Numerical ranges of weighted shifts
Kuo-Zhong Wang*, Pei Yuan Wu

Department of Applied Mathematics, National Chiao Tung University, Hsinchu 30010, Taiwan

**A R T I C L E   I N F O**

Article history:
Received 23 February 2011
Available online 9 April 2011
Submitted by J.A. Ball

Keywords:
Numerical range
Numerical radius
Numerical contraction
Unilateral weighted shift
Bilateral weighted shift

**A B S T R A C T**

Let $A$ be a unilateral (resp., bilateral) weighted shift with weights $w_n$, $n \geq 0$ (resp., $-\infty < n < \infty$). Eckstein and Rácz showed before that $A$ has its numerical range $W(A)$ contained in the closed unit disc if and only if there is a sequence $\{a_n\}_{n=0}^{\infty}$ (resp., $\{a_n\}_{n=-\infty}^{\infty}$) in $[-1, 1]$ such that $|w_n|^2 = (1 - a_n)(1 + a_{n+1})$ for all $n$. In terms of such $a_n$’s, we obtain a necessary and sufficient condition for $W(A)$ to be open. If the $w_n$’s are periodic, we show that the $a_n$’s can also be chosen to be periodic. As a result, we give an alternative proof for the openness of $W(A)$ for an $A$ with periodic weights, which was first proven by Stout. More generally, a conjecture of his on the openness of $W(A)$ for $A$ with split periodic weights is also confirmed.

© 2011 Elsevier Inc. All rights reserved.

**1. Introduction**

Let $A$ be a bounded linear operator on a complex Hilbert space $H$ with inner product $\langle \cdot, \cdot \rangle$ and its associated norm $\| \cdot \|$. The numerical range $W(A)$ of $A$ is, by definition, the subset $\{\langle Ax, x \rangle : x \in H, \|x\| = 1\}$ of the complex plane. Its numerical radius $w(A)$ is sup$|z| : z \in W(A)$. $A$ is said to be a numerical contraction if $W(A)$ is contained in $D$ ($D \equiv \{z \in \mathbb{C} : |z| < 1\}$) or, equivalently, if $w(A) \leq 1$.

The purpose of this paper is to study the numerical ranges of unilateral and bilateral weighted shifts. Recall that the unilateral (resp., bilateral) weighted shift with weights $w_n$, $n \geq 0$ (resp., $-\infty < n < \infty$), is the operator with matrix

$$A = \begin{pmatrix} 0 & w_0 & 0 & \cdots \\ w_1 & 0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots \\ \end{pmatrix}$$

on $\ell^2 = \{(x_0, x_1, \ldots) : \sum_{n=0}^{\infty} |x_n|^2 < \infty\}$ (resp., $\ell^2(\mathbb{Z}) = \{(\ldots, x_{-1}, x_0, x_1, \ldots) : \sum_{n=-\infty}^{\infty} |x_n|^2 < \infty\}$). Here, in the bilateral case, we underline the $(0,0)$-entry of a matrix and the 0th component of a vector. We will also consider the finite weighted shift with weights $w_j$, $1 \leq j \leq n - 1$:

$$A = \begin{pmatrix} 0 & \cdots & \cdots & \cdots \\ w_1 & 0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots \\ w_{n-1} & \cdots & \cdots & 0 \\ \end{pmatrix}$$
on $\mathbb{C}^n$.

* Corresponding author.
E-mail addresses: kwang@math.nctu.edu.tw (K.-Z. Wang), pywu@math.nctu.edu.tw (P.Y. Wu).

