A FAST ELLIPSE/CIRCLE DETECTOR USING GEOMETRIC
SYMMETRY

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Abstract—Through the use of a global geometric symmetry, a fast ellipse/circle detector is proposed in this
paper. Based on the geometric symmetry, the proposed method first locates candidates of ellipse and circle
centers. In the meantime, according to these candidate centers, all feature points in an input image are
grouped into several subimages. Then, for each subimage, by using geometric symmetry again, all ellipses
and circles are obtained. The method significantly reduces the time required to evaluate all possible
parameters without using edge direction information. Experimental results are given to show the correctness
and effectiveness of the proposed method.

2. PROPOSED METHOD

The proposed method consists of two phases: symmetric center location and parameter estimation. In
Phase 1, we will locate the candidates of ellipse and circle centers and classify the feature points in an input
image into different subimages according to these candidate centers. In Phase 2, for each subimage, based
on the geometric symmetry of ellipses and circles all ellipses and circles will be extracted. Before describing
these two phases in detail, we will first introduce some theorems which will be used in the proposed method.

2.1. Properties for ellipses and circles

Theorem 1. Let E be an ellipse or circle that is scanned
from left to right and top to bottom (rightward). Assume
that each horizontal scan line HS intersects E at XL and
XR. Let XM be the midpoint of XL and XR. Then
each XM lies on the same straight line l, which will be
Fig. 1. The two symmetric axes generated by horizontal and vertical scanning transform: (a) the middle point $X_M$a of $X_L$ and $X_R$ lies on the same straight line $L_v$. (b) The middle point $Y_M$b of $Y_T$ and $Y_B$ lies on the same straight line $L_h$.

This theorem has been proven by Yin and Chen.\(^{17}\) Note that in the sequel, $X_R(X_L)$ will be referred to as the symmetric point of $X_L(X_R)$ relative to $I_v$.

**Theorem 2.** Let $E$ be an ellipse or circle that is scanned from top to bottom and left to right (downward). Assume that each vertical scan line $V_S_i$ intersects $E$ at $Y_T_i$ and $Y_B_i$. Let $Y_M_i$ be the midpoint of $Y_T_i$ and $Y_B_i$. Then each $Y_M_i$ lies on the same straight line $L_h$, which will be referred to below as the symmetric horizontal axis [see Fig. 1(b)]. The proof of Theorem 2 is similar to that of Theorem 1. In the sequel, $Y_T_i(Y_B_i)$ will be referred to as the symmetric point of $Y_B_i(Y_T_i)$ relative to $L_h$.

**Theorem 3.** Let $E$ be an ellipse or a circle and let $I_v$, $I_h$ be its two symmetric axes generated by rightward and downward scanning. Then the cross-point of $I_v$ and $I_h$ is the center of $E$.

**Proof.** Without loss of generality, suppose that the center point of $E$ is $(0,0)$. Then $E$ can be expressed by the following equation:

$$dx^2 + exy + fy^2 = 1.$$

Consider the horizontal scanning line $Y = 0$. This line will intersect $E$ at $X_L(-1/\sqrt{d},0)$ and $X_R(1/\sqrt{d},0)$, and the middle point $X_M$ of $X_L$ and $X_R$ is $(0,0)$. Similarly, the vertical scanning line $X = 0$ will intersect $E$ at $Y_T(0,1/\sqrt{f})$ and $Y_B(0,-1/\sqrt{f})$, and the midpoint $Y_M$ of $Y_T$ and $Y_B$ is $(0,0)$. Since $(0,0)$ is on $E$, it follows that $E$ is an ellipse.
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Fig. 3. Boundary point A and its symmetric point C relative to the center of ellipse E and their symmetric points relative to the symmetric vertical and horizontal axes of E form two parallelograms with their vertexes on the ellipse: (a) AB1CD1, (b) AB2CD2.

both line $l_v$ and $l_h$, $(0, 0)$ must be the cross-point of $l_v$ and $l_h$.

Note that if the center of ellipse (or circle) E is $(x_0, y_0)$ and E is translated by $(-x_0, -y_0)$, for any point A(x, y) on the translated E, its symmetric point C(-x, -y) relative to (0,0) should also be on the translated E (see Fig. 2).

