An initial boundary value problem for one-dimensional shallow water magnetohydrodynamics in the solar tachocline

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ABSTRACT

In this article we investigate the shallow water magnetohydrodynamic equations in space dimension one with Dirichlet boundary conditions only for the velocity. This model has been proposed to study the phenomena in the solar tachocline. In this article, the local well-posedness in time of the model is proven by constructing the approximate solutions and showing the strong convergence of the approximate solutions.

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1. Introduction

In geophysical fluid dynamics, when the thickness of the shallow layer of fluid considered is much smaller than the horizontal scale, the Shallow Water (SW) equations are often commonly used to describe the evolution of such a system (see [1]). In general, SW equations are considered as a simplified version of the primitive equations and are mainly applied to model the large-scale dynamics of thinly stratified atmospheric or oceanic flows under the influence of Coriolis force and thermal forcing. From the mathematical point of view, SW equations have been extensively studied in the literature, in particular, on the questions such as the existence and uniqueness of solutions and related issues. We refer the interested reader to e.g. [2–9] and references therein.

In the context of astrophysical dynamics, the shallow water magnetohydrodynamic (SWMHD) equations for studying the phenomena in the solar tachocline were (to the best of our knowledge) first introduced by Gilman [10] and the word tachocline was introduced for the first time in [11]. In the literature, there are some theoretical and numerical works available for studying the shallow water magnetohydrodynamic equations (for theoretical works, see [12–14]; for numerical works, see [15–20]). However, as far as we know, there has been very few work on the subject of the well-posedness of the shallow water magnetohydrodynamic equations addressed in the literature. In the article, we are interested in studying the existence and uniqueness of solutions to the shallow water magnetohydrodynamic equations.

In this article, the one-dimensional shallow water magnetohydrodynamic equations are considered. For the Cauchy problem of the equations, the well-posed result follows from the general theory of quasilinear hyperbolic systems (see e.g. [21]). As for the periodic boundary condition or Dirichlet boundary condition, to the best of our knowledge, these problems have not been solved yet. More specifically, no general theory of quasilinear hyperbolic systems can be applied directly. In this work, we deal with the case of the Dirichlet boundary condition. The case of the periodic boundary condition will be addressed elsewhere.

In the presented work, we consider the one-dimensional shallow water magnetohydrodynamic equations with Dirichlet boundary conditions only imposed on the velocity. Under the assumption that the height remains strictly positive, as the ideas used in [9], we are able to show the existence and uniqueness of the strong solutions to the system on a certain time...
interval depending on the initial data. Indeed, we consider a new system derived from expressing the magnetic field in terms of the height due to the one-dimensional structure. Then, we construct approximate solutions by iteratively solving some linearized equations and show that these sequences are Cauchy in suitable Hilbert spaces. The difficulty of the estimates comes from the nonlinearity of the magnetic field, which can be solved under the suitable assumption on the height. At the end, we show that the sequences strongly converge to the limits which are the solutions of our problem. As for other higher dimensions, various boundary conditions or related mathematical issues to the shallow water magnetohydrodynamic equations, these problems will be studied elsewhere.

The rest of the article is organized as follows. In Section 2, we recall the equations and state our main results. In Section 3, we construct the approximate linear system, make various a priori estimates, derive the uniform boundedness in $t$ and show the positivity of the approximate height. Finally, we conclude the proof of the existence and uniqueness of strong solutions in Section 4.

2. The shallow water magnetohydrodynamic equations: main results

In this section, we consider the one-dimensional shallow water magnetohydrodynamic equations on the interval $I = [0, 1]$ as [10, 12]:

\[
\begin{aligned}
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} - B \frac{\partial B}{\partial x} + g \frac{\partial h}{\partial x} &= 0, \\
\frac{\partial B}{\partial t} + u \frac{\partial B}{\partial x} - B \frac{\partial u}{\partial x} &= 0, \\
\frac{\partial h}{\partial t} + \frac{\partial}{\partial x} (uh) &= 0, \\
\frac{\partial}{\partial x} (hB) &= 0.
\end{aligned}
\]

Here the unknown functions $u$, $B$ and $h$ represent the fluid velocity, the magnetic field and the height of the conducting fluid, respectively. The constant $g$ is the magnitude of the gravitational acceleration. Equations

\[
u(0, 0) = u_0(x), \quad B(0, 0) = B_0(x), \quad h(0, 0) = h_0(x),
\]

and with the following boundary conditions

\[
u(0, t) = u(1, t) = 0.
\]

We assume that the initial conditions satisfy the compatibility conditions. Namely, the functions $u_0(x)$, $B_0(x)$, and $h_0(x)$ satisfy

\[
u_0(x = 0) = u_0(x = 1) = 0, \quad \frac{\partial}{\partial x} (h_0B_0) = 0.
\]

We also assume that $h_0(x) \geq 2H_0 > 0$ for $x \in I$ and $H_0$ is a positive constant. In this article, no boundary conditions are imposed on the magnetic field and the height.

In what follows we show the following main result.

**Theorem 1.** Given $(u_0, B_0, h_0)$ in $H^2(0, 1)$ which satisfy (4) and $h_0 \geq 2H_0 > 0$. Then there exist a constant $T^* > 0$ depending on the initial data $|u_0|_{L^2(0, 1)}$, $|B_0|_{L^2(0, 1)}$, and $|h_0|_{H^2(0, 1)}$, and a unique solution $(u, B, h)$ of the problem (1) on the time interval $(0, T^*)$ with the initial conditions (2) and boundary conditions (3) such that

\[
u(0, t) \in L^\infty(0, T^*; H^2(0, 1)^3).
\]

Moreover, $h(x, t) \geq H_0$ for $t \in [0, T^*]$.

From (1), we obtain

\[
\frac{\partial (hB)}{\partial t} = \frac{\partial h}{\partial t} B + \frac{\partial B}{\partial t} h = -B \left( u \frac{\partial h}{\partial x} + h \frac{\partial u}{\partial x} \right) + h \left( \frac{\partial u}{\partial x} - u B \frac{\partial u}{\partial x} \right) = -u \frac{\partial (hB)}{\partial x} = 0.
\]

Combined with (1)$_3$, we can infer that

\[
hB \equiv C = h_0B_0
\]

where $C$ is assumed to be a nonzero constant due to the presence of the magnetic field.
Thus, we observe that Eq. (1) is equivalent to

\[
\begin{align*}
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + g \frac{\partial h}{\partial x} + \frac{C^2}{h^3} \frac{\partial h}{\partial x} &= 0, \\
\frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} + h \frac{\partial u}{\partial x} &= 0, \\
hB &= C.
\end{align*}
\]

(6)

Therefore, to prove Theorem 1, we only need to show the following theorem.

**Theorem 2.** Given \((u_0, h_0)\) in \(H^2(0, 1)\) which satisfy (3) and \(h_0 \geq 2H_0 > 0\). Then there exist a constant \(T^* > 0\) depending on the initial data \(|u_0|_{H^2(0, 1)}\) and \(|h_0|_{H^2(0, 1)}\) and a unique solution \((u, h)\) of the problem (6) on the time interval \((0, T^*)\) with the initial conditions (2) and boundary conditions (3) such that

\[
(u, h) \in L^\infty(0, T^*; H^2(0, 1)^2).
\]

Moreover, \(h(x, t) \geq H_0\) for \(t \in [0, T^*]\).

Throughout this article, we denote \(c\) and \(c_i\) by various constants depending on the initial data and the size of the domain only and \(c\) can be different at different occurrences.

### 3. Approximate linear systems

In this section we prove the local existence of strong solutions of (6). The proofs are based on works related to shallow water equations in [9], which do not take the magnetic field into account. Similarly, we construct approximate linear systems of the problem (6), obtain various a priori estimates for them and then prove the uniform boundedness in \(t\) and the positivity of the approximate height.

