Spectral radius and degree sequence of a graph
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Abstract

Let G be a simple connected graph of order n with degree sequence d_1, d_2, ..., d_n in non-increasing order. The spectral radius ρ(G) of G is the largest eigenvalue of its adjacency matrix. For each positive integer ℓ at most n, we give a sharp upper bound for ρ(G) by a function of d_1, d_2, ..., d_ℓ, which generalizes a series of previous results.

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1. Introduction

Let G be a simple connected graph of n vertices and m edges with degree sequence d_1 ⩾ d_2 ⩾ ... ⩾ d_n. The adjacency matrix A = (a_{ij}) of G is a binary square matrix of order n with rows and columns indexed by the vertex set VG of G such that for any i, j ∈ VG, a_{ij} = 1 if i, j are adjacent in G. The spectral radius ρ(G) of G is the largest eigenvalue of its adjacency matrix, which has been studied by many authors.

The following theorem is well-known [6, Chapter 2].

Theorem 1.1. If A is a nonnegative irreducible n × n matrix with largest eigenvalue ρ(A) and row-sums r_1, r_2, ..., r_n, then

\[ \rho(A) \leq \max_{1 \leq i \leq n} r_i \]

with equality if and only if the row-sums of A are all equal.
In 1985 [1, Corollary 2.3], Brauldi and Hoffman showed the following result.

**Theorem 1.2.** If \( m \leq k(k - 1)/2 \), then
\[
\rho(G) \leq k - 1
\]
with equality if and only if \( G \) is isomorphic to the complete graph \( K_n \) of order \( n \).

In 1987 [8], Stanley improved Theorem 1.2 and showed the following result.

**Theorem 1.3.**
\[
\rho(G) \leq -1 + \sqrt{1 + 8m}
\]
with equality if and only if \( G \) is isomorphic to the complete graph \( K_n \) of order \( n \).

In 1998 [3, Theorem 2], Yuan Hong improved Theorem 1.3 and showed the following result.

**Theorem 1.4.**
\[
\rho(G) \leq \sqrt{2m - n + 1}
\]
with equality if and only if \( G \) is isomorphic to the star \( K_{1,n-1} \) or to the complete graph \( K_n \).

In 2001 [4, Theorem 2.3], Hong et al. improved Theorem 1.4 and showed the following result.

**Theorem 1.5.**
\[
\rho(G) \leq \frac{d_n - 1 + \sqrt{(d_n + 1)^2 + 4(2m - nd_n)}}{2}
\]
with equality if and only if \( G \) is regular or there exists \( 2 \leq t \leq n \) such that \( d_1 = d_{t-1} = n - 1 \) and \( d_t = d_n \).

In 2004 [7, Theorem 2.2], Jinlong Shu and Yarong Wu improved Theorem 1.1 in the case that \( A \) is the adjacency matrix of \( G \) by showing the following result.

**Theorem 1.6.** For \( 1 \leq \ell \leq n \),
\[
\rho(G) \leq \frac{d_\ell - 1 + \sqrt{(d_\ell + 1)^2 + 4(\ell - 1)(d_1 - d_\ell)}}{2}
\]
with equality if and only if \( G \) is regular or there exists \( 2 \leq t \leq \ell \) such that \( d_1 = d_{t-1} = n - 1 \) and \( d_t = d_n \).

Moreover, they also showed in [7, Theorem 2.5] that if \( p + q \geq d_1 + 1 \) then Theorem 1.6 improves Theorem 1.5 where \( p \) is the number of vertices with the largest degree \( d_1 \) and \( q \) is the number of vertices with the second largest degree. The special case \( \ell = 2 \) of Theorem 1.6 is reproved [2].

In this research, we present a sharp upper bound of \( \rho(G) \) in terms of the degree sequence of \( G \), which improves Theorem 1.2 to Theorem 1.6.
Theorem 1.7. For 1 ≤ ℓ ≤ n,
\[ \rho(G) ≤ φ_\ell := \frac{d_\ell - 1 + \sqrt{(d_\ell + 1)^2 + 4 \sum_{i=1}^{\ell-1} (d_i - d_\ell)}}{2}, \]
with equality if and only if G is regular or there exists 2 ≤ t ≤ ℓ such that d_1 = d_\ell = n - 1 and d_t = d_n.

