1}, i_2, i_{n-1}\}\) such that \(|\{u \in V_{1}(SG_{n}^{[^i_1]}))(u_1) = i_{k+1}\) and \(d_{SG_{n}}(u, (x_{k}^n)) = 2\} = \(n - 3 < (n - 2)!/2\) if \(n \geq 5\). Thus, we can choose a vertex \(y_{n-2}\) of \(V_{1}(SG_{n}^{[^i_{n-1}]}))\) such that \(d_{SG_{n}}(y_{n-2}, (x_{n-3}^n)) > 2, (y_{n-2})_1 = i_1\) and \((y_{n-2}, (y_{n-2}^n)) \notin F\). By Theorem 2, there exists a Hamiltonian path \(Q_{n-1}\) of \(SG_{n}^{[^i_1]} - F_{i_1}\) joining \((x_{n-2}^n)\) to \((e^{i-1})\). Then we set \(C_1 := e R w (w^n) P_1 x_1 (x_1^n) P_2 x_2 (x_2^n) \cdots x_{n-2}^n P_{n-1} ((e^{i-1})^n) (e^{i-1}) e\).

Next, we can pick out a vertex \(y_{n-1}\) of \(V_1(SG_{n}^{[^i_{n-1}]}))\) such that \((y_{n-1})_1 = i_2\) and \((y_{n-1}, (y_{n-1}^n)) \notin F\). For any \(2 \leq k \leq n - 3\), we have \(|\{u \in V_{1}(SG_{n}^{[^i_1]}))(u_1) = i_{k+1}\) and \(d_{SG_{n}}(u, (x_{k}^n)) = 2\} = \(n - 3\). By Lemma 1, there are \((n - 2)!/2\) pairwise disjoint edges joining vertices of \(V_0(SG_{n}^{[^i_1]}))\). It is noticed that \((n - 2)!/2 > (n - 3) + (n - 5) = 2n - 8\) if \(n \geq 5\). Thus, we can choose a vertex \(y_{n-2}\) of \(V_{1}(SG_{n}^{[^i_{n-1}]}))\) such that \(d_{SG_{n}}(y_{n-2}, (x_{n-3}^n)) > 2, (y_{n-2})_1 = i_{k+1}\) and \((y_{n-2}, (y_{n-2}^n)) \notin F\) for each \(2 \leq k \leq n - 3\). Since \((n - 2)!/2 > |\{u \in V_{1}(SG_{n}^{[^i_1]}))(u_1) = i_1\) and \(d_{SG_{n}}(u, (x_{k}^n)) = 2\} = \(n - 3\) + (n - 5) = 2n - 8\) if \(n \geq 5\), we can choose a vertex \(y_{n-2}\) of \(V_{1}(SG_{n}^{[^i_{n-1}]}))\) such that \(d_{SG_{n}}(y_{n-2}, (x_{n-3}^n)) > 2, (y_{n-2})_1 = i_1\) and \((y_{n-2}, (y_{n-2}^n)) \notin F\). By Theorem 2, there exists a Hamiltonian path \(Q_1\) of \(SG_{n}^{[^i_1]} - F_{i_1}\) joining \((y_{n-2})\) to \((w^n)\). Again, there exists a Hamiltonian path \(Q_2\) of \(SG_{n}^{[^i_1]} - F_{i_1}\) joining \((y_{n-2})\) to \((y_{n}^n)\). Then we set \(C_1 := e R w (w^n) P_1 x_1 (x_1^n) P_2 x_2 (x_2^n) \cdots x_{n-2}^n P_{n-1} ((e^{i-1})^n) (e^{i-1}) e\).
a Hamiltonian path \( Q_k \) of \( SG_n^{[k]} - F_k \) joining \((y_{k-1})^n\) to \( y_k \) for \( 3 \leq k \leq n - 2 \). We set \( C_2 := e(e)^{n-1} \sqrt{e(e)^{n-1}} Q_{n-1} - y_{n-1} (y_{n-1})^n Q_{n-1} y_{n-1} (y_{n-1})^n \cdot (y_{n-2})^n Q_1 (w)^n w R^{-1} e. \)