Now, we define \(u^0(x, t) = u_0(x)\), \(h^0(x, t) = h_0(x)\) and then iteratively construct \(u^{k+1}, h^{k+1}\) as the solutions of the following linear problem:

\[
\begin{align*}
&u^{k+1} + u_ku^{k+1} + \left(g + \frac{C^2}{(h^k)^3}\right)h^{k+1} = 0, \\
h^{k+1} + u_kh^{k+1} + u^kh^k = 0,
\end{align*}
\]

(8)

with the initial conditions

\[
u^{k+1}(x, 0) = u_0(x), \quad h^{k+1}(x, 0) = h_0(x).
\]

(9)

and with the boundary conditions

\[
u^{k+1}(0, t) = u^{k+1}(1, t) = 0.
\]

(10)

Now we assume that \(u^k, h^k \in L^\infty(0, T; H^2(0, 1)), u^0, h^0 \in L^\infty(0, T; H^1(0, 1))\) and \(h^k(x, t) \geq H_0\). We need to show that \(u^{k+1}, h^{k+1}\) enjoy the same properties.

#### 3.1. Estimates \(u^{k+1}, h^{k+1}\) in \(L^\infty(0, T; L^2(0, 1))\)

We multiply (8) by \(h^ku^{k+1}\) and (8) by \((g + C^2/(h^k)^3)h^{k+1}\), add the resulting equations, integrate over the domain \(I\) and find that

\[
\begin{align*}
&\int_0^1 h^ku^{k+1}_t u^{k+1} \, dx + \int_0^1 h^ku^{k+1}_x u^{k+1} \, dx + \int_0^1 \left(g + \frac{C^2}{(h^k)^3}\right) h^k(u^{k+1}h^{k+1})_x \, dx \\
&\quad + \int_0^1 \left(g + \frac{C^2}{(h^k)^3}\right) h^{k+1}h^{k+1}_x \, dx + \int_0^1 \left(g + \frac{C^2}{(h^k)^3}\right) u^kh^kh^{k+1} \, dx = 0.
\end{align*}
\]

(11)

The first term in (11) can be written as

\[
\int_0^1 h^ku^{k+1}_t u^{k+1} \, dx = \frac{1}{2} \frac{d}{dt} \int_0^1 h^k(u^{k+1})^2 \, dx - \frac{1}{2} \int_0^1 h^k_t(u^{k+1})^2 \, dx.
\]

(12)

To bound the last term in (12), due to the fact that \(h^k \geq H_0\) and the Sobolev embedding theorem \(H^1(0, 1) \subset L^\infty(0, 1)\), we have

\[
\left|\int_0^1 h^k_t(u^{k+1})^2 \, dx\right| \leq c|h^k_t|_{H^1(0, 1)} \int_0^1 h^k(u^{k+1})^2 \, dx.
\]

(13)
To bound the second term in (11), we have
\[
\left| \int_0^1 h^k u^k u^{k+1} u_x^{k+1} \, dx \right| = \left| -\frac{1}{2} \int_0^1 (h^k u_x^k) (u^{k+1})^2 \, dx \right|
\leq c (|u^k|_{L^\infty(0,1)} |h_x^k|_{L^\infty(0,1)} + |u_x^k|_{L^\infty(0,1)}) \int_0^1 h^k (u^{k+1})^2 \, dx
\leq c (|u^k|_{H^2(0,1)} |h_x^k|_{H^2(0,1)} + |u_x^k|_{H^2(0,1)}) \int_0^1 h^k (u^{k+1})^2 \, dx.
\]
(14)

To bound the third term in (11), we have
\[
\left| \int_0^1 \left( g + \frac{C^2}{(h^k)^3} \right) h^k (h^k u^{k+1})_x \, dx \right|
\leq c \left| \int_0^1 h_x^k (h^k u^{k+1})_x \, dx \right| + c \int_0^1 \left( g + \frac{C^2}{(h^k)^3} \right) h_x^k (h^k u^{k+1}) \, dx
\leq c |h_x^k|_{H^2(0,1)} \left( \int_0^1 \left( g + \frac{C^2}{(h^k)^3} \right) (h^{k+1})^2 \, dx \right)^{1/2} \left( \int_0^1 h^k (u^{k+1})^2 \, dx \right)^{1/2}.
\]
(15)

To estimate the fourth term in (11), we obtain
\[
\int_0^1 \left( g + \frac{C^2}{(h^k)^3} \right) h^{k+1} h_t^k \, dx = \frac{1}{2} \frac{d}{dt} \int_0^1 \left( g + \frac{C^2}{(h^k)^3} \right) (h^{k+1})^2 \, dx - \frac{3}{2} \int_0^1 \frac{C^2}{(h^k)^4} h_t^k (h^{k+1})^2 \, dx.
\]
(16)

Similarly, the last term in (16) can be bounded by
\[
\left| \int_0^1 \frac{C^2}{(h^k)^4} h_t^k (h^{k+1})^2 \, dx \right| \leq c |h_t^k|_{H^1(0,1)} \int_0^1 \left( g + \frac{C^2}{(h^k)^3} \right) (h^{k+1})^2 \, dx.
\]
(17)

Combining all the previous estimates, we find that
\[
\frac{d}{dt} l_0^{k+1}(t) \leq c_0 l_0^k(t) l_0^{k+1}(t),
\]
(18)

where the function \( l^{k+1}(t) \) is defined as
\[
l_0^{k+1}(t) = \int_0^1 h^k (u^{k+1})^2 \, dx + \int_0^1 \left( g + \frac{C^2}{(h^k)^3} \right) (h^{k+1})^2 \, dx,
\]
(19)

and
\[
\xi_0^k(t) = |u^k|_{H^2(0,1)} + |u_x^k|_{H^1(0,1)} |h_x^k|_{H^2(0,1)} + |h_x^k|_{H^2(0,1)} + |h_t^k|_{H^1(0,1)}.
\]
(20)

Then applying the Gronwall inequalities for (18), we obtain
\[
l_0^{k+1}(t) \leq l_0^{k+1}(0) e^{\sigma_0 \int_0^t \xi_0^k(s) \, ds}.
\]
(21)

3.2. *Estimates \( u^{k+1}, h^{k+1} \) in \( L^\infty(0, T; H^1(0, 1)) \)*

To make a priori estimates for \( u^{k+1}, h^{k+1} \) in \( L^\infty(0, T; H^1(0, 1)) \), we differentiate the approximate equations (8) with respect to the variable \( x \) and find that
\[
\begin{align*}
\left\{ \begin{array}{l}
 u_x^{k+1} + u^k u_x^{k+1} + u_x^{k+1} + \left( g + \frac{C^2}{(h^k)^3} \right) h_x^{k+1} - \frac{3C^2}{(h^k)^4} h_t^k h_x^{k+1} = 0, \\
 h_x^{k+1} + u_x^{k+1} h_t^k + h_x^k u_x^{k+1} + u_x^k h_x^{k+1} + u_x^k h_x^{k+1} = 0.
\end{array} \right. 
\end{align*}
\]
(22)

