A New Dual-type Method Used in Solving Optimal Power Flow Problems

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Abstract—In the framework of SQP method for OPF problems, we propose a new dual-type method for solving the QP subproblems induced in the SQP method. Our method achieves some attractive features; it is computationally efficient and numerically stable. The computational formulae of our method are simple, concise and easy to be programmed. We have tested our method for OPF problems on several power systems including a 2500-bus system.

I. INTRODUCTION

Numerous numerical techniques [1]–[10] have been developed for solving optimal power flow (OPF) problems. These methods are based on various mathematical programming techniques such as successive linear programming (SLP) method [1]–[3], successive quadratic programming (SQP) method [4]–[6], Lagrangian Newton method [7]–[9] or the newly developed interior point (IP) method [10]. Each of the above methods has its special features and advantages. Observing the SQP method which possesses a quadratic convergence rate, however, the reduced Hessian is dense. The innovative Lagrangian Newton method [7], [8] successfully exploits the sparsity structure of the system; however, efforts are needed to cope with the difficulties of identifying the binding inequality constraints and the possibility of singular Hessian matrix as pointed out by Monticelli and Liu in [9], and they provided remedied strategies to overcome those pitfalls. Nonetheless, the method in [9] as well as the method in [7] and [8] require sophisticated software programming skill.

In this paper, we use the framework of SQP method and propose a new dual-type method to solve the QP subproblems. Our method intends to achieve the following features: (i) good convergence rate, (ii) no need to identify the binding constraints, (iii) computational efficiency, (iv) easy programming and (v) numerical stability.

In the framework of SQP method, our method will inherit the advantage of fast convergence as demonstrated in Section V. Features (ii)–(iv) will be achieved by the proposed dual-type method as explained in Section III. To address feature (v), we provide a mathematical proof for the convergence of the proposed dual-type method in the Appendix.

II. STATEMENT OF THE OPTIMAL POWER FLOW PROBLEM

Throughout this paper, if not specifically explained, we assume the following notations:

e, f: state variables represent the real and imaginary part of the complex voltage.

u: control variables including real and reactive power generation, P_G and Q_G, transformer tap ratio, switching capacitor banks, etc..

x = (u, e, f) denotes the vector of all variables.

F(z): objective function which can be total generation cost, pollution cost, system losses, etc.

g(x): real and reactive power mismatch.

h(e, f): functional inequality constraints such as security constraints on line flows for specified lines.

V: vector of voltage magnitude, \( V_i \equiv \sqrt{e_i^2 + f_i^2} \).

\( u, u' \): upper and lower limits of voltage magnitude.

\( P_G, Q_G, P_G', Q_G' \): upper and lower limits of control variables u, such as \( P_G, Q_G, P_G', Q_G' \), etc.

h, h': upper and lower limits of functional inequality constraints.

k, t: iteration index.

(\cdot): step-size.

diag[\cdot]: a diagonal matrix formed by the diagonal terms of the matrix \cdot.

\Delta: : the increment of the vector.

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\( P(z) \): penalty function for the violations of constraints.
\( w \): penalty coefficient.
\( \lambda \): the Lagrange multiplier vector.
\( \phi(\lambda) \): the dual function of the QP subproblem.
\( \phi^*(\lambda) \): unconstrained dual function.
\( \Omega \): the set formed by the inequality constraints of the QP subproblems.
\( \gamma, \eta, \sigma_p, \sigma_d \): positive real numbers.
\( \tau_p, \tau_d \in (0, 1) \).

The OPF problem can be stated as follows:

\[
\begin{align*}
\min_x & \quad F(x) \\
g(x) &= 0 \\
V \leq x \leq \tilde{V} \\
u \leq u \leq \tilde{u}
\end{align*}
\]  
(1)

**Remark 1** For the purpose of explanation, we do not include the functional inequality constraints, \( h(e, f) \leq h \), in (1), however, this will be treated afterwards.

### III. SOLUTION METHOD

**A. The SQP Method**

The SQP method uses the following iterations to solve the OPF problem given in (1):

\[ x(k+1) = x(k) + \alpha(k)\Delta x(k) \]  
(2)

where \( \alpha(k) \) is a step-size, and \( \Delta x(k) \) is the solution of the following QP subproblem:

\[
\begin{align*}
\min_{\Delta x} & \quad \frac{\partial F(x(k))}{\partial x} \Delta x + \frac{1}{2} \Delta x^T H \Delta x \\
g(x(k)) + \frac{\partial g(x(k))}{\partial x} \Delta x &= 0, \\
V \leq x(k) + \frac{\partial V(k)}{\partial e} \Delta e + \frac{\partial V(k)}{\partial f} \Delta f \leq \tilde{V}, \\
u \leq u(k) + \Delta u \leq \tilde{u},
\end{align*}
\]  
(3)

in which the diagonal matrix \( H \) is defined by

\[ H = \text{diag} \left[ \frac{\partial^2 F(x(k))}{\partial x^2} \right] + \frac{1}{2} \eta I \]  
(4)

where \( I \) is an identity matrix, and \( \eta \) is a small positive real number but enough to make \( H \) positive definite.