Theorem 4. Let E be an ellipse or a circle with center $(0, 0)$, A be a point on E, and C be the symmetric point of A relative to $(0,0)$. Let $B_1$ and $D_1$ be the symmetric points of A and C relative to $l_v$, respectively. Let $B_2$ and $D_2$ be the symmetric points of C and A relative to $l_h$, respectively. Then the quadrangles $AB_1CD_1$ and $AB_2CD_2$ are parallelograms (see Fig. 3).

Proof. Let the coordinates of A be $(u, v)$. Then C will be on E with coordinates $(-u, -v)$. Since $B_1$ is the symmetric point of A relative to $l_v$, $B_1$ will be on the same horizontal scanning line as A and have the coordinates $(u', v)$. Similarly, $D_1$ will have the coordinates $(u', -v)$. Since the symmetric point SB(-$u'$, -$v$) of $B_1$ relative to $(0,0)$ is on E and the scanning line $Y = -v$ can intersect E at only two points, $SB$ will be $D_1$ or C. Since $u' \neq u$, $SB$ must be $D_1$ and $u'' = -u$. Thus the lengths of line segments $AB_1$ and $CD_1$ are both $|u' - u|$ and the quadrangle $AB_1CD_1$ is a parallelogram. In a similar way, we can prove that $AB_2CD_2$ is also a parallelogram.

Note that if E is a circle or an ellipse without rotation, $B_1$ and $B_2$ will be the same point, $D_1$ and $D_2$ are also the same.

2.2. Phase 1: symmetric center location

In this subsection, based on Theorems 1–3 above, we will present the first phase of the proposed method, in which all possible centers of circles and ellipses will be located.

Let $f$ be an input image. Before applying the proposed method, we first use an edge extractor to find the boundary points of objects in $f$ and store the result on an image $F$. Then Phase 1 is carried out. First $F$ is classified into several subimages by a horizontal scanning transform procedure, described as follows.

2.2.1. Horizontal scanning transform procedure

Initialize a blank image $G$

Scan $F$ from left to right and top to bottom

For each boundary point $(i, j)$ on $F$

If there exists another boundary point $(k, j)$ on $F$

then for each such a point $(k, j)$, set

$u = (i + k)/2$

$G(u, j) = 1$

End (scan)

Apply Hough transform to $G$ to extract all lines in $G$

Consider each extracted line to be a candidate symmetric vertical axis of an ellipse or a circle

For each extracted line $l_v$

Group all symmetric points in $F$ relative to $l_v$ into a subimage $F_h$.

By Theorem 1, we know that if there exists an ellipse (or a circle) E in image $F$, through the above procedure, each point in the left part of E must produce one point, which is on the symmetric vertical axis $l_v$ of E, in image $G$. Thus, after the above procedure, many points on $l_v$ should appear in $G$. Based on this fact, by applying Hough transform to $G$, we can extract $l_v$. Also, all points in E must be put in the same subimage. Note that ellipses with different symmetric vertical axes will be put in different subimages, which will facilitate the detection later.

Next, a vertical scanning transform is applied to each $F_h$ produced by the above procedure, as described below.

2.2.2. Vertical scanning transform procedure

Initialize a blank image $G$

Scan $F_h$ from top to bottom and left to right

For each boundary point $(i, j)$

If there exists another boundary point $(i, k)$ in the same vertical scanning line

then for each point $(i, k)$, set

$u = [(j + k)/2]$

$G(i, u) = 1$

End (scan)

Apply Hough transform to $G$ to extract all lines in $G$

Consider each extracted line to be a candidate sym-
Fig. 4. An illustration of the symmetric center location phase. (a) A synthetic image. (b) A candidate symmetric vertical axis $I_v$ is extracted by applying the horizontal scanning transform to (a). (c) The corresponding symmetric boundary points relative to $I_v$. (d) A candidate symmetric horizontal axis $I_h$ is extracted by applying the vertical scanning transform to (c). (e) The corresponding symmetric boundary points relative to $I_h$. (f) The crosspoint of $I_h$ and $I_v$ is the center of the ellipse.

metric horizontal axis of an ellipse or a circle
For each extracted line $I_h$
Group all symmetric points in $F_h$ relative to $I_h$
into a subimage $F_{hv}$.
Consider the cross point of $I_h$ and $I_v$ to be a
candidate center of an ellipse or a circle.