We multiply (22)_1 by \( h^k u_x^{k+1} \) and (22)_2 by \( (g + C^2/(h^k)^3) h_x^{k+1} \), add the resulting equations, integrate over the domain \( I \) and find that
\[
\int_0^1 h^k u_x^{k+1} u_x^{k+1} \, dx + \int_0^1 h^k u_x^{k+1} (u_x^{k+1})^2 \, dx + \int_0^1 u^k h^k u_x^{k+1} u_x^{k+1} \, dx
+ \int_0^1 \left( g + \frac{C^2}{(h^k)^3} \right) h^k (u_x^{k+1} h_x^{k+1}) \, dx - 3 \int_0^1 \frac{C^2}{(h^k)^4} h_x^k u_x^{k+1} u_x^{k+1} \, dx
\]
\[
\int_0^1 h^k (u_x^{k+1})^2 \, dx
= 0.
\]
Let six terms can be written as
\[ u^k(0, t) = u^k(1, t) = 0. \]
Then, it can be inferred from (8), and (29) that
\[ h_x^{k+1}(0, t) = h_x^{k+1}(1, t) = 0. \]
Applying (30) in (28), we have
\[ \int_0^1 u^k h_x^k u_x^{k+1} u_x^k dx \leq c |h^k|_{H^2(0, 1)} \left( \int_0^1 \left( g + \frac{C^2}{(h^3)^3} \right) \left( h_x^{k+1} \right)^2 dx \right)^{1/2} \left( \int_0^1 h^k(u_x^k)^2 dx \right)^{1/2}. \]
Similarly, the fifth term in (23) can be bounded by
\[ \int_0^1 \frac{C^2}{(h^3)^3} h_x^k u_x^{k+1} h_x^{k+1} dx \leq c |h^k|_{H^2(0, 1)} \left( \int_0^1 \left( g + \frac{C^2}{(h^3)^3} \right) \left( h_x^{k+1} \right)^2 dx \right)^{1/2} \left( \int_0^1 h^k(u_x^k)^2 dx \right)^{1/2}. \]
The sixth term in (23) can be written as
\[ \int_0^1 \left( g + \frac{C^2}{(h^3)^3} \right) h_x^{k+1} h_x^k dx = \frac{1}{2} \frac{d}{dt} \int_0^1 \left( g + \frac{C^2}{(h^3)^3} \right) (h_x^{k+1})^2 dx + \frac{3}{2} \int_0^1 \frac{C^2}{(h^4)^4} h_x^k (h_x^{k+1})^2 dx. \]
The last term in (33) can be bounded by
\[ \int_0^1 \frac{C^2}{(h^3)^4} h_t^4 (h_x^{k+1})^2 dx \leq c |h^t|_{H^1(0, 1)} \int_0^1 \left( g + \frac{C^2}{(h^3)^3} \right) (h_x^{k+1})^2 dx. \]
Similarly, the seventh and eighth terms can be bounded by
\[ \int_0^1 \left( g + \frac{C^2}{(h^3)^3} \right) h_x^k h_x^{k+1} u_x^{k+1} dx \leq c |h^k|_{H^2(0, 1)} \left( \int_0^1 \left( g + \frac{C^2}{(h^3)^3} \right) (h_x^{k+1})^2 dx \right)^{1/2} \left( \int_0^1 h^k(u_x^k)^2 dx \right)^{1/2}, \]
and
\[ \left| \int_0^1 \left( g + \frac{C^2}{(h^3)^2} \right) u^k(h^k+1)^2 dx \right| \leq c |u^h|_{H^2(0,1)} \int_0^1 \left( g + \frac{C^2}{(h^3)^2} \right) (h^{k+1})^2 dx. \] (36)

To estimate the ninth term in (23), we obtain
\[ \left| \int_0^1 \left( g + \frac{C^2}{(h^3)^2} \right) h^k u^k h^{k+1} dx \right| = \left| \frac{1}{2} \int_0^1 \left[ \left( g + \frac{C^2}{(h^3)^2} \right) u^k - 3 \frac{C^2}{(h^3)^2} u^k h^k \right] (h^{k+1})^2 dx \right| \leq c (|u^h|_{H^2(0,1)} + |u^k|_{H^1(0,1)} |h^k|_{H^2(0,1)} |h^{k+1}|_{H^1(0,1)}) \int_0^1 \left( g + \frac{C^2}{(h^3)^2} \right) (h^{k+1})^2 dx. \] (37)

Combining all the previous estimates, we find that
\[ \frac{d}{dt} l^{k+1}_1(t) \leq c_1 \xi^k(t) l^{k+1}_1(t), \] (38)

where the function \( l^{k+1}_1(t) \) is defined as
\[ l^{k+1}_1(t) = \int_0^1 h^k (u^{k+1})^2 dx + \int_0^1 \left( g + \frac{C^2}{(h^3)^2} \right) (h^{k+1})^2 dx, \] (39)

and
\[ \xi^k(t) = \xi^k_0(t) + (1 + |u^h|_{H^1(0,1)}) |h^k|_{H^2(0,1)} |h^{k+1}|_{H^1(0,1)}. \] (40)

Using the Gronwall inequalities to (38), we obtain
\[ l^{k+1}_1(t) \leq l^{k+1}_1(0)e^{c_1 \int_0^t \xi^k(s) ds}. \] (41)

3.3. Estimates \( u^{k+1}, h^{k+1} \) in \( L^\infty(0, T; H^2(0, 1)) \)

Again, to make a priori estimates for \( u^{k+1}, h^{k+1} \) in \( L^\infty(0, T; H^2(0, 1)) \), we differentiate (22) with respect to the variable \( x \) and obtain
\[ \begin{cases} u^{k+1}_{xx} + u^h u^{k+1}_{xx} + 2u^k u^{k+1}_{xx} + u^k u^{k+1}_{xx} - 6C^2 \frac{1}{(h^3)^4} h^k u^{k+1}_{xx} + \left( g + \frac{C^2}{(h^3)^2} \right) h^{k+1}_{xx} \\ + \frac{12C^2}{(h^3)^5} (h^k) h^{k+1}_x = 0, \\ h^{k+1}_{xx} + 2h^k u^{k+1}_{xx} + h^k u^{k+1}_{xx} + h^k u^{k+1}_{xx} + u^k h^{k+1}_{xx} + u^k h^{k+1}_{xx} + 2u^k h^k_{xx} + u^k h^{k+1}_{xx} = 0. \end{cases} \] (42)

We multiply (42)1 by \( h^k u^{k+1}_{xx} \) and (42)2 by \( (g + C^2/(h^3)^2) h^{k+1}_{xx} \), add the resulting equations, integrate over the domain \( I \), and find that
\[ \begin{align*} \int_0^1 h^k u^{k+1}_{xx} u^{k+1}_{xx} dx &+ \int_0^1 h^k u^{k+1}_{xx} u^k u^{k+1}_{xx} dx + 2 \int_0^1 h^k u^k u^{k+1}_{xx}^2 dx \\ &+ \int_0^1 h^k u^k u^{k+1}_{xx} u^{k+1}_{xx} dx - \int_0^1 6C^2 \frac{1}{(h^3)^4} h^k u^{k+1}_{xx} h^{k+1}_{xx} dx + \int_0^1 \left( g + \frac{C^2}{(h^3)^2} \right) h^k (u^{k+1}_{xx} h^{k+1}_{xx}) dx \\ &+ \int_0^1 \frac{12C^2}{(h^3)^5} (h^k) h^{k+1}_x dx - \int_0^1 3C^2 \frac{1}{(h^3)^3} h^k u^{k+1}_{xx} u^{k+1}_{xx} dx + \int_0^1 \left( g + \frac{C^2}{(h^3)^2} \right) h^{k+1}_x h^{k+1}_{xx} dx \\ &+ 2 \int_0^1 \left( g + \frac{C^2}{(h^3)^2} \right) h^k u^{k+1}_x h^{k+1}_x dx + \int_0^1 \left( g + \frac{C^2}{(h^3)^2} \right) h^k h^{k+1}_x u^{k+1}_x dx \\ &+ \int_0^1 \left( g + \frac{C^2}{(h^3)^2} \right) u^k h^{k+1}_x h^{k+1}_x dx + 2 \int_0^1 \left( g + \frac{C^2}{(h^3)^2} \right) u^k h^{k+1}_x h^{k+1}_x dx \\ &+ \int_0^1 \left( g + \frac{C^2}{(h^3)^2} \right) u^k h^{k+1}_x h^{k+1}_x dx = 0. \end{align*} \] (43)