As a result, \( \{ C_1, C_2 \} \) turns out to be a set of two-mutually independent Hamiltonian cycles rooted at \( e \) in \( SG_n - F \). Figure 2(c) illustrates \( C_1 \) and \( C_2 \) in \( S_5 \).

\[ \Box \]

**Lemma 10** Let \( f \) be any integer of \( \mathbb{N} \) for \( n \geq 5 \). Suppose that \( F \subset E(SG_n) \) with \( |F| = f \), and \( u \) is any vertex of \( SG_n \). Then there exist \((n - 1 - f)\)-mutually independent Hamiltonian cycles rooted at \( u \) in \( SG_n - F \).

**Proof** As explained in the proof of Lemma 9, there exists an automorphism of \( SG_n \) that can map any edge in \( F \) into an edge of dimension \( n \) and map \( u \) to \( e \) simultaneously. Hence, we can assume that \( u = e \), and \( F \) contains at least one edge of dimension \( n \). Let \( F_k = F \cap E(SG_n^{[k]}) \) for every \( k \in \mathbb{N} \). Thus, we have \( |F_k| \leq n - 5 \) for every \( k \in \mathbb{N} \). Moreover, let \( A_1 = E^{1,n} - \{ (e, (e)^n) \} \), and let \( A_i = E^{i,n} \cup \{ (e, (e)^i) \} \) for \( 2 \leq i \leq n - 1 \).

**Case 1** Suppose that \( (e, (e)^n) \in F \). It is noticed that there are at least \( n - 1 - f \) elements of \( |F \cap A_2|, |F \cap A_3|, \ldots, |F \cap A_{n-4}| \) equal to 0. Without loss of generality, we can assume that \( |F \cap (\cup_{i=0}^{n-4} A_i)| = 0 \). Thus, at least one of \( |F \cap A_1|, \ldots, |F \cap A_{n-1}| \) equals to 0.

**Subcase 1.1** Suppose that \( |F \cap A_1| = 0 \). Let \( B = (b_{i,j}) \) be an \((n - 1 - f) \times n\) matrix with

\[
 b_{i,j} = \begin{cases} 
 f + i + j & \text{if } f + i + j \leq n, \\
 f + i + j - n & \text{otherwise.}
 \end{cases}
\]

It is noticed that \( b_{i,n-f-i} = n \) for every \( 1 \leq i \leq n - 1 - f \). Then we will construct a set of \((n - 1 - f)\)-mutually independent Hamiltonian cycles \( \{ C_1, C_2, \ldots, C_{n-1-f} \} \) rooted at \( e \) in \( SG_n - F \).

Let \( i \in \mathbb{N} \). We set \( t_i = n - f - i \). By Lemma 7, there exists a Hamiltonian path \( Q_i \) of \( (SG_n^{[t_i]} - F_{t_i}) \) of \( (e, (e)^{t_i}, (e)^{t_i}) \) joining two vertices \( x_i \) and \( x_i \) in \( V_0(SG_n^{[t_i]}) - \{ e \} \) such that \( (x_i)^{t_i} = b_{i,t_i-1} \) and \( (y_i)^{t_i} = b_{i,t_i+1} \). By Lemma 3, there exists a Hamiltonian path \( P_i \) of \( SG_n^{[t_i]} - F \) joining \((e)^{t_i}\) to \((e)^{t_i}\). Similarly, there exists a Hamiltonian path \( R_i \) of \( SG_n^{[t_i]} - F \) joining \((e)^{t_i}\) to \((e)^{t_i}\). Then we set \( C_i := e (e)^{t_i} ((e)^{t_i})^{t_i} P_i (x_i)^{t_i} x_i Q_i y_i (y_i)^{t_i} R_i ((e)^{t_i})^{t_i} (e)^{t_i}\).