**Step-size determination.** Concerning the step-size determination rule, a cubic fit or quadratic fit method [14] is popular especially for the unconstrained Lagrangean formulation in the Lagrangian Newton method [7]. However, in the SQP method, while reducing the value of objective function \( F(x) \), we should prevent \( x(k+1) \) being too far away from the nonlinear constraints in (1). Therefore, we employ Armijo’s rule [11], which considers the penalty of violating constraints, for the determination of step-size \( \alpha(k) \) as follows:

\[
F(x(k) + \tau_p^2 \sigma_p \Delta x(k)) + w P(x(k) + \tau_p^2 \sigma_p \Delta x(k)) - F(x(k)) \leq \frac{\gamma}{2} \tau_p^2 \sigma_p \Delta x^T(k) H \Delta x(k)
\]  
(5)

where the penalty function \( P(x) \) represents the penalty for the violations on the constraints and is defined by

\[
P(x) \equiv \max \left\{ \max_i \{g_i(x)\}, \max_i \{V_i - \tilde{V}_i, \tilde{V}_i - V_i\}, \right. \\
\max \{u_i - \bar{u}_i, u_i - \bar{u}_i\} \left. \right\},
\]  
(6)

\( w \) is a weighting penalty coefficient, and \( \gamma \in (0, \frac{1}{2}) \). Although Armijo’s rule seems inefficient, in most of our test results, the inequality test (5) is passed for \( m = 0 \) most of the times. Convergence of the SQP method (2) with \( \alpha(k) \) determined according to (5) has been shown in [11].

**Treatment of discrete control variables.** In the QP subproblem (3), we treat all the incremental variables \( \Delta x \) as continuous variables. However, the updated formula (2) may make the updated discrete control variables not having the exact discrete values. To remedy this pitfall, we apply an approximation rule for the update of discrete control variables as follows:

Let \( u_d \) be the subvector of \( u \) denote the discrete control variables, such as switching capacitor banks, transformer tap ratio, etc., then the continuous-value \( \Delta u_d(k) \) is the increment of \( u_d(k) \) obtained from solving (3). The approximation rule for the update of \( u_d(k+1) \) is

\[ u_d(k + 1) = [u_d(k) + \alpha(k)\Delta u_d(k)] \]  
(7)

where \([\cdot]\) denote the closest discrete-value to the value of (\( \cdot \)). Then \( u_d(k+1) \) obtained from (7) will be the closest discrete-value of \( u_d \) to the value of \( u_d(k) + \alpha(k)\Delta u_d(k) \).

**Comment 1** When there exist integer variables in a nonlinear programming problem, the computation is very involved. Therefore, heuristic methods are developed to handle integer variables in most of practical applications such as the approximation rule presented here. Though our heuristic rule works well in our problem as shown in Section V, there is no guarantee that this rule will obtain satisfactory solutions in general nonlinear programming problems consisting of integer variables.

**B. The Proposed Dual-type Method.**

Since in (3), all variables \( \Delta x \) are continuous variables, the objective function is strictly convex, and the constraints are linear, we can solve the dual problem of (3)
instead of solving (3) directly provided that the solution of (3) exists. This is well-known Duality Theory [14].

The dual problem of the QP subproblem (3) is

$$\max_\lambda \phi(\lambda)$$

where the dual function

$$\phi(\lambda) = \min_{\Delta x \in \Omega} \frac{\partial F(x(k))}{\partial x} \Delta x + \frac{1}{2} \Delta x^T H \Delta x + \lambda^T [g(x(k)) + \frac{\partial g(x(k))}{\partial x} \Delta x],$$

in which the set \( \Omega \) denotes the set of inequality constraints in (3) such that \( \Omega \equiv \{ \Delta x | V \leq V(k) + \Delta \delta_{\text{e}} + \frac{\partial V(k)}{\partial \delta} \Delta \delta \leq \bar{V}, \ u(k) \leq u(k) + \Delta \nu \leq \bar{u} \} \).