The first term in (43) can be written as
\[ \int_0^1 h^k u^{k+1}_{xx} u^{k+1}_{xx} dx = \frac{1}{2} \frac{d}{dt} \int_0^1 h^k (u^{k+1}_{xx})^2 dx - \frac{1}{2} \int_0^1 h^k u^{k+1}_{xx} dx, \] (44)
and the last term can be bounded by
\[
\left| \int_0^1 h_t^k(u_{xx}^{k+1}) \, dx \right| \leq c |h_t^k|_{H^1(0,1)} \int_0^1 h^k(u_{xx}^{k+1})^2 \, dx. \tag{45}
\]
To bound the second, seventh, eighth, tenth and twelfth terms in (42), we obtain
\[
\left| \int_0^1 h^k u_{xx}^{k+1} u_{xx} u_{xx}^{k+1} \, dx \right| \leq |h^1|_{L^\infty(0,1)} |u_{xx}^{k+1}|_{L^\infty(0,1)} |u^k_{xx}|_{L^2(0,1)} |u_{xx}^{k+1}|_{L^2(0,1)}
\leq c |h^1|_{H^1(0,1)} |u^t|_{H^2(0,1)} \left( \int_0^1 h^k(u_{xx}^{k+1})^2 \, dx + \int_0^1 h^k(u_{xx}^{k+1})^2 \, dx \right). \tag{46}
\]
\[
\left| \int_0^1 \frac{12 C^2}{(h^k)^2} (h_{xx}^k)^2 u_{xx}^{k+1} h_{xx}^{k+1} \, dx \right| \leq c |h^k|_{H^2(0,1)} |h_{xx}^k|_{H^1(0,1)} \left( \int_0^1 h^k(u_{xx}^{k+1})^2 \, dx \right)^{1/2}
\times \left( \int_0^1 \left( g + \frac{C^2}{(h^k)^2} \right) (h_{xx}^{k+1})^2 \, dx \right)^{1/2}. \tag{47}
\]
\[
\left| \int_0^1 \frac{3 C^2}{(h^k)^3} h_{xx}^k u_{xx}^{k+1} h_{xx}^{k+1} \, dx \right| \leq c |h^k|_{H^2(0,1)} |h_{xx}^k|_{H^1(0,1)} \left( \int_0^1 h^k(u_{xx}^{k+1})^2 \, dx \right)
+ \int_0^1 \left( g + \frac{C^2}{(h^k)^3} \right) (h_{xx}^{k+1})^2 \, dx + \int_0^1 \left( g + \frac{C^2}{(h^k)^3} \right) (h_{xx}^{k+1})^2 \, dx. \tag{48}
\]
\[
\left| \int_0^1 \left( g + \frac{C^2}{(h^k)^3} \right) h_{xx}^k u_{xx}^{k+1} h_{xx}^{k+1} \, dx \right| \leq c |u^k|_{H^2(0,1)} |h_{xx}^k|_{H^1(0,1)} \left( \int_0^1 h^k(u_{xx}^{k+1})^2 \, dx \right)
+ \int_0^1 h^k(u_{xx}^{k+1})^2 \, dx + \int_0^1 \left( g + \frac{C^2}{(h^k)^3} \right) (h_{xx}^{k+1})^2 \, dx. \tag{49}
\]
and
\[
\left| \int_0^1 \left( g + \frac{C^2}{(h^k)^3} \right) u_{xx}^k h_{xx}^{k+1} h_{xx}^{k+1} \, dx \right| \leq c |u^k|_{H^2(0,1)} |h_{xx}^k|_{H^1(0,1)} \left( \int_0^1 h^k(u_{xx}^{k+1})^2 \, dx \right)
+ \int_0^1 \left( g + \frac{C^2}{(h^k)^3} \right) (h_{xx}^{k+1})^2 \, dx + \int_0^1 \left( g + \frac{C^2}{(h^k)^3} \right) (h_{xx}^{k+1})^2 \, dx. \tag{50}
\]
To estimate the third and thirteenth terms in (42), we have
\[
\left| \int_0^1 h^k u_{xx}^k (u_{xx}^{k+1})^2 \, dx \right| \leq c |u^k|_{H^2(0,1)} \int_0^1 h^k(u_{xx}^{k+1})^2 \, dx. \tag{51}
\]
and
\[
\left| \int_0^1 \left( g + \frac{C^2}{(h^k)^3} \right) u_{xx}^k (h_{xx}^{k+1})^2 \, dx \right| \leq c |u^k|_{H^2(0,1)} \int_0^1 \left( g + \frac{C^2}{(h^k)^3} \right) (h_{xx}^{k+1})^2 \, dx. \tag{52}
\]
To bound the fourth and fourteenth terms in (42), due to integration by parts, we find that
\[
\left| \int_0^1 h^k u_{xx} u_{xx}^{k+1} u_{xx} \, dx \right| \leq c (|u^k|_{H^2(0,1)} + |h^k|_{H^2(0,1)} |u^t|_{H^1(0,1)}) \int_0^1 h^k(u_{xx}^{k+1})^2 \, dx. \tag{53}
\]
and
\[
\left| \int_0^1 \left( g + \frac{C^2}{(h^k)^3} \right) u^k h_{xx}^{k+1} h_{xx}^{k+1} \, dx \right| \leq c (|u^k|_{H^2(0,1)} + |h^k|_{H^2(0,1)} |u^t|_{H^1(0,1)}) \int_0^1 \left( g + \frac{C^2}{(h^k)^3} \right) (h_{xx}^{k+1})^2 \, dx. \tag{54}
\]
To estimate the fifth and eleventh terms in (42), we obtain
\[
\left| \int_0^1 \frac{C^2}{(h^k)^2} h_{xx}^k u_{xx}^{k+1} h_{xx}^{k+1} \, dx \right| \leq c |h_{xx}^k|_{L^2(0,1)} \left( \int_0^1 h^k(u_{xx}^{k+1})^2 \, dx \right)^{1/2} \left( \int_0^1 \left( g + \frac{C^2}{(h^k)^3} \right) (h_{xx}^{k+1})^2 \, dx \right)^{1/2}. \tag{55}
\]
Applying the Gronwall lemma to (63), we have
\[ \left| \int_0^1 \left( g + \frac{C^2}{(h^3)^3} \right) h^k u^{k+1}_{xx} u_{xx}^{k+1} \, dx \right| \leq c|h|^3 \|H^2_0,1\| \left( \int_0^1 h^k(u_{xx}^{k+1})^2 \, dx \right)^{1/2} \left( \int_0^1 \left( g + \frac{C^2}{(h^3)^3} \right) (h_x^{k+1})^2 \, dx \right)^{1/2}. \] (56)

The sixth term in (42) can be written as
\[ \int_0^1 \left( g + \frac{C^2}{(h^3)^3} \right) h^k(u_{xx}^{k+1} h_{xx}^{k+1})_x \, dx = \left( g + \frac{C^2}{(h^3)^3} \right) h^k u_{xx}^{k+1} h_{xx}^{k+1} |_{t=0} - \int_0^1 \left( g - 2\frac{C^2}{(h^3)^3} \right) h^k u_{xx}^{k+1} h_{xx}^{k+1} \, dx. \] (57)

Thus, we need to calculate the boundary terms from (57). Indeed, we can infer from (30) that
\[ h_{xx}^{k+1}(0, t) = h_{xx}^{k+1}(1, t) = 0. \] (58)

Due to (22), (58) implies that
\[ u_{xx}^{k+1}(0, t) = u_{xx}^{k+1}(1, t) = 0. \] (59)

We can now estimate (57) as
\[ \int_0^1 \left( g + \frac{C^2}{(h^3)^3} \right) h^k(u_{xx}^{k+1} h_{xx}^{k+1})_x \, dx \leq c|h|^3 \|H^2_0,1\| \int_0^1 \left( g + \frac{C^2}{(h^3)^3} \right) (h^{k+1})^2 \, dx^{1/2} \times \left( \int_0^1 h^k(u^{k+1})^2 \, dx \right)^{1/2}. \] (60)