By Lemma 6, \( (SG_n^{[n-1-f]} - F_{n-1-f}) - \{ e \} \) has a Hamiltonian path \( T \) joining \((e)^{b_{i,n}}\) to a vertex \( z \) of \( V_0(SG_n^{[n-1-f]}) - \{ e \} \) with \( (z_1)^{n-1} = b_{n-1,n-f} - 2 \). By Lemma 3, there exists a Hamiltonian path \( W \) of \( SG_n^{[n-1-f]} - F \) joining \((z)^{n-1}\) to \((e)^{b_{i,n-1-f}}\). Then we set \( C_{n-1-f} := e (e)^{b_{i,n}} T Z (z)^{n} W ((e)^{b_{i,n-1-f}})^{n} (e)^{b_{i,n-1-f}} e \).

As a result, \( \{ C_1, \ldots, C_{n-2-f}, C_{n-1-f} \} \) turns out to be a set of \((n - 1 - f)\)-mutually independent Hamiltonian cycles rooted at \( e \) in \( SG_n - F \). Figure 3 illustrates \( \{ C_1, C_2, C_3, C_4 \} \) in \( SG_6 - F \) with \( |F| = f = 1 \).

**Subcase 1.2** Suppose that \( |F \cap A_1| > 0 \). It is noticed that \( f \geq 2 \) in this subcase. Thus, at least one of \( |F \cap A_2|, \ldots, |F \cap A_{n-1}| \) equals to 0. Without loss of generality, we can assume that...
\[ |F \cap A_2| = 0 \text{. Let } B = (b_{i,j}) \text{ be an } (n - 1 - f) \times n \text{ matrix with} \]
\[
b_{i,j} = \begin{cases} 
  f + i + j & \text{if } f + i + j \leq n, \\
  2 & \text{if } f + i + j = n + 1, \\
  1 & \text{if } f + i + j = n + 2, \\
  f + i + j - n & \text{otherwise.} 
\end{cases}
\]

Using a similar manner to that of Subcase 1.1, we can construct a set of \((n - 1 - f)\)-mutually independent Hamiltonian cycles \(\{C_1, C_2, \ldots, C_{n-1-f}\}\) rooted at \(e\) in \(SG_n - F\).

**Case 2** Suppose that \(\{e, (e)^n\} \notin F\). It is noticed that there are at least \(n - 2 - f\) elements of \(|F \cap A_2|, |F \cap A_3|, \ldots, |F \cap A_{n-1}|\) equal to 0. Without loss of generality, we can assume that \(|F \cap (\cup_{j=f+2}^{n-1} A_i)| = 0\). Thus, at least one of \(|F \cap A_1|, \ldots, |F \cap A_{f+1}|\) is 0.

**Subcase 2.1** Suppose that \(|F \cap A_1| = 0\). Let \(B_n = (b_{i,j})\) be an \((n - 1 - f) \times n\) matrix with
\[
B_5 = \begin{bmatrix}
  1 & 2 & 3 & 4 & 5 \\
  4 & 5 & 1 & 2 & 3 \\
  5 & 4 & 2 & 3 & 1
\end{bmatrix},
\]
and for \(n \geq 6\)
\[
b_{i,j} = \begin{cases} 
  j & \text{if } i = 1, \\
  f + i + j & \text{if } 2 \leq i \leq n - 2 - f \text{ and } f + i + j \leq n, \\
  f + i + j - n & \text{if } 2 \leq i \leq n - 2 - f \text{ and } f + i + j > n, \\
  n & \text{if } i = n - 1 - f \text{ and } j = 1, \\
  3 & \text{if } i = n - 1 - f \text{ and } j = 2, \\
  2 & \text{if } i = n - 1 - f \text{ and } j = 3, \\
  n - 1 & \text{if } i = n - 1 - f \text{ and } j = 4, \\
  j - 1 & \text{if } i = n - 1 - f \text{ and } 5 \leq j \leq n - 1, \\
  1 & \text{if } i = n - 1 - f \text{ and } j = n.
\end{cases}
\]

Then we construct a set of \((n - 1 - f)\)-mutually independent Hamiltonian cycles \(\{C_1, C_2, \ldots, C_{n-1-f}\}\) rooted at \(e\) in \(SG_n - F\) as follows.