The proposed dual-type method uses the following iterations to solve (8):

$$\lambda(t + 1) = \lambda(t) + \beta(t) \Delta \lambda(t),$$

where \( \beta(t) \) is a step-size, and \( \Delta \lambda(t) \) is obtained from solving the linear equations

$$\left[ \frac{\partial^2 \phi(\lambda(t))}{\partial \lambda^2} - \delta I \right] \Delta \lambda(t) + \frac{\partial \phi(\lambda(t))}{\partial \lambda} = 0,$$

in which \( \delta > 0 \), \( I \) is an identity matrix and the unconstrained dual function \( \phi^u \) is defined by deleting the primal-variable constraints \( \Delta x \in \Omega \) in \( \phi(\lambda) \) shown in (9) such that

$$\phi^u(\lambda) = \min_{\Delta x} \frac{\partial F(x(k))}{\partial x} \Delta x + \frac{1}{2} \Delta x^T H \Delta x + \lambda^T [g(x(k)) + \frac{\partial g(x(k))}{\partial x} \Delta x]$$

Phase 1: Obtain the solution \( \Delta \bar{x} \) of \( \phi^u(\lambda(t)) \) for a given \( \lambda(t) \), that is the unconstrained minimization problem on the RHS of (9) with \( \lambda = \lambda(t) \) which can be solved in two phases using Projection Theory.

Phase 2: Project \( \Delta \bar{x} \) onto the constraint set \( \Omega \), and the resulting projection is \( \Delta \bar{x} \).

The first derivative \( \frac{\partial \phi(\lambda(t))}{\partial \lambda} \) and the approximate Hessian matrix \( \frac{\partial^2 \phi(\lambda(t))}{\partial \lambda^2} \) can be computed based on the formula given in [14] as follows:

$$\frac{\partial \phi(\lambda(t))}{\partial \lambda} = g(x(k)) + \frac{\partial g(x(k))}{\partial x} \Delta \bar{x}$$

$$\frac{\partial^2 \phi(\lambda(t))}{\partial \lambda^2} = -\frac{\partial g(x(k))}{\partial x} \|H^{-1} \frac{\partial g(x(k))}{\partial x}\|^2$$

(14)

where \( \Delta \bar{x} \) in (13) is the solution of \( \phi(\lambda(t)) \) for a given \( \lambda(t) \), that is the constrained minimization problem on the RHS of (9) with \( \lambda = \lambda(t) \). We will present the method using Projection Theory to solve \( \Delta \bar{x} \) later.

Since \( \frac{\partial^2 \phi(\lambda(t))}{\partial \lambda^2} \) is at least negative semidefinite, \( \frac{\partial^2 \phi(\lambda(t))}{\partial \lambda^2} - \delta I \) is negative definite. This ensures that \( \Delta \lambda(t) = \left[ \frac{\partial^2 \phi(\lambda(t))}{\partial \lambda^2} - \delta I \right]^{-1} \frac{\partial \phi(\lambda(t))}{\partial \lambda} \) is an ascent direction to maximize \( \phi(\lambda) \). However, to guarantee the updated point \( \lambda(t + 1) \) will increase the value of \( \phi(\lambda) \), we develop an Armijo’s rule to determine the step-size \( \beta(t) \) as follows:

Let \( \beta(t) = \frac{m(t)}{\tau_D^2} \sigma_D \), where \( 0 < \tau_D < 1, \sigma_D > 0 \), and \( m(t) \) is the smallest nonnegative integer \( m \) such that

$$\phi(\lambda(t)) + \frac{\delta \tau_D^2 \sigma_D}{2} \|\Delta \lambda(t)\|^2 \geq \phi(\lambda(t)) + \frac{\delta \tau_D^2 \sigma_D}{2} \|\Delta \lambda(t)\|^2.$$ (15)

A sketch of the mathematical proof for the justification of (15) and the convergence of (10) is given in the Appendix.

Remark 2 Since the objective function \( \phi(\lambda) \) in (8) is continuous and quadratic, it is practically suitable to use a cubic fit or quadratic fit method to determine the step-size \( \beta(t) \). On account of giving a rigorous mathematical proof, we prefer to use Armijo’s rule here.

Applicability of sparse matrix technique. The non-zero elements of the fixed-dimension, constant matrix \( \frac{\partial^2 \phi^u(\lambda(t))}{\partial \lambda^2} \) in (14) as well as \( \frac{\partial \phi^u(\lambda(t))}{\partial \lambda} - \delta I \) have the same structure as the bus admittance matrix of the power network. Therefore, we may employ a sparse matrix technique to solve linear equations (11).

However, to set up \( \frac{\partial \phi^u(\lambda(t))}{\partial \lambda} \) in (11), we need to compute \( \Delta \bar{x} \) first as shown in (13).

Applicability of Projection Theory. \( \Delta \bar{x} \) is the solution of the constrained minimization problem on the RHS of (9) with \( \lambda = \lambda(t) \) which can be solved in two phases using Projection Theory.

Phase 1: Obtain the solution \( \Delta \bar{x} \) of \( \phi^u(\lambda(t)) \) for a given \( \lambda(t) \), that is the unconstrained minimization problem on the RHS of (12) with \( \lambda = \lambda(t) \).