This result improves Theorem 1.5 and Theorem 1.6 since φ_n is exactly the upper bounds in Theorem 1.5 and is at most the upper bound appearing in Theorem 1.6. Additionally, generalized from this research, a similar upper bound of the spectral radius in terms of the average 2-degree sequence of a graph will be presented in [5].

Notice that the number φ_\ell defined in Theorem 1.7 is at least d_\ell. The sequence φ_1, φ_2, ..., φ_n is not necessary to be non-increasing. We show that this sequence is first non-increasing and then non-decreasing, and determine its lowest value in Section 3.

2. Proof of Theorem 1.7

Proof. Let the vertices be labeled by 1, 2, ..., n with degrees d_1 ≥ d_2 ≥ ... ≥ d_n, respectively. For each 1 ≤ i ≤ ℓ - 1, let x_i ≥ 1 be a variable to be determined later. Let \( U = diag(x_1, x_2, ..., x_{\ell-1}, 1, 1, ..., 1) \) be a diagonal matrix of size n × n. Then \( U^{-1} = diag(x_1^{-1}, x_2^{-1}, ..., x_{\ell-1}^{-1}, 1, 1, ..., 1) \).

Let B = U^{-1}AU. Notice that A and B have the same eigenvalues.

Let r_1, r_2, ..., r_n be the row-sums of B. Then for 1 ≤ i ≤ ℓ - 1 we have
\[
\begin{align*}
    r_i &= \sum_{k=\ell+1}^{n} x_k a_{ik} + \sum_{k=1}^{\ell-1} \frac{1}{x_i} a_{ik} = \frac{1}{x_i} \sum_{k=1}^{n} a_{ik} + \frac{1}{x_i} \sum_{k=1}^{\ell-1} (x_k - 1) a_{ik} \\
    &≤ \frac{1}{x_i} d_i + \frac{1}{x_i} \left( \sum_{k=1, k \neq i}^{\ell-1} x_k - (\ell - 2) \right),
\end{align*}
\]
and for ℓ ≤ j ≤ n we have
\[
\begin{align*}
    r_j &= \sum_{k=1}^{\ell-1} x_k a_{jk} + \sum_{k=\ell}^{n} a_{jk} = \sum_{k=\ell}^{n} a_{jk} + \sum_{k=1}^{\ell-1} (x_k - 1) a_{jk} \\
    &≤ d_\ell + \left( \sum_{k=1}^{\ell-1} x_k - (\ell - 1) \right).
\end{align*}
\]

For 1 ≤ i ≤ ℓ - 1 let
\[
x_i = 1 + \frac{d_i - d_\ell}{φ_\ell + 1} ≥ 1,
\]

where φ_\ell is defined in Theorem 1.7. Then for 1 ≤ i ≤ ℓ - 1 we have
\[
\begin{align*}
  r_i &≤ \frac{1}{x_i} d_i + \frac{1}{x_i} \left( \sum_{k=1, k \neq i}^{\ell-1} x_k - (\ell - 2) \right) = φ_\ell,
\end{align*}
\]
and for \( \ell \leq j \leq n \) we have
\[
r_j \leq d_\ell + \left( \sum_{k=1}^{\ell-1} x_k - (\ell - 1) \right) = \phi_\ell.
\]

Hence by Theorem 1.1,
\[
\rho(G) = \rho(B) \leq \max_{1 \leq i \leq n} \{ r_i \} \leq \phi_\ell.
\]  
(2.4)

The first part of Theorem 1.7 follows.

The sufficient condition of \( \phi_\ell = \rho(G) \) follows from the fact that
\[
\phi_\ell \leq \frac{d_\ell - 1 + \sqrt{(d_\ell + 1)^2 + 4(\ell - 1)(d_1 - d_\ell)}}{2}
\]

and applying the second part in Theorem 1.6.

To prove the necessary condition of \( \phi_\ell = \rho(G) \), suppose \( \phi_\ell = \rho(G) \). Then the equalities in (2.1) and (2.2) all holds. If \( d_1 = d_\ell \), then \( d_1 = \phi_1 = \phi_\ell = \rho(G) \), and \( G \) is regular by the second part of Theorem 1.1. Suppose \( 2 \leq t \leq \ell \) such that \( d_{t-1} > d_t = d_\ell \). Then \( x_i > 1 \) for \( 1 \leq i \leq t - 1 \) by (2.3). For each \( 1 \leq i \leq \ell - 1 \), the equality in (2.1) implies that \( d_{jk} = 1 \) for \( 1 \leq k \leq t - 1 \), \( k \neq i \). For each \( \ell \leq j \leq n \), the equality in (2.2) implies that \( a_{jk} = 1 \) for \( 1 \leq k \leq t - 1 \) and \( d_j = d_\ell \). Hence \( n - 1 = d_1 = d_{t-1} > d_t = d_\ell = d_n \).