Note that if there exists an ellipse $E$ in $F$ with
symmetric axes $I_h$ and $I_v$, then through the above two
procedures, $I_h$ and $I_v$ will be extracted and all points in
$E$ will be put in the same subimage. Furthermore, by
Theorem 3, we know that the cross-point of $I_h$ and $I_v$
must be the center of $E$. Based on this fact, each
cross-point of the lines $I_h$ and $I_v$ can be considered to be a
candidate center of an ellipse or a circle.

In Fig. 4 an example is given to illustrate the sym-
metric center location phase. Applying the horizontal
scanning transform procedure to Fig. 4(a), one candi-
date symmetric vertical axis $I_v$ is extracted and shown
in Fig. 4(b), the corresponding symmetric boundary
points with respect to $I_v$ are shown in Fig. 4(c). Figure
4(d) shows a candidate symmetric horizontal axis $I_h$
extracted by applying the vertical scanning transform
procedure to Fig. 4(c). Figure 4(e) shows the corres-
ponding symmetric boundary points with respect to $I_h$.
The crosspoint of $I_h$ and $I_v$ is considered to be a candi-
date for the center of the ellipse, as shown in Fig. 4(f).

2.3. Phase 2: parameter estimation
In Phase 2, based on the result of Phase 1 and
Theorems 3–4, the remaining three parameters ($a$: half-
length of the major axis, $b$: half-length of the minor
axis, $\theta$: the orientation angle) for each ellipse or the
single parameter (radius) for a circle in each subimage
$F_{hv}$ will be found. Initially, the proposed method es-
establishes an array $AR(a, b, \theta)$ with $AR(a, b, \theta) = 0$ for each entry $(a, b, \theta)$. Then, for each subimage $F_h$, corresponding to a candidate symmetric center $O(x_0, y_0)$, each point $A$ in $F_h$ is checked to see if its symmetric point $C$ relative to $O$ exists in $F_h$. If it does, then $B_1$ and $D_1$, the points in $F_h$ that are symmetric to $A$ and $C$, respectively, with respect to $l$, are located. $B_2$ and $D_2$, the points symmetric to $A$ and $C$ with respect to $l$, are located. Then the two quadrangles $AB_1CD_1$ and $AB_2CD_2$ are checked to see whether they are parallelograms (by testing if the length of $AB_1$ equals that of $CD_1$ and if the length of $AD_2$ equals that of $B_2C$). If they are, points $A, C, B_1, B_2, O_1,$ and $D_2$ are considered to lie on an ellipse. Then $A, B_1,$ and $B_2$ are shifted $(-x_0, -y_0)$ to move the ellipse center to $(0, 0)$.

Since an ellipse with center $(0, 0)$ can be expressed by equation (1), by substituting the coordinates of the shifted $A, B_1,$ and $B_2$ into equation (1), we can obtain three equations with three unknown parameters $(d, e, f)$. Thus once these three parameters $(d, e, f)$ are solved, the three parameters $(a, b, \theta)$ for the ellipse can be obtained by the following formulas:

\[
\theta = \frac{\tan^{-1} \left[ e/(d - f) \right]}{2}
\]
\[
a = \sqrt{\frac{1}{(d \times \cos^2 \theta + e \times \sin \theta \times \cos \theta + f \times \sin^2 \theta)}}
\]
\[
b = \sqrt{\frac{1}{(f \times \cos^2 \theta - e \times \sin \theta \times \cos \theta + d \times \sin^2 \theta)}}.
\]

The $(a, b, \theta)$ obtained is then considered to be a set of possible parameters for an ellipse and the corresponding entry $AR(a, b, \theta)$ is increased by one. After all points in $F_h$ are processed, a local peak-finding algorithm is applied to $AR$, and the peaks found are considered to be the parameters of ellipses.