Phase 2: Project \( \Delta \bar{x} \) onto the constraint set \( \Omega \), and the resulting projection is \( \Delta \bar{x} \).

The validity of this two-phase method is justified based on Projection Theory in [12] and is shown in Theorem 1 and Theorem 2 in the Appendix. In the following, we will describe the detailed computational formulae of this two-phase method.

From (12), the solution of the unconstrained minimization problem \( \Delta \bar{x} \) which is \( \Delta \bar{u}, \Delta \bar{\delta}, \Delta \bar{f} \), can be analytically derived by

$$\Delta \bar{x} = -H^{-1} \left[ \frac{\partial F(x(k))}{\partial x} + \frac{\partial g(x(k))}{\partial x} \lambda \right]$$

(16)

Since \( H \) is a diagonal positive definite matrix, no extra effort is needed to compute \( H^{-1} \) in (16).

The inequality constraints for \( \Delta u, \Delta \delta, \Delta f \) and \( \Delta u \) are decoupled, and these inequality constraints are also decoupled for different buses; thus, the projection can be treated separately for each individual bus. The projection of \( \Delta \bar{u} \) onto the set \( \Omega \) is trivial and can be computed in the following: Let \( \Delta \bar{u}_i \) be the projection of \( \Delta \bar{u}_i \) onto the subset \( \{ \Delta u_i \leq u_i(k) + \Delta u_i \leq \bar{u}_i \} \), then

$$\Delta \bar{u}_i = \begin{cases} u_i(k) + \Delta \bar{u}_i > \bar{u}_i, \\ u_i(k) < u_i(k), \text{ if } u_i(k) + \Delta \bar{u}_i < u_i(k), \\ \Delta \bar{u}_i, \text{ otherwise.} \end{cases}$$

(17)
Though the projection of \((\Delta \dot{e}, \Delta \dot{f})\) onto the set \(\Omega\) is more complicated, by simple geometrical calculation, we can obtain the following: Let \((\Delta \dot{e}, \Delta \dot{f})\) be the projection of \((\Delta \dot{e}, \Delta \dot{f})\) onto the subset \(\{((\Delta \dot{e}, \Delta \dot{f}) | \sqrt{\epsilon_1(k)^2 + f_1(k)^2} + \frac{\epsilon_1(k)\Delta \dot{e}}{\sqrt{\epsilon_1(k)^2 + f_1(k)^2}} + \frac{f_1(k)\Delta \dot{f}}{\sqrt{\epsilon_1(k)^2 + f_1(k)^2}} \leq V_1\}\) of \(\Omega\). Let \(\tau_1 = |V_1\sqrt{\epsilon_1(k)^2 + f_1(k)^2} - (\epsilon_1(k)^2 + f_1(k)^2)|\), \(\tau_2 = |V_1\sqrt{\epsilon_1(k)^2 + f_1(k)^2} - (\epsilon_1(k)^2 + f_1(k)^2)|\), and \(\tau_3 = f_1(k)\Delta \dot{e} - \epsilon_1(k)\Delta \dot{f}\) then

\[
\Delta \dot{e}_i = \begin{cases} 
(\epsilon_1(k)\tau_1 + f_1(k)\tau_3)/(\epsilon_1^2(k) + f_1^2(k)) & \text{if } \epsilon_1(k)\Delta \dot{e}_i + f_1(k)\Delta \dot{f}_i > \tau_1, \\
(\epsilon_1(k)\tau_2 + f_1(k)\tau_3)/(\epsilon_1^2(k) + f_1^2(k)) & \text{if } \epsilon_1(k)\Delta \dot{e}_i + f_1(k)\Delta \dot{f}_i < \tau_2, \\
\Delta \dot{e}_i & \text{otherwise}.
\end{cases}
\] (18)

\[ \Delta \dot{f}_i = \begin{cases} 
(f_1(k)\tau_2 - \epsilon_1(k)\tau_3)/(\epsilon_1^2(k) + f_1^2(k)), & \text{if } \epsilon_1(k)\Delta \dot{e}_i + f_1(k)\Delta \dot{f}_i > \tau_1, \\
(f_1(k)\tau_2 - \epsilon_1(k)\tau_3)/(\epsilon_1^2(k) + f_1^2(k)), & \text{if } \epsilon_1(k)\Delta \dot{e}_i + f_1(k)\Delta \dot{f}_i < \tau_2, \\
\Delta \dot{f}_i & \text{otherwise}.
\end{cases}
\] (19)

**Remark 3** The reason that we do not use polar coordinate for bus voltage is the projection of phase angle onto the range \((-2\pi, 2\pi)\) will lose validity.