We can pick out a vertex \(v\) of \(V_1(SG_n^{(b_{1n})}) - \{(e)^{b_{n-2-f,1}}\}\) with \((v)_1 = b_{1,n-1}\). By Theorem 2, there exists a Hamiltonian path \(W\) of \((SG_n^{(b_{1n})}) - F - b_{1,n-1}\) joining \(v\) to \((e)^{b_{n-2-f,1}}\). By Lemma 3, there exists a Hamiltonian path \(D\) of \(SG_n^{(b_{1,j})} - F\) joining \((e)^n\) to \((v)^n\). We set \(C_1 := e (e)^n D (v)^n v W (e)^{b_{n-2-f,1}} e\).
Let $i \in \mathbb{N}_{n-2-f-1}$. We set $t_i = n - f - i$. By Lemma 7, there exists a Hamiltonian path $Q_i$ of $(SG_n^{(b_{i,j})} - F_{b_{i,j}}) - \{e, (e)^{b_{i,j}}\}$ joining two vertices $x_i$ and $y_i$ in $V_0(SG_n^{(b_{i,j})} - \{e\})$ such that $(x_i)_1 = b_{i_1,1}$ and $(y_i)_1 = b_{i_1,1}$. By Lemma 3, there exists a Hamiltonian path $P_i$ of $SG_n^{(j=1,b_{i,j})}_{[j=1,b_{i,j}]} - F$ joining $((e)^{b_{i,j}})j$ to $(x_i)^j$. Similarly, there exists a Hamiltonian path $R_i$ of $SG_n^{(j=1,b_{i,j})}_{[j=1,b_{i,j}]} - F$ joining $(y_i)^j$ to $((e)^{b_{i,j}})j$. Then we set $C_i := e (e)^{b_{i,j}} ((e)^{b_{i,j}})j P_i (x_i)^j x_i Q_i y_i (y_i)^j R_i ((e)^{b_{i,j}})j (e)^{b_{i,j}} e$.

By Lemma 1, there are $(n-2)!/2 > n-3$ pairwise disjoint edges joining vertices of $V_0(SG_n^{(b_{1,j})})$ to vertices of $V_1(SG_n^{(b_{1,j})})$ for $3 \leq k \leq n-1$. Thus, we can choose a vertex $z_k$ of $V_0(SG_n^{(b_{1,j})})$ such that $(z_k)_1 = b_{n-1,f,k-1}$, $(z_k)_n \notin F$, and $z_k \notin C_i((k-1)(n-1)! + 1)$. By Lemma 4, there exists a Hamiltonian path $T$ of $(SG_n^{(j=1,b_{1,j} - f-1)})$ - $F$ - $\{e\}$ joining $((e)^{b_{1,j}})$ to $(z_k)^n$. By Theorem 2, there exists a Hamiltonian path $H_k$ of $SG_n^{(b_{1,j} - f)} - F_{b_{1,j} - f}$ joining $z_k$ to $(z_k+1)^n$ for $3 \leq k \leq n-2$. By Lemma 3, there exists a Hamiltonian path $H_{n-1}$ of $SG_n^{(j=1,b_{1,j} - f)} - F$ joining $z_{n-1}$ to $e^n$. Then we set $C_{n-1-f} := e((e)^{b_{1,j}}) T(z_k)^n z_k H(z_k)^n \cdots z_{n-2} H_{n-2}(z_{n-1}) H_{n-1}(e)^n$.

Consequently, $\{C_1, C_2, \ldots, C_{n-2-f}, C_{n-1-f}\}$ is a set of $(n-1-f)$-mutually independent Hamiltonian cycles rooted at $e$ in $SG_n - F$. Figure 4(a) illustrates $\{C_1, C_2, C_3, C_4\}$ in $SG_6 - F$ with $|F| = f = 1$.