Note that if the detected shape is a circle (i.e. $a = b$, $\theta = 0$) or an ellipse with $\theta = 0$, we will find that $B_1 = B_2$, as mentioned previously, and the above procedure cannot be used to solve for the parameters. In this situation, the shifted circle or ellipse with center $(0, 0)$ can be expressed by

\[dx^2 + fy^2 = 1.\] (3)

To solve $(d, f)$, for point $A(x, y)$, take each point $A'$ (see Fig. 5) in $F_h$ with

\[A' = (x', x) \quad \text{if } |x'| < |y|\]
\[A' = (y, y') \quad \text{otherwise}.
\]

By substituting the coordinates of the shifted $A$ and $A'$ into equation (3), we can solve $(d, f)$. After we have $(d, f)$, the parameters $(a, b)$ can be obtained by

\[a = \sqrt{\frac{1}{d}}\]
\[b = \sqrt{\frac{1}{f}}.
\]

The obtained parameters $(a, b)$ are then considered to be a set of possible parameters for an ellipse without any orientation or a circle. The remaining steps are similar to the procedure used for an ellipse with orientation except that array $AR(a, b)$ is replaced by $AR_1(a, b)$.

### 3. Experimental Results

This section presented the results of experiments in which the proposed method was applied to several images. Figure 6(a) shows a synthetic image including four objects: two overlapping ellipses, a circle, and an ellipse without orientation. In the symmetric center location phase, four symmetric centers were extracted and their corresponding symmetric boundary points were put into four different subimages, as shown in Fig. 6(b)–(e). The parameter estimation phase was then applied to each subimage to find the remaining parameters. Figure 6(f) presents the final result super-
imposed on the original image with a set of two orthogonal-cross line segments standing for the major and minor axes of the ellipses. Figure 7(a) shows a real image including a circle, an ellipse, and some other types of objects. Figure 7(b) shows the result of boundary extraction. Applying the proposed method to Fig. 7(b) yielded two subimages corresponding to the two centers, as shown in Fig. 7(c)-(d). The final result is shown in Fig. 7(e). From these experimental results, we can see that separated and partially occluded ellipses (circles) can be located successfully. For comparison purposes, we also apply the Tsuji method\(^2\) to Fig. 7(a). By using a SUN SPARC-10 workstation, the running time for our method is 7.9 s, and for the Tsuji method is 8.9 s. The proposed method is little faster than the Tsuji method. But the Tsuji method needs the edge direction information, it is a hard work to get accurate edge direction in a noisy image.

Figure 8(a) shows another synthetic image including four ellipses: (1) intact, (2) intermittent, (3) with 12.5\%
Fig. 7. Results of applying the proposed method to a real image. (a) A real image. (b) The result of boundary extraction. (c) and (d) The results of applying symmetric center location phase to (b); two subimages with their corresponding symmetric centers are obtained. (e) The final result of the proposed method.
Fig. 8. Results of applying the proposed method to defective ellipses. (a) Four ellipses: intact, intermittent, 12.5% defect and 25% defect. (b) Three ellipses are detected, the ellipse with 25% defect is not located.

defect, (4) with 25% defect. The detecting results are shown in Fig. 8(b). From this figure, we can see that ellipse with 25% defect is not detected. This is due to that many points on the boundary of the ellipse cannot be used to form parallelograms of the type employed in Phase 2. In general, if the defectiveness of an ellipse exceeds 25%, the proposed method may not work well.

4. CONCLUSION

In this paper, we have proposed a fast method for locating circles and ellipses. The symmetry of ellipses and circles is used as a basis for classifying boundary points into several subimages, each of which is then treated separately. In Phase 2, because we again take advantage of global geometric symmetry, for each boundary point A, we only need to evaluate one set of parameters \((a, b, \theta)\) of an ellipse that possibly passes through A. Since the edge direction is not used, the estimation accuracy will not be lost. Thus, the proposed method is faster and more precise than existing Hough-based algorithms for ellipse and circle detection. In addition, the proposed method is simple and easy to implement.

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