The ninth term in (42) can be written as
\[ \int_0^1 \left( g + \frac{C^2}{(h^3)^3} \right) h_{xx}^{k+1} h_{txx}^{k+1} \, dx = \frac{1}{2} \frac{d}{dt} \int_0^1 \left( g + \frac{C^2}{(h^3)^3} \right) (h^{k+1})^2 \, dx + \frac{1}{2} \int_0^1 \frac{3C^2}{(h^4)^3} h^k (h_{xx}^{k+1})^2 \, dx. \] (61)

The last term in (61) can be bounded by
\[ \int_0^1 \frac{3C^2}{(h^4)^3} h^k (h_{xx}^{k+1})^2 \, dx \leq c|h|^3 \|H^2_0,1\| \int_0^1 \left( g + \frac{C^2}{(h^3)^3} \right) (h^{k+1})^2 \, dx. \] (62)

Combining all the previous estimates, we find that
\[ \frac{d}{dt} l_{2+1}^{k+1}(t) \leq c_2 \xi^k_2(t) l_{2+1}^{k+1}(t) + c_2 \xi^k_2(t) l_{1+1}^{k+1}(t), \] (63)

where the function \( l_{2+1}^{k+1}(t) \) is defined as
\[ l_{2+1}^{k+1}(t) = \int_0^1 h^k(u_{xx}^{k+1})^2 \, dx + \int_0^1 \left( g + \frac{C^2}{(h^3)^3} \right) (h^{k+1})^2 \, dx, \] (64)

and
\[ \xi^k_2(t) = \xi^k_1(t) + |h|^3 \|H^2_0,1\| |u|^k \|H^2_0,1\|. \] (65)

Applying the Gronwall lemma to (63), we have
\[ l_{2+1}^{k+1}(t) \leq l_{2+1}^{k+1}(0) e^{\int_0^t \xi^k_2(s) \, ds} + \int_0^t c_2 \xi^k_2(s) l_{2+1}^{k+1}(s) e^{-\int_0^t \xi^k_2(s) \, ds} \, ds. \]

Due to (41), the above inequality implies that
\[ l_{2+1}^{k+1}(t) \leq \left( l_{2+1}^{k+1}(0) + c_1 l_{2+1}^{k+1}(0) \int_0^t \xi^k_2(s) \, ds \right) e^{\int_0^t \xi^k_2(s) \, ds}. \] (66)
3.4. Estimates $u^{k+1}_t$, $h^{k+1}_t$ in $L^\infty(0, T; L^2(0, 1))$

From (8), we obtain

$$|u^{k+1}_t|_{L^2(0, 1)} \leq |u^k|_{L^\infty(0, 1)}|u^{k+1}_x|_{L^2(0, 1)} + c|h^{k+1}_x|_{L^2(0, 1)}$$

$$\leq c|u^k|_{H^1(0, 1)} \left( \int_0^1 h^k(u^{k+1}_x)^2 \, dx \right)^{1/2} + c|h^{k+1/2}_x|_{H^1(0, 1)} \left( \int_0^1 \left( g + \frac{C^2}{(h^k)^3} \right) (h^{k+1}_x)^2 \, dx \right)^{1/2}, \quad (67)$$

and

$$|h^{k+1}_t|_{L^2(0, 1)} \leq c|h^{k+1}_x|_{H^1(0, 1)} \left( \int_0^1 h^k(u^{k+1}_x)^2 \, dx \right)^{1/2} + c|h^{k+1/2}_x|_{H^1(0, 1)} |u^k|_{H^1(0, 1)} \left( \int_0^1 \left( g + \frac{C^2}{(h^k)^3} \right) (h^{k+1}_x)^2 \, dx \right)^{1/2}. \quad (68)$$

The inequalities (67) and (68) imply that

$$|u^{k+1}_t|^2_{L^2(0, 1)} \leq c_5 (|u^k|^2_{H^2(0, 1)} + |h^k|^2_{H^2(0, 1)}) (I_1^{k+1})^{1/2}(t) + c(|u^k|_{H^1(0, 1)} + 1)(I_2^{k+1})^{1/2}(t), \quad (69)$$

$$|h^{k+1}_t|^2_{L^2(0, 1)} \leq c_5 (|h^k|^2_{H^2(0, 1)} + |u^k|^2_{H^2(0, 1)}) |h^k|^{3/2}_{H^1(0, 1)} (I_1^{k+1})^{1/2}(t) + c(|h^k|_{H^1(0, 1)} + |u^k|_{H^1(0, 1)}) |h^k|^{3/2}_{H^1(0, 1)} (I_2^{k+1})^{1/2}(t). \quad (70)$$

3.5. Estimates $u^{k+1}_t$, $h^{k+1}_t$ in $L^\infty(0, T; H^1(0, 1))$

To derive a priori estimates for $u^{k+1}_t$ and $h^{k+1}_t$ in $L^\infty(0, T; H^1(0, 1))$, we need to estimate $u^{k+1}_x$ and $h^{k+1}_x$ in $L^\infty(0, T; L^2(0, 1))$. For that purpose, it can be inferred from (22) that

$$|u^{k+1}_x|_{L^2(0, 1)} \leq c(|u^k|_{H^2(0, 1)} + |h^k|_{H^2(0, 1)})(I_1^{k+1})^{1/2}(t) + c_6 (|u^k|_{H^1(0, 1)} + 1)(I_2^{k+1})^{1/2}(t), \quad (71)$$

and

$$|h^{k+1}_x|_{L^2(0, 1)} \leq c(|h^k|_{H^2(0, 1)} + |u^k|_{H^2(0, 1)}) |h^k|^{3/2}_{H^1(0, 1)} (I_1^{k+1})^{1/2}(t) + c(|h^k|_{H^1(0, 1)} + |u^k|_{H^1(0, 1)}) |h^k|^{3/2}_{H^1(0, 1)} (I_2^{k+1})^{1/2}(t). \quad (72)$$

Using (41) and (66), we have

$$|u^{k+1}_x|^2_{L^2(0, 1)} \leq c_6 (|u^k|^2_{H^2(0, 1)} + |h^k|^2_{H^2(0, 1)}) (I_1^{k+1})^{1/2}(0) e^{\int_0^t 2h^k \xi_2(s) \, ds} + c_6 (|u^k|_{H^2(0, 1)} + 1)(I_2^{k+1})^{1/2}(0) e^{\int_0^t 2h^k \xi_2(s) \, ds}, \quad (73)$$

and

$$|h^{k+1}_x|^2_{L^2(0, 1)} \leq c_6 (|h^k|^2_{H^2(0, 1)} + |u^k|^2_{H^2(0, 1)}) |h^k|^{3/2}_{H^1(0, 1)} (I_1^{k+1})^{1/2}(0) e^{\int_0^t 2h^k \xi_2(s) \, ds} + c_6 (|h^k|_{H^2(0, 1)} + |u^k|_{H^2(0, 1)}) |h^k|^{3/2}_{H^1(0, 1)} (I_2^{k+1})^{1/2}(0) e^{\int_0^t 2h^k \xi_2(s) \, ds}. \quad (74)$$

3.6. Uniform estimates on $u^{k+1}$ and $h^{k+1}$

Now, we would like to obtain uniform estimates on $u^{k+1}$ and $h^{k+1}$ by induction. This also gives us the existence of the solutions for the approximate linear system (8)–(10). For that purpose, let two constants $\alpha$ and $\beta$ satisfy

$$H_0 |u^0|_{H^2(0, 1)} = \frac{\alpha}{12}, \quad \left( g + \frac{C^2}{H_0} \right) |h^0|_{H^2(0, 1)} = \frac{\beta}{12}. \quad (75)$$