Subcase 2.2 Suppose that $|F \cap A_1| > 0$. Thus, at least one of $|F \cap A_2|, \ldots, |F \cap A_{f+1}|$ equals to 0. Without loss of generality, we can assume that $|F \cap A_2| = 0$. Let $B_n = (b_{i,j})$ be an $(n-1-f) \times n$ matrix with

$$b_{i,j} = \begin{cases} n & \text{if } i = 1 \text{ and } j = 1, \\ n + j & \text{if } i = 1 \text{ and } 2 \leq j \leq n-2, \\ 2 & \text{if } i = 1 \text{ and } j = n-1, \\ 1 & \text{if } i = 1 \text{ and } j = n, \\ f + i + j & \text{if } 2 \leq i \leq n-2-f \text{ and } f + i + j \leq n, \\ 2 & \text{if } 2 \leq i \leq n-2-f \text{ and } f + i + j = n + 1, \\ 1 & \text{if } 2 \leq i \leq n-2-f \text{ and } f + i + j = n + 2, \\ f + i + j - n & \text{if } 2 \leq i \leq n-2-f \text{ and } f + i + j \geq n + 3, \\ j & \text{if } i = n-1-f. \end{cases}$$

By Lemma 1, there are $(n-2)!/2 > n-3$ pairwise disjoint edges joining vertices of $V_0(SG_n^{(b_{1,j})})$ to vertices of $V_1(SG_n^{(b_{1,j})})$. Thus, we can choose a vertex $z$ of $V_0(SG_n^{(b_{1,j})})$ such that $(z)_1 = b_{1,1}$, $(z) \notin F$, and $(z)^n \notin ((e)^{b_{1,j}})$. By Theorem 2, there exists a Hamiltonian path $T$ of $(SG_n^{(j=1,b_{1,j} - f-1)}) - \{e\}$ joining $((e)^{b_{1,j}})$ to $(z)^n$. By Lemma 3, there exists a Hamiltonian path $H$ of $SG_n^{(j=1,b_{1,j})}_{[j=1,b_{1,j}]} - F$ joining $z$ to $(e)^n$. Then we set $C_1 := e((e)^{b_{1,j}}) T(z)^n z H(e)^n$.

Let $i \in \mathbb{N}_{n-2-f-1}$. We set $t_i = n - f - i$. By Lemma 7, there exists a Hamiltonian path $Q_i$ of $(SG_n^{(b_{i,j})} - F_{b_{i,j}}) - \{e, (e)^{b_{1,j}}\}$ joining two vertices $x_i$ and $y_i$ in $V_0(SG_n^{(b_{i,j})} - \{e\})$ such that $(x_i)_1 = b_{i_1,1}$ and $(y_i)_1 = b_{i_1,1}$. By Lemma 3, there exists a Hamiltonian path $P_i$ of $SG_n^{(j=1,b_{i,j})}_{[j=1,b_{i,j}]} - F$ joining $((e)^{b_{i,j}})j$ to $(x_i)^j$. Similarly, there exists a Hamiltonian path $R_i$ of $SG_n^{(j=1,b_{i,j})}_{[j=1,b_{i,j}]} - F$ joining $(y_i)^j$ to $((e)^{b_{i,j}})j$. Then we set $C_i := e(e)^{b_{i,j}} ((e)^{b_{i,j}})j P_i (x_i)^j x_i Q_i y_i (y_i)^j R_i ((e)^{b_{i,j}})j (e)^{b_{i,j}} e$.