We complete the proof. \( \square \)

3. The sequence \( \phi_1, \phi_2, \ldots, \phi_n \)

The sequence \( \phi_1, \phi_2, \ldots, \phi_n \) is not necessarily non-increasing. For example, the path \( P_n \) of \( n \) vertices has \( 2 = d_1 = d_{n-2} > d_{n-1} = d_n = 1 \), and it is immediate to check that if \( n \geq 6 \) then \( \phi_1 = \phi_2 = 2 < \sqrt{n - 1} = \phi_{n-1} = \phi_n \).

Clearly that for all \( 1 \leq s < t \leq n \), \( d_s = d_t \) implies that \( \phi_s = \phi_t \). However, \( \phi_s = \phi_t \) does not imply \( d_s = d_t \). For example, in the graph with degree sequence \( (4, 3, 3, 2, 1) \), one can check that \( \phi_4 = \phi_5 = 3 \) but \( d_4 > d_5 \).

Recall that \( d_s = d_{s+1} \) implies \( \phi_s = \phi_{s+1} \) for \( 1 \leq s \leq n - 1 \). The following proposition describes the shape of the sequence \( \phi_1, \phi_2, \ldots, \phi_n \).

**Proposition 3.1.** Suppose \( d_s > d_{s+1} \) for \( 1 \leq s \leq n - 1 \), and let \( \geq \in \{ >, = \} \). Then
\[
\phi_s \geq \phi_{s+1} \iff \sum_{i=1}^{s} d_i \geq s(s - 1).
\]

**Proof.** Recall that
\[
\phi_s = \frac{d_s - 1 + \sqrt{(d_s + 1)^2 + 4 \sum_{i=1}^{s-1} (d_i - d_s)}}{2}.
\]

The proposition follows from the following equivalent relations step by step:
\[
\phi_s \geq \phi_{s+1} \iff d_s - d_{s+1} + \sqrt{(d_s + 1)^2 + 4 \sum_{i=1}^{s-1} (d_i - d_s)}
\]
and deleting the common term $d_s - d_{s+1}$. Notice that if $2s - (d_s + 1) < 0$ in (3.1) then in the case that $\geq s = n$, all statements fails, and in the case that $\geq s > n$ the left hand side of (3.1) is at least $d_s + 1$, which is greater than $|2s - (d_s + 1)|$, so the equivalent relation in the next step holds. □

Corollary 3.2. Let $3 \leq \ell \leq n$ be the smallest integer such that $\sum_{i=1}^{\ell} d_i < \ell(\ell - 1)$. Then for $1 \leq j \leq n$ we have

$$\phi_j = \min\{\phi_k \mid 1 \leq k \leq n\}$$

if and only if $d_j = d_\ell$, or $d_j = d_{\ell-1}$ with $\sum_{i=1}^{\ell-1} d_i = (\ell - 1)(\ell - 2)$.

Proof. From Proposition 3.1, $\sum_{i=1}^{\ell} d_i = (\ell - 1)(\ell - 2)$ implies $\phi_{\ell-1} = \phi_\ell$. Also, clearly that $d_j = d_\ell$ implies $\phi_j = \phi_\ell$. We show that $\phi_\ell = \min\{\phi_k \mid 1 \leq k \leq n\}$ in the following.

For $1 \leq s \leq \ell - 1$, from Proposition 3.1 we have $\phi_s \geq \phi_{s+1}$ since $\sum_{i=1}^{s} d_i \geq s(s - 1)$. For $\ell - s \leq n - 1$, notice that $\sum_{i=1}^{\ell} d_i < t(t - 1)$ implies $d_s < t - 1$, and hence $\sum_{i=1}^{\ell} d_i < t(t - 1) + (t - 1) < t(t + 1)$. From Proposition 3.1 we have $\phi_\ell \leq \phi_{\ell+1} \leq \cdots \leq \phi_n$ since $\sum_{i=1}^{\ell} d_i < \ell(\ell - 1)$. The result follows. □

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References