We assume that

$$H_0 |u^k|_{L^\infty(0, T; H^2(0, 1))} + g |h^k|_{L^\infty(0, T; H^2(0, 1))} \leq \alpha + \beta, \quad (76)$$

$$|u^{k+1}_t|_{L^\infty(0, T; H^1(0, 1))} \leq (c_5 + c_6) \left( 1 + \frac{\alpha + \beta}{H_0} \right) \frac{\alpha + \beta}{2}, \quad (77)$$

$$|h^{k+1}_t|_{L^\infty(0, T; H^1(0, 1))} \leq (c_5 + c_6) \left( \frac{1}{g} + \frac{1}{H_0} \right) \frac{(\alpha + \beta)^2}{2}. \quad (78)$$
We will show that the inequalities (76)–(78) hold for all \( k \) on a small time \( T > 0 \) that does not depend on \( k \) by induction. It is easy to see that inequalities (76)–(78) hold for \( k = 0 \). Now we suppose that the inequalities (76)–(78) are true for \( k = i \). Then, we can infer that there exists a constant \( \gamma \) independent of \( k \) such that

\[
|\xi^k_t(t)|_{\mathcal{L}^2((0, 1))} \leq \gamma (\alpha, \beta).
\]

We will now show that for a small time \( T > 0 \) that depends only on \( \alpha \) and \( \beta \), we still have

\[
H_0 |u^{i+1}_{t}|_{\mathcal{L}^2((0, T; \mathcal{H}^2(0, 1)))} + g |h^{i+1}_{t}|_{\mathcal{L}^2((0, T; \mathcal{H}^2(0, 1)))} \leq \alpha + \beta,
\]

(79)

\[
|u^{i+1}_{t}|_{\mathcal{L}^2((0, T; \mathcal{H}^1(0, 1)))} \leq (c_5 + c_6) \left( 1 + \frac{\alpha + \beta}{H_0} \right),
\]

(80)

\[
|u^{i+1}_{t}|_{\mathcal{L}^2((0, T; \mathcal{H}^1(0, 1)))} \leq (c_5 + c_6) \left( \frac{1}{g} + \frac{1}{H_0} \right) (\alpha + \beta)^2.
\]

(81)

By using the inequalities for \( h^{i+1}_{0} \), \( h^{i+1}_{1} \) and \( h^{i+1}_{2} \), and the fact that \( h^i \geq H_0 \), we obtain

\[
H_0 |u^{i+1}_{t}|_{\mathcal{L}^2((0, 1))} + g |h^{i+1}_{t}|_{\mathcal{L}^2((0, 1))} \leq \frac{\alpha + \beta}{12} e^{(c_0 \gamma) T},
\]

(82)

\[
H_0 |u^{i+1}_{t}|_{\mathcal{L}^2((0, 1))} + g |h^{i+1}_{t}|_{\mathcal{L}^2((0, 1))} \leq \frac{\alpha + \beta}{12} e^{(c_1 \gamma) T},
\]

(83)

\[
H_0 |u^{i+1}_{t}|_{\mathcal{L}^2((0, 1))} + g |h^{i+1}_{t}|_{\mathcal{L}^2((0, 1))} \leq \frac{\alpha + \beta}{12} e^{(c_2 \gamma) T} (1 + c_3 \gamma T).
\]

(84)

Adding all the inequalities (82)–(84) and choosing suitable \( T \) so that \( c_3 \gamma T < 1 \), we find that

\[
H_0 |u^{i+1}_{t}|_{\mathcal{L}^2((0, 1))} + g |h^{i+1}_{t}|_{\mathcal{L}^2((0, 1))} \leq \frac{\alpha + \beta}{2} e^{(\text{max}(c_0, c_1, c_2) \gamma) T}.
\]

By choosing \( T_1 \) small enough such that \( e^{(\text{max}(c_0, c_1, c_2) \gamma) T} < 2 \), we obtain that

\[
H_0 |u^{i+1}_{t}|_{\mathcal{L}^2((0, 1))} + g |h^{i+1}_{t}|_{\mathcal{L}^2((0, 1))} \leq \alpha + \beta, \quad \forall t \leq T_1.
\]

To find \( T_2 \) small enough such that the inequalities (80) and (81) are true, we use the inequalities (69), (70), (73) and (74) which yield

\[
|u^{i+1}_{t}|_{\mathcal{L}^2((0, 1))} \leq (c_5 + c_6) \left( \frac{\alpha + \beta}{H_0} + 1 \right) \frac{\alpha + \beta}{12} e^{(\text{max}(c_1, c_2) \gamma) T},
\]

and

\[
|u^{i+1}_{t}|_{\mathcal{L}^2((0, 1))} \leq (c_5 + c_6) \left( \frac{1}{H_0} + \frac{1}{g} \right) \frac{\alpha + \beta}{12} e^{(\text{max}(c_1, c_2) \gamma) T}.
\]

By taking \( T_2 \) small enough such that \( e^{(\text{max}(c_1, c_2) \gamma) T_2} \leq 2 \), the inequalities (80) and (81) are true for \( t \leq T_2 \).

To complete the induction, we take \( T^* = \min(T_1, T_2) \) and find that all the inequalities (82)–(84) are satisfied for \( t \leq T^* \) where \( \alpha, \beta, \gamma \) and \( T^* \) do not depend on \( k \).

Now, we need to show that the relation that \( h^{k+1} \geq H_0 \) is still true since we assume the same relation satisfied by \( h^k \). This can be proven by showing that there exists a small time (independent of \( k \) and possibly smaller than \( T^* \)) such that

\[
|h^{k+1}(x, t) - h_0(x)| < H_0,
\]

which implies that \( h^{k+1} \geq H_0 \). Note that we still denote this small time as \( T^* \). Similar proofs can be found in [9].

4. Existence and uniqueness of strong solutions

In this section, we will show the existence and uniqueness of the strong solutions for Eq. (6) for a small time \( T^* \).

4.1. Existence

To prove the existence of the solutions, the main key is to verify that the sequences \( \{u^k\}_{k=1}^\infty \) and \( \{h^k\}_{k=1}^\infty \) constructed by solving the approximate linear systems (8)–(10) are Cauchy in \( C((0, T^*); \mathcal{H}^1(0, 1)) \). For that purpose, we write
\[ u^{k+1} = u^{k+1} - u^k, \]
\[ \Phi^{k+1} = h^{k+1} - h^k, \]
and find that \( v^{k+1} \) and \( \Phi^{k+1} \) satisfy
\[
\begin{aligned}
&v_i^{k+1} + u^k v_x^{k+1} + v^k u_x^{k+1} + (g + \frac{C^2}{(h^k)^3}) \Phi_x^{k+1} + C^2 \left( \frac{1}{(h^k)^3} - \frac{1}{(h^k - \Phi^k)^3} \right) h_x^k = 0, \\
&\phi_i^{k+1} + u^k \phi_x^{k+1} + v^k \phi_x^{k+1} + h^k \phi_x^{k+1} + u_x^{k} \Phi^k = 0,
\end{aligned}
\]
with the initial conditions
\[ v^{k+1}(x, 0) = 0, \quad \Phi^{k+1} = 0, \]
and the boundary conditions
\[ v^{k+1}(0, t) = v^{k+1}(1, t) = 0. \]