**C. Summary of the Overall Method.**

Our method for solving OPF problem (1) is using the SQP method (2) where \(\Delta x(k)\) is the solution of the QP subproblem (3). The proposed iterative dual-type method uses (10) to solve (8), the dual problem of the QP subproblem, instead of solving (3) directly. The \(\Delta \lambda(t)\) in (10) is obtained from solving (11) using sparse matrix technique, in which the \(\Delta \lambda\) needed to set up \(\frac{\delta \phi(\lambda(t))}{\delta \lambda}\) can be computed using the simple two-phase method. Consequently, the iterative dual-type method converges to optimal solution \(\lambda^*\), and the solution \(\Delta \dot{x}\) of the constrained minimization problem on the RHS of (9) with \(\lambda = \lambda^*\) is \(\Delta x(k)\), the solution of (3).

**D. The Advantageous Features of the Proposed Dual-type Method.**

In the following, we will describe how the proposed dual-type method achieves the four attractive features (ii)–(v) we claimed in Section 1.

In the dual function (9), we put the set of inequality constraints \(\Omega\) as the domain of primal variables \(\Delta x\) so that we can apply the Projection Theory to circumvent the need of identifying the binding inequality constraints. This address feature (ii).

All the computational requirements of our method for solving OPF problems almost lie in solving the linear equations (11) and the calculations of \(\Delta \dot{x}\) in (16)–(19). Equations (16)–(19) are as simple as they show. The approximate Hessian matrix \(\frac{\delta^2 \phi^*(\lambda(t))}{\delta \lambda^2} - \delta I\) is a sparse constant matrix; then the optimal ordering for the setup of memory locations for non-zero elements and fill-ins need only be done once. Therefore, the computational efficiency of our method can be expected. This address feature (iii).

Fig. 1 shows the flow chart of our method. Since all the computational formulae of our method are simple and concise, easy to be programmed is a natural result. This address feature (iv).

Convergence of the SQP method with step-size \(\alpha(k)\) determined according to (5) has been shown in [11]. Convergence of the proposed dual-type method for solving the dual problem of QP subproblem is shown in the Appendix. These rigorous mathematical justifications address feature (v).

**E. The Inclusion of Functional Inequality Constraints.**

For the nonlinear inequality constraints such as security constraints on line flows

\[
\bar{h} \leq h(e, f) \leq \tilde{h},
\] (20)

we can convert them into equality constraints by using surplus variable vector \(\vec{z}\) such that

\[
h(e, f) = \vec{z}
\]

\[
h \leq \vec{z} \leq \tilde{h}
\] (21)
Then (21) has the same form of constraints as in (1). Although the inclusion of equality constraints in (21) will increase the dimension of \( \frac{\partial \tilde{\mathcal{G}}(\mathbf{z})}{\partial \mathbf{z}} \), in our application, the sparsity is still retained; thus the sparse matrix technique can still apply. Consequently, the five attractive features of our method are still valid.

**Comment 2** If there are too many functional inequality constraints, the increase in the dimension of approximate Hessian matrix and the number of surplus variables will cause extra computational complexity even through the five attractive features still exist. However, the functional inequality constraints in OPF problems are mostly the line flow constraints on specific transmission lines which are generally not too many. Therefore, our approach is suitable for the problem considered in this paper.

### IV. SOME REMARKS

**A. Remark on Our Method**

There are many dual-type methods in the literature; for example, the dual LP method [2], the Lagrangian relaxation method [14], and the interior point method of primal-dual approach [10], etc. The proposed method for solving the dual problem of QP subproblem is also a dual-type method but differs from all the existing methods. Our method has similarity with the Lagrange relaxation method. However, in the dual function we defined in (9), we put the set of inequality constraints, \( \Omega \), as the domain of the primal variables instead of using Lagrange multiplier \( \mu \) to associate with the inequality constraints in Lagrangian relaxation approaches. This trick enables our method to have a constant sparse approximate Hessian matrix and apply Projection Theory to deal with the difficulties encountered by binding inequality constraints. Consequently, the four attractive features can be achieved as described in Section III.D.

**B. Remark on the Objective Function of OPF**

Observing from the objective function of (3), if the considered OPF problem is an economic dispatch control problem, the SQP method (2) is a Newton-type method. However, if the criterion is to minimize the system losses, the SQP method (2) is a Jacobi-type method. The Jacobi-type method associated with our dual-type method for solving the OPF problems is still very computationally efficient as we will demonstrate by numerical examples in next section.