By Lemma 1, there are $(n-2)!/2 > n-3$ pairwise disjoint edges joining vertices of $V_0(SG_n^{(b_{1,j})})$ to vertices of $V_1(SG_n^{(b_{1,j})})$. Thus, we can pick out a vertex $w$ of $V_0(SG_n^{(b_{1,j})})$
such that $(w_1) = b_{n-1-f,3}, \{w, (w)^n\} \notin F,$ and $d_{SG_n}(w, (y_{n-2-f})^n) > 1.$ Moreover, we choose a vertex $v$ of $V_1(SG_n[b_{n-1-f,n}])$ such that $(v_1) = b_{n-1-f,n-1}$ and $\{v, (v)^n\} \notin F.$ By Lemma 3, there exists a Hamiltonian path $D_1$ of $SG_n[\bigcup_{j=1}^{j=n-1} \{b_{n-1-f,j}\}] - F$ joining $(e)^n$ to $w.$ Similarly, there exists a Hamiltonian path $D_2$ of $SG_n[\bigcup_{j=1}^{j=n-1} \{b_{n-1-f,j}\}] - F$ joining $(w)^n$ to $(v)^n.$ By Theorem 2, there exists a Hamiltonian path $W$ of $(SG_n[\bigcup_{j=1}^{j=n-1} \{b_{n-1-f,j}\}] - F)$ joining $v$ to $(e)^{b_{n-2-f,1}}.$ Then we set $C_{n-1-f} := e \{(e)^n D_1 w (w)^n D_2 (v)^n \} \cup W (e)^{b_{n-2-f,1}} e.$

Hence, $\{C_1, C_2, \ldots, C_{n-2-f}, C_{n-1-f}\}$ forms a set of $(n-1-f)$-mutually independent Hamiltonian cycles rooted at $e$ in $SG_n - F.$ Figure 4(b) illustrates $\{C_1, C_2, C_3, C_4\}$ in $SG_6 - F$ with $|F| = f = 1.$

Combining Theorem 3 and Lemmas 8–10, we summarize those results as follows.

**Theorem 4** Let $F \subset E(SG_n)$ with $|F| \leq n - 3$ for $n \geq 3,$ and let $u \in V(SG_n).$ Then there exist $(n - 2 - |F|)$-mutually independent Hamiltonian cycles rooted at $u$ in $SG_n - F$ if $n \in \{3, 4\},$ and there exist $(n - 1 - |F|)$-mutually independent Hamiltonian cycles rooted at $u$ in $SG_n - F$ if $n \geq 5.$

5. Conclusion

In this paper, we study the problem of finding mutually independent Hamiltonian cycles in a faulty star graph. That is, given a set of faulty edges $F \subset E(SG_n)$ with $|F| \leq n - 3,$ we show that $SG_n - F$ has a set of $(n - 2 - |F|)$-mutually (respectively, $(n - 1 - |F|)$-mutually) independent Hamiltonian cycles rooted at any vertex if $n \in \{3, 4\}$ (respectively, $n \geq 5$). We believe that a similar result could be obtained for the graph generated by any transposition tree of order $n [1]$ when at most $n - 3$ edges fail. On the other hand, we also believe that our current result can be further refined; to be precise, we would like to show that $IHC(SG_n - F) = \delta(SG_n - F),$ where $\delta(SG_n - F)$ denotes the minimum degree of graph $SG_n - F.$

The edge faults considered in this work are random and independent. To guarantee that such a faulty star graph remains Hamiltonian in this situation, the maximum number of faulty edges
cannot exceed \( n - 3 \). For this reason, we can make a more general condition on the nature of faulty edges such that every vertex still has at least two neighbours in a faulty star graph. This kind of sets of faulty edges is called conditionally faulty. Let \( F \subset E(\text{SG}_n) \) be conditionally faulty. Then we believe that mutually independent Hamiltonian cycles can be constructed in \( \text{SG}_n \) if \( |F| \leq 3n - 10 \). These results convince us that the star graph is really robust enough to interconnect computing units in parallel and distributed systems.

Acknowledgements

The authors express the most immense gratitude to Editor-in-Chief, Editor, and the anonymous referees for their careful reading and constructive comments. They greatly improve the quality of the paper. This work was supported in part by the National Science Council of the Republic of China under Contract NSC 98-2218-E-468-001-MY3. J.J.M. Tan was supported in part by the National Science Council of the Republic of China under Contract NSC 96-2221-E-009-134-MY3 and in part by the Aiming for the Top University and Elite Research Center Development Plan.

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