**A priori estimates for** \( v^{k+1} \) and \( \Phi^{k+1} \) in \( L^\infty(0, T^*; H^1(0, 1)) \). Multiplying (85) by \( h^k u^{k+1} \), (85) by \( (g + \frac{C^2}{(h^k)^3}) \Phi^{k+1} \), adding the resulting equations, integrating the equation over the domain \( I \) and applying the integration by parts and boundary conditions, we find that
\[
\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \int_0^1 h^k (v^{k+1})^2 + (g + \frac{C^2}{(h^k)^3}) (\Phi^{k+1})^2 \, dx - \frac{1}{2} \int_0^1 h_x^k (v^{k+1})^2 \, dx \\
&- \frac{1}{2} \int_0^1 (u^k h_x^k) v^{k+1} \, dx + \int_0^1 h^k v^k u_x^{k+1} \, dx \\
&- \int_0^1 \left( (g + \frac{C^2}{(h^k)^3}) h_x^k \right) v^{k+1} \Phi^{k+1} \, dx + \int_0^1 C^2 h^k v^k h_x^k \left( \frac{1}{(h^k)^3} - \frac{1}{(h^k - \Phi^k)^3} \right) \, dx \\
&+ \frac{3}{2} \int_0^1 \frac{C^2}{(h^k)^3} h_x^k (\Phi^{k+1})^2 \, dx - \frac{1}{2} \int_0^1 \left( (g + \frac{C^2}{(h^k)^3}) u^k \right) (\Phi^{k+1})^2 \, dx \\
&+ \int_0^1 \left( g + \frac{C^2}{(h^k)^3} \right) v^k h_x^k \Phi^{k+1} \, dx + \int_0^1 \left( g + \frac{C^2}{(h^k)^3} \right) h^k u_x^{k+1} \Phi^{k+1} \, dx \\
&+ \int_0^1 \left( g + \frac{C^2}{(h^k)^3} \right) u_x^{k} \Phi^{k+1} \Phi^{k+1} \, dx = 0.
\end{aligned}
\]

To estimate Eq. (88), we point out how to deal with the sixth term in (88) since other terms can be estimated as before. Due to the condition that \( h^k - \Phi^k = h^k - \Phi^{k-1} \) is bounded below, we obtain
\[
\left| \int_0^1 C^2 h^k v^{k+1} h_x^k \left( \frac{1}{(h^k)^3} - \frac{1}{(h^k - \Phi^k)^3} \right) \, dx \right|
\leq c |h_h^{k+1/2}_{H^1(0,1)}| |h^{k-1/2}_{H^1(0,1)}| \left( \int_0^1 h^k (v^{k+1})^2 \, dx \right)^{1/2} \left( \int_0^1 \left( g + \frac{C^2}{(h^k)^3} \right) (\Phi^{k+1})^2 \, dx \right)^{1/2}.
\]

We have the following inequality:
\[
\frac{d}{dt} j^{k+1}_0 (t) \leq c_T \xi^k + c_8 (\xi^k + |h^{k-1/2}_{H^1(0,1)}|)^2 (t).
\]

where the term \( j^{k+1}_0 \) is defined as
\[
j_0^{k+1} (t) = \int_0^1 h^k (v^{k+1})^2 \, dx + \int_0^1 \left( g + \frac{C^2}{(h^k)^3} \right) (\Phi^{k+1})^2 \, dx.
\]

Since we have that the term \( |h^{k-1/2}_{H^1(0,1)}| \) is bounded and independent of all \( k \), we set
\[
|h^{k-1/2}_{H^1(0,1)}| \leq \gamma_1 = \gamma_1 (\alpha, \beta),
\]
where \( \gamma_1 \) is a constant that does not depend on \( k \).

Then, using the Gronwall inequality and the fact that \( j^{k+1}_0 (0) = 0 \), we obtain
\[
j_0^{k+1} (t) \leq c_8 \int_0^t (\xi^k + \gamma_1) j_0^k (s) \, ds \leq c_8 T^* \sup_{0 \leq s \leq T^*} j_0^k (s).
\]
By taking $T^*$ small enough so that $c_8(y + \gamma_1)e^{\gamma T^*}T^* \leq \frac{1}{2}$, we find that

$$H_0|v^{k+1}|^2_{L^2(0,1)} + g|\Phi^{k+1}|^2_{L^2(0,1)} \leq j_0^{k+1}(t) \leq \frac{c_9}{2k+1}, \quad \forall 0 \leq t \leq T^* \text{ and } k.$$  

(90)

Thus, we show that the sequences $\{u^k\}$ and $\{h^k\}$ are Cauchy in $C([0, T^*]; L^2(0, 1))$, which implies that the strong convergence of $u^k$ and $h^k$ in $C([0, T^*]; L^2(0, 1))$ is verified.

To estimate $v^{k+1}$ and $\Phi^{k+1}$ in $C([0, T^*]; H^1(0, 1))$, we differentiate (85) with respect to the variable $x$, multiply the first equation by $h^kv^{k+1}$, the second equation by $(g + C^2/(h^k)^4)\Phi^{k+1}$, add the resulting equation and integrating the equation over the domain $I$, we find that, after using the integration by parts and boundary conditions,

$$\frac{1}{2} \left\{ \int_0^1 h^k(v^{k+1})^2 \, dx + \int_0^1 \left( g + \frac{C^2}{(h^k)^3} \right) (\Phi^{k+1})^2 \, dx \right\} - \frac{1}{2} \int_0^1 h^k(v^{k+1})^2 \, dx$$

$$+ \int_0^1 h^k u_x^{k+1}v_x^{k+1} \, dx - \frac{1}{2} \int_0^1 (h^k u_x^{k+1}v_x^{k+1}) \, dx + \int_0^1 h^k u_x^{k+1}v_x^{k+1} \, dx$$

$$+ \int_0^1 h^k u_x^{k+1}v_x^{k+1} \, dx - \frac{1}{2} \int_0^1 (h^k u_x^{k+1}v_x^{k+1}) \, dx + \int_0^1 h^k u_x^{k+1}v_x^{k+1} \, dx$$

$$+ C^2 \int_0^1 h^k v_x^{k+1} \left( \frac{3h_x^k - \Phi_x^k}{(h^k)^4} \right) h^k \, dx + C^2 \int_0^1 \left( \frac{1}{(h^k)^3} - \frac{1}{(h^k - \Phi^k)^3} \right) h^k \, dx$$

$$\leq \frac{3C^2}{2} \int_0^1 h^k v_x^{k+1} \left( \frac{3h_x^k - \Phi_x^k}{(h^k)^4} \right) h^k \, dx$$

$$+ \int_0^1 \left( g + \frac{C^2}{(h^k)^3} \right) u_x^{k+1} h_x v_x^{k+1} \, dx + \int_0^1 \left( g + \frac{C^2}{(h^k)^3} \right) u_x^{k+1} h_x \, dx$$

$$+ \int_0^1 (g + C^2/(h^k)^4)\Phi^{k+1} h^k \, dx + \int_0^1 (g + C^2/(h^k)^4)\Phi^{k+1} h_x \, dx$$

$$+ \int_0^1 (g + C^2/(h^k)^4)\Phi^{k+1} h_x \, dx + \int_0^1 (g + C^2/(h^k)^4)\Phi^{k+1} h_x \, dx = 0.$$  

(91)

To estimate the terms in (91), we only present how to estimate the ninth and tenth terms in (91) since other terms can be estimated by applying previous techniques. To bound the ninth term in (91), we have

$$\left| 3C^2 \int_0^1 h^k v_x^{k+1} \left( \frac{3h_x^k - \Phi_x^k}{(h^k)^4} \right) h^k \, dx \right|$$

$$\leq \left| 3C^2 \int_0^1 h^k v_x^{k+1} \left( \frac{1}{(h^k)^4} + \frac{1}{(h^k - \Phi^k)^4} \right) h^k \, dx \right| + \left| \int_0^1 h^k(v_x^{k+1})^2 \, dx \right|^{1/2}$$

$$+ \left| \int_0^1 (g + \frac{C^2}{(h^k)^3}) \Phi_x^k \, dx \right|^{1/2}.$$  

(92)

As for the tenth term in (91), we find that

$$\left| \int_0^1 (g + C^2/(h^k)^3)\Phi_x \, dx \right|$$

$$\leq \left| \int_0^1 (g + C^2/(h^k)^3)\Phi_x \, dx \right|$$

$$\leq \left| (g + \frac{C^2}{(h^k)^3}) \Phi_x \right| h_x \, dx$$

$$\leq \left| (g + \frac{C^2}{(h^k)^3}) \Phi_x \right| h_x \, dx$$

$$\leq \left| (g + \frac{C^2}{(h^k)^3}) \Phi_x \right| h_x \, dx$$

$$\left| \int_0^1 \left( g + \frac{C^2}{(h^k)^3} \right) \Phi_x^2 \, dx \right| + \left( \int_0^1 (g + C^2/(h^k)^3) \Phi_x \, dx \right) \left( \int_0^1 (g + C^2/(h^k)^3) \Phi_x \, dx \right)^{1/2}.$$  