**C. Remark on No Feasible Solution**

It is possible that the QP subproblem (3) does not have any feasible solution. If so, the objective value of the dual problem (8) will be unbounded. This is owing to the magnitude of some components of \( \Delta \lambda(t) \) increase as iteration \( t \) increases; in other words, the magnitude of some components of \( \frac{\partial \tilde{\mathcal{G}}(\mathbf{z})}{\partial \mathbf{z}} \) do not decrease as \( t \) increases as can be observed from (11). Investigating further, we found from (13) and (16) that components of \( \Delta \tilde{\mathcal{G}} - \Delta \mathcal{G} \) with irreducible magnitude should be the major factor causing the above problem. This implies that if we push all the primal variables \( x \) to satisfy the inequality constraints in \( \Omega \), the objective value of (8) will be unbounded. To remedy this infeasible situation, we may release the constraints with larger magnitude of \( \Delta \tilde{\mathcal{G}} - \Delta \mathcal{G} \) or \( |h(x(k) + \Delta \mathbf{z}) - h(x(k))| \) when \( \max (|\Delta \lambda_i(t)|) \) does not decrease. In fact, the above reasoning is similar to the way of handling infeasible solution in [8].

### V. TEST RESULTS

We tested our method for three cases of OPF problems on several power systems using a Spark-10 workstation.

**Case (i):** We consider the OPF with economic criterion with fixed transformer tap ratio, without switching capacitor banks, and no security constraints on line flows. We use total generation cost \( \sum \alpha_i P_i^2 + \beta_i P_i + \gamma_i \) as the objective function of the OPF problem. The coefficients \( \alpha_i, \beta_i, \) and \( \gamma_i \) of the generation cost curve are various for different generation buses. The parameters we select are as follows: \( \epsilon = 10^{-6}, w = 100, \sigma_D = \sigma_P = 1, \tau_D = \tau_P = 0.9, \delta = \eta = 1.0, \) and \( \gamma = 0.1 \). We have tested the OPF problems in this case on eight systems. All computer runs begin from a flat start with initial voltages being \( e_i = 1.0 \) and \( f_i = 0.0 \) for all buses \( i \)’s. Table I shows the final objective value and the CPU times consumption of each OPF problem in Case (i).

<table>
<thead>
<tr>
<th>IEEE system</th>
<th>no. of lines</th>
<th>no. of G-buses</th>
<th>final obj. value (100 MVA base)</th>
<th>CPU time (seconds)</th>
</tr>
</thead>
<tbody>
<tr>
<td>6-bus</td>
<td>11</td>
<td>3</td>
<td>432.711 (433.444)</td>
<td>0.08</td>
</tr>
<tr>
<td>9-bus</td>
<td>9</td>
<td>3</td>
<td>321.093 (321.490)</td>
<td>0.09</td>
</tr>
<tr>
<td>11-bus</td>
<td>17</td>
<td>3</td>
<td>620.384 (620.000)</td>
<td>0.15</td>
</tr>
<tr>
<td>30-bus</td>
<td>41</td>
<td>6</td>
<td>154.694 (154.000)</td>
<td>0.41</td>
</tr>
<tr>
<td>57-bus</td>
<td>86</td>
<td>29</td>
<td>4845.123 (4845.000)</td>
<td>0.57</td>
</tr>
<tr>
<td>118-bus</td>
<td>179</td>
<td>54</td>
<td>2638.720 (2681.937)</td>
<td>0.72</td>
</tr>
<tr>
<td>244-bus</td>
<td>445</td>
<td>46</td>
<td>36819.537 (36819.537)</td>
<td>6.11</td>
</tr>
<tr>
<td>2500-bus</td>
<td>3152</td>
<td>124</td>
<td>50577.994 (578.757)</td>
<td>578.75</td>
</tr>
</tbody>
</table>
Figure 2(a): The detailed progression of our method for solving the OPF problem on IEEE 224-bus system.

Figure 2(b): The detailed progression of our method for solving the OPF problem on IEEE 2500-bus system.

Table II: The final objective value and CPU time consumption of the tested OPF problems with system losses criterion in Case (ii).

<table>
<thead>
<tr>
<th>IEEE system</th>
<th>no. of lines</th>
<th>no. of G-buses</th>
<th>final object value (100MVA base)</th>
<th>CPU time (seconds)</th>
</tr>
</thead>
<tbody>
<tr>
<td>6-bus</td>
<td>11</td>
<td>3</td>
<td>9.99</td>
<td>0.07</td>
</tr>
<tr>
<td>9-bus</td>
<td>9</td>
<td>3</td>
<td>0.37</td>
<td>0.20</td>
</tr>
<tr>
<td>11-bus</td>
<td>17</td>
<td>3</td>
<td>6.50</td>
<td>0.25</td>
</tr>
<tr>
<td>30-bus</td>
<td>41</td>
<td>6</td>
<td>5.47</td>
<td>0.49</td>
</tr>
<tr>
<td>57-bus</td>
<td>86</td>
<td>29</td>
<td>8.82</td>
<td>0.56</td>
</tr>
<tr>
<td>118-bus</td>
<td>179</td>
<td>54</td>
<td>13.35</td>
<td>0.67</td>
</tr>
<tr>
<td>244-bus</td>
<td>445</td>
<td>46</td>
<td>37.82</td>
<td>5.57</td>
</tr>
<tr>
<td>2500-bus</td>
<td>3152</td>
<td>124</td>
<td>461.50</td>
<td>441.23</td>
</tr>
</tbody>
</table>
Table III: The final objective value and CPU time consumption of the tested OPF problems with economic criterion, consisting of switch capacitor banks and security constraints on line flows in Case (iii).

<table>
<thead>
<tr>
<th>IEEE system</th>
<th>no. of sw. cap. banks installed</th>
<th>no. of secur. constr. on line flows</th>
<th>final objective value (100 MVA base)</th>
<th>CPU time (seconds)</th>
</tr>
</thead>
<tbody>
<tr>
<td>6-bus</td>
<td>2</td>
<td>2</td>
<td>432.87 (432.90)</td>
<td>0.05 (0.11)</td>
</tr>
<tr>
<td>9-bus</td>
<td>2</td>
<td>2</td>
<td>321.02 (321.58)</td>
<td>0.11 (0.10)</td>
</tr>
<tr>
<td>11-bus</td>
<td>2</td>
<td>2</td>
<td>620.59 (620.21)</td>
<td>0.14 (0.16)</td>
</tr>
<tr>
<td>30-bus</td>
<td>4</td>
<td>4</td>
<td>155.39 (152.54)</td>
<td>0.57 (0.49)</td>
</tr>
<tr>
<td>57-bus</td>
<td>5</td>
<td>10</td>
<td>4850.73 (4849.17)</td>
<td>0.53 (0.54)</td>
</tr>
<tr>
<td>118-bus</td>
<td>10</td>
<td>20</td>
<td>2636.94 (2636.82)</td>
<td>1.26 (1.10)</td>
</tr>
<tr>
<td>244-bus</td>
<td>15</td>
<td>20</td>
<td>36819.41 (36815.08)</td>
<td>9.53 (11.81)</td>
</tr>
</tbody>
</table>

The Special Structure of H.

According to [13], almost all the cost criteria can be formulated as a functions of real power generation. Thus, the diagonal terms of the diagonal positive definite matrix H in (4) corresponding to e and f have the same values as $\frac{1}{2} \eta$.

**Theorem 1** The solution $\Delta \tilde{x}$ of the constrained minimization problem on the RHS of (9) can be solved in two phases:

- **Phase 1:** Compute $\Delta \tilde{x} = -H^{-\frac{1}{2}} \left[ \frac{\partial F(x(k))}{\partial x} + \frac{\partial g(x(k))}{\partial x} \lambda \right]$ which is (16).
- **Phase 2:** Project $\Delta \tilde{x}$ onto $\Omega$, the resulting projection is $\Delta \hat{x}$.

Proof: Since the square terms of $\Delta x$ contains a scaling matrix $H$, the basic idea of this proof is using a coordinate transformation to transform the minimization problem into a projection problem as follows.

Neglecting constant terms $\lambda^T g(x(k))$ and letting $\Delta y = H^{-\frac{1}{2}} \Delta x$, where the diagonal positive definite matrix $H^{-\frac{1}{2}}$ is defined by $H^{-\frac{1}{2}} \cdot H^{-\frac{1}{2}} = H$, we can rewrite the constrained minimization problem on the RHS of (9) as

$$h^{-\frac{1}{2}} \Delta y \in \Omega \Rightarrow \frac{1}{2} \|H^{-\frac{1}{2}} \Delta y - [H^{-\frac{1}{2}} \frac{\partial F(x(k))}{\partial x} + H^{-\frac{1}{2}} \frac{\partial g(x(k))}{\partial x} \lambda] \|^2_2$$

Since the constraints $H^{-\frac{1}{2}} \Delta y \in \Omega$ is equivalent to $\Delta y \in H^{-\frac{1}{2}} \Omega$, where $H^{-\frac{1}{2}} \Omega$ is defined as $\{H^{-\frac{1}{2}} \Delta x | \Delta x \in \Omega \}$. Thus, we can rewrite (22) as

$$\min_{\Delta y \in H^{-\frac{1}{2}} \Omega} \frac{1}{2} \|H^{-\frac{1}{2}} \Delta y - [H^{-\frac{1}{2}} \frac{\partial F(x(k))}{\partial x} + H^{-\frac{1}{2}} \frac{\partial g(x(k))}{\partial x} \lambda] \|^2_2$$