(93)

Therefore, combining (92) and (93), we obtain that

$$\frac{d}{dt} |j_1^{k+1}(t)| \leq c\xi_2(t)(j_1^{k+1}(t) + j_1^k(t) + j_0^k(t)).$$  

(94)
where the function $f_{k+1}^{t}(t)$ is defined as

$$f_{1}^{t}(t) = \int_{0}^{1} h^{t} f_{x}^{t} x^{2} dx + \int_{0}^{1} \left( g + \frac{C^{2}}{(h^{t})^{3}} \right) (\Phi_{x}^{t})^{2} dx,$$

and

$$\xi_{3}(t) = \xi_{2}^{k}(t) + (1 + |h_{1}^{k}|) |h^{k}_{1}| + |h^{k-1}_{1}|^{3/2}.$$  

Adding (89) and (94), we find that

$$\frac{d}{dt} (f_{0}^{k+1}(t) + f_{1}^{k+1}(t)) \leq c_{10} \xi_{3}(t) (f_{0}^{k+1}(t) + f_{1}^{k+1}(t)) + c_{11} (\xi_{3}(t) + |h^{k-1}_{1}|^{7} + f_{0}^{k}(t) + f_{1}^{k}(t)).$$  

Since $\xi_{3} + |h^{k-1}_{1}|^{7}$ is uniformly bounded in $[0, T^{*}]$, after using the Gronwall inequality for (95) and the fact that $f_{0}^{k+1}(0) = f_{1}^{k+1}(0) = 0$, we obtain that

$$f_{0}^{k+1}(t) + f_{1}^{k+1}(t) \leq c_{11} \int_{0}^{t} (\xi_{3}(s) + |h^{k-1}_{1}|^{7} + f_{0}^{k}(s) + f_{1}^{k}(s)) ds e^{c_{10} \xi_{3}(s)}.$$

(96) where $\gamma_{2}$ is a constant independent of $k$ and satisfying $\xi_{3}(t) \leq \gamma_{2}$.  

By taking $T^{*}$ small enough such that $c_{11} T^{*} (\gamma_{1} + \gamma_{2}) e^{c_{10} T^{*}} < 1/2$, we find that for all $k$

$$H_{0} |u^{k+1}|^{2} h^{k}_{1} + g + |h^{k+1}_{1}|^{2} \leq f_{0}^{k+1}(t) + f_{1}^{k+1}(t) \leq c_{k+1}, \quad \text{for } t \in [0, T^{*}).$$  

(97) Thanks to (97), we can deduce that the sequences $\{u^{k}\}$ and $\{h^{k}\}$ are Cauchy in $C([0, T^{*}]; \mathcal{H}^{1}(0, 1))$. Thus, there exist functions $(u, h) \in C([0, T^{*}]; \mathcal{H}^{1}(0, 1))$ such that $u^{k}$ and $h^{k}$ strongly converge to $u$ and $h$ in $C([0, T^{*}]; \mathcal{H}^{1}(0, 1))$, respectively. To show that $u$ and $h$ are the solutions to the system (6), we can indeed pass to the limit in the approximate linear system (8) and find that $u$ and $h$ satisfy the system (6). Furthermore, since $|u^{k}|$ and $|h^{k}|$ are bounded in $L^{\infty}(0, T^{*}; H^{2}(0, 1))$, we can deduce that the solutions $u$ and $h$ also belong to $L^{\infty}(0, T^{*}; H^{2}(0, 1))$. Then, the existence of the solution is established.

4.2. Uniqueness

To show the uniqueness of the solution to (6), we assume that two solutions $(u_{1}, h_{1})$ and $(u_{2}, h_{2}) \in L^{\infty}(0, T^{*}; H^{2}(0, 1))$ satisfy (6). Then we write

$$\begin{cases}
u = u_{1} - u_{2}, \\
\Phi = h_{1} - h_{2},
\end{cases}$$

and find that $\nu$ and $\Phi$ satisfy

$$\begin{cases}
u_{t} + u_{1}x \nu_{x} + H(x \nu + u_{2} v_{x} + \left( g + \frac{C^{2}}{(h^{1}_{1})^{3}} \right) \Phi_{x} + C^{2} \left( \frac{1}{(h^{1}_{1})^{3}} - \frac{1}{(h_{1} - \Phi)^{3}} \right) h_{2,x} = 0, \\
\Phi_{t} + h_{1}x \nu + u_{2} \Phi_{x} + h_{1}v_{x} + u_{2} \nu \Phi = 0,
\end{cases}$$

(98) with the initial conditions

$$\nu(x, 0) = \Phi(x, 0) = 0,$$

(99) and with the boundary conditions

$$\nu(0, t) = \nu(1, t) = 0, \quad \forall t \in [0, T^{*}].$$

(100) Multiplying (98) by $h_{1} \nu, \Phi_{x},$ we have $g + \frac{C^{2}}{(h^{1}_{1})^{3}} \Phi$, integrating over the domain $I$, adding the resulting equations and using the integration by parts and the boundary conditions, we obtain

$$\begin{align*}
\frac{1}{2} \int_{0}^{1} h_{1} \nu^{2} + \left( g + \frac{C^{2}}{(h^{1}_{1})^{3}} \right) \Phi^{2} \nu \, dx - \frac{1}{2} \int_{0}^{1} h_{1} \nu^{2} \, dx + \frac{1}{2} \int_{0}^{1} h_{1} \nu^{2} \, dx \\
- \frac{1}{2} \int_{0}^{1} (h_{1}u_{2}) \nu^{2} \, dx + \int_{0}^{1} \left( g + \frac{C^{2}}{(h^{1}_{1})^{3}} \right) h_{1}(\nu \Phi)_{x} \, dx + \int_{0}^{1} C^{2} \left( \frac{1}{(h^{1}_{1})^{3}} - \frac{1}{(h_{1} - \Phi)^{3}} \right) h_{1} \nu \, dx \\
+ \frac{3}{2} \int_{0}^{1} C^{2} \left( h_{1} \Phi \right)_{x} \nu^{2} \, dx + \frac{1}{2} \int_{0}^{1} C^{2} \left( \frac{1}{(h^{1}_{1})^{3}} - \frac{1}{(h_{1} - \Phi)^{3}} \right) \nu \, dx \\
+ \int_{0}^{1} \left( g + \frac{C^{2}}{(h^{1}_{1})^{3}} \right) u_{2} \Phi \, dx = 0.
\end{align*}$$

(101)
As before, using similar techniques, we derive the following inequality:

\[
\frac{d}{dt} \left[ \int_0^1 \left( h_1 v^2 + \left( g + \frac{C^2}{(h_1)^3} \right) \Phi^2 \right) \, dx \right] \leq c \left( |h_1|_{H^1(0,1)} + |u_1|_{H^2(0,1)} + |h_1|_{H^2(0,1)} |u_2|_{H^1(0,1)} + |u_2|_{H^2(0,1)} \left( |h_1|_{H^2(0,1)} \right)^{3/2} \right) 
+ \int_0^1 \left( h_1 v^2 + \left( g + \frac{C^2}{(h_1)^3} \right) \Phi^2 \right) \, dx . \tag{102}
\]

Applying the Gronwall inequality for (102) and using the initial conditions, we obtain that

\[
\int_0^1 (H_0 v^2 + g \Phi^2) \, dx \leq \int_0^1 \left( h_1 v^2 + \left( g + \frac{C^2}{(h_1)^3} \right) \Phi^2 \right) \, dx = 0, \quad \forall \ t \in [0, T^*],
\]

which implies that the solution is unique.

Finally, the proof of Theorem 2 is complete.

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References