The minimization problem in (23) is simply a projection problem of projecting $\Delta \hat{y} = H^{-\frac{1}{2}} \left[ \frac{\partial F(x(k))}{\partial x} + \frac{\partial g(x(k))}{\partial x} \lambda \right]$ onto the set $H^{-\frac{1}{2}} \Omega$. Let $\Delta \hat{y}$ be the projection of $\Delta \hat{y}$, then $\Delta \hat{x} = H^{-\frac{1}{2}} \Delta \hat{y}$. In fact, the above projection process is equivalent to project $\Delta \tilde{x} = H^{-\frac{1}{2}} \Delta \tilde{y} = H^{-\frac{1}{2}} \left[ \frac{\partial F(x(k))}{\partial x} + \frac{\partial g(x(k))}{\partial x} \lambda \right]$ onto the set $\Omega$ and the projection is $\Delta \hat{x}$ as we stated in the two-phase procedures of this Theorem. In the following, we will prove this claim. The set $\Omega$ are decoupled for each individual bus $i$, and the simple bounded inequality constraints (such as the constraints for real power

VI. CONCLUSION

The proposed dual-type method for solving the OPF subproblems in the framework of SQP method is a new method in OPF literature and also a new dual-type method in nonlinear programming methodologies. This method is general, theoretically sound, and computationally efficient. The exploitation of the sparsity structure of power system network and capability of coping with difficulties encountered by inequality constraints make this method attractive for applications on other power system optimization problems.
and reactive power generations) $u_i \leq u_i(k) + \Delta u_i \leq u_i$ are decoupled from the constraints on voltage magnitude $V_i - V_i(k) \leq \frac{\delta V_e}{\delta e_i} \Delta e_i + \frac{\delta V_f}{\delta f_i} \Delta f_i \leq V_i - V_i(k)$. Since $H^\frac{1}{2}$ is diagonal, and the diagonal terms of $H^\frac{1}{2}$ corresponding to $e_i$ and $f_i$ are the same as $\frac{1}{\sqrt{2}} \eta_i$, Fig. 3 and Fig. 4 geometrically show the equivalence of projecting $\Delta \hat{x}$ onto the set $\Omega$ and $\Delta \hat{y}$ onto the set $H^\frac{1}{2} \Omega$ using coordinate transformation. This proves our claim. □

**Remark 5** In general, if $H$ does not possess special structure, more complicated formula are needed to obtain $\Delta \hat{x}$ and $\Delta \hat{y}$, however, the simplicity of the two-phase procedures still hold.

**Theorem 2** The $\Delta \hat{x}$ obtained from (17)-(19) is the projection of $\Delta \hat{x}$ onto the set $\Omega$.

Proof: The result is trivial by inspection from Fig. 3 and Fig. 4. □

**Theorem 3** The dual-type method (10) with $\beta(t)$ determined according to (15) is an ascent method.

Proof: First, we can rewrite (8) as $\min[-\phi(\lambda)]$. From (14), $-\bigg[\nabla^2 \phi(\lambda(t)) - \delta I\bigg]$ is positive definite. Using Decent Lemma in [12] and by simple calculations, we can set $\beta(t) = -\frac{1}{\sigma_D} \frac{\delta^2 m(t)}{m(t)} \sigma_D$, where $m(t)$ is the smallest nonnegative integer that the following inequality holds

$$-\phi(\lambda(t) + \frac{1}{\sqrt{2}} \frac{m(t)}{D} \sigma_D \Delta \lambda(t)) \leq -\phi(\lambda(t)) - \frac{\delta^2 m(t)}{2 \sigma_D} \|\Delta \lambda(t)\|_2^2,$$

which is (15). We then have

$$\phi(\lambda(t) + \frac{1}{\sqrt{2}} \frac{m(t)}{D} \sigma_D \Delta \lambda(t)) \geq \phi(\lambda(t)) + \frac{\delta \beta(t)}{2} \|\Delta \lambda(t)\|_2^2. \quad (24)$$

This shows that (10) is an ascent method as long as $\|\Delta \lambda(t)\|_2^2 \neq 0$. In fact, the condition $\|\Delta \lambda(t)\|_2^2 = 0$ implies $\frac{\delta \phi(\lambda(t))}{\delta \lambda(t)} = 0$ which is the necessary condition when $\phi(\lambda)$ achieves its maximum. Thus, (10) is an ascent method to maximize $\phi(\lambda)$.

Combining Theorem 3 with the two-phase method shown in Theorems 1 and 2, and also by Duality Theory [14], we have the following theorem which is the main theoretical result of the proposed dual-type method. □

**Theorem 4** The dual-type method (10) converges to a point $\lambda^*$ such that $\frac{\delta \phi(\lambda)}{\delta \lambda} = 0$ and maximize $\phi(\lambda)$. Furthermore, $\Delta \hat{x}$, the solution of the constrained minimization problem on the RHS of (9) with $\lambda = \lambda^*$, equals $\Delta \hat{x}(k)$, the optimal solution of (3).

Proof: the proof can be similarly developed from the proof of Proposition 2.1 of Section 3.2.2 in [12]. □

**VIII. ACKNOWLEDGMENT**

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References


BIography