0022-247X/$ – see front matter © 2011 Elsevier Inc. All rights reserved.
doi:10.1016/j.jmaa.2011.04.010
Since a weighted shift $A$ (unilateral, bilateral or finite) is unitarily equivalent to $e^{i\theta} A$ for any real $\theta$, its numerical range is an open or closed circular disc centered at the origin. Therefore, the study of their numerical ranges boils down to two things: (1) to determine whether $W(A)$ is open or closed, and (2) to find its radius $\|w(A)\|$. In the present paper, we are mainly concerned with the first problem. A prototypical example of the shifts we consider is the simple unilateral (resp., bilateral) shift, that is, the one with weights $1, 1, \ldots$ (resp., $\ldots, 1, 1, \ldots$). It is known that its numerical range equals the open unit disc (cf. [6, Solution 212(2)]). In Section 2 below, we first prove some preliminary results on the openness (or closedness) of the numerical ranges of the weighted shifts by using only their basic properties. For example, we show that if $A$ is a unilateral (resp., bilateral) weighted shift with weights $w_n$ convergent to a nonzero $a$ and satisfying $|w_n| \geq |a|$ for all $n$, then $W(A)$ is open if and only if $w(A)$ equals $|a|$ (Proposition 2.3). Then, in Section 3, we consider the parametric representation, due to Eckstein and Rácz [3], of the weights of a weighted shift $A$ with $w(A) \leq 1$, namely, a unilateral (resp., bilateral) weighted shift $A$ with weights $w_n$, $n \geq 0$ (resp., $-\infty < n < \infty$), is a numerical contraction if and only if there is a sequence $(a_n)_{n=0}^\infty$ (resp., $(a_n)_{n=-\infty}^{\infty}$) in $[-1, 1]$ such that $|a_n|^2 = (1 - a_n)(1 + a_{n+1})$ for all $n$. In terms of such $a_n$’s, we obtain a necessary and sufficient condition for $W(A)$ to be open (or closed) (Theorem 3.3). If the weights $w_n$ are periodic, we show that the corresponding $a_n$’s can also be chosen to be periodic (Lemma 4.1). As a consequence of these results, we can give an alternative proof for the openness of the numerical ranges of such operators (Proposition 4.2), first proven by Stout [13, Proposition 6]. More generally, a conjecture of his on the openness of shifts is [11].

2. Generalities

We start with a result on the attaining vectors for the numerical radius of a weighted shift. For any operator $A$, let $\text{Re } A = (A + A^*)/2$ denote its real part.

Proposition 2.1. Let $A$ be a unilateral (resp., bilateral) weighted shift with weights $w_n > 0$ for all $n$. If $W(A)$ is closed, then

(a) the set $\{x \in \ell^2, w_2(z): \langle Ax, x \rangle = W(A) \|x\|^2\}$ is a subspace of dimension one, and
(b) there is a unique unit vector $x = (x_n)$ in $\ell^2$ (resp., $\ell^2(z)$) with $x_n > 0$ for all $n$ such that $\langle Ax, x \rangle = W(A)$. 

Proof. We only prove for the unilateral weighted shifts; the proof for the bilateral case is similar.

Let $M = \{x \in \ell^2: \langle Ax, x \rangle = W(A) \|x\|^2\}$. For any vector $x$, we have

$$\langle (w(A)I - Re A)x, x \rangle = W(A) \|x\|^2 - \text{Re} \langle Ax, x \rangle \\
\geq W(A) \|x\|^2 - \|Ax\| \\
\geq 0,$$

which implies that $w(A)I - Re A \geq 0$. In particular, if $x$ is in $M$, then $\langle (w(A)I - Re A)x, x \rangle = 0$ and hence $\text{Re} \langle Ax, x \rangle = W(A)$. Conversely, if $(\text{Re} Ax)x = w(A)x$, then $\text{Re} \langle Ax, x \rangle = W(A) \|x\|^2$. Since $W(A)$ is closed, we have $W(A) = \{z \in \mathbb{C}: |z| \leq w(A)\}$. It thus follows from $\langle A(x/\|x\|), x/\|x\| \rangle \in W(A)$ for $x \neq 0$ that $\langle Ax, x \rangle = W(A) \|x\|^2$. Therefore, $M = \ker(w(A)I - Re A)$ is a subspace.

For any unit vector $x = (x_n)$ in $M$, we have

$$w(A) = \|Ax, x\| \leq \|A|x, x\| \leq W(A),$$

where $|x|$ denotes the (unit) vector $(x_n)$. It follows that $\langle A|x|, |x| \rangle = w(A)$. Hence we may assume from the outset that $x_n \geq 0$ for all $n$. If $x_0 = 0$ for some $n_0 \geq 1$, then, from $\langle Ax, x \rangle = w(A)x$, we have $\langle w_{n_0-1}x_{n_0-1} + w_{n_0}x_{n_0+1} \rangle/2 = w(A)x_0 = 0$, which implies that $x_{n_0-1} = x_{n_0+1} = 0$. Repeating these arguments with $x_{n_0-1}$ and $x_{n_0+1}$ replacing $x_{n_0}$ yields $x_n = 0$ for all $n$. Similarly for $x_0 = 0$. This shows that $x = 0$ contradicting our assumption on $x$. We hence must have $x_n > 0$ for all $n$.

To show that $\text{dim } M = 1$, let $x = (x_n)$ and $y = (y_n)$ be any two vectors in $M$. There are scalars $a$ and $b$, not both zero, such that $ax_0 + by_0 = 0$. Then $ax + by$ in $M$ has its 0th component equal to zero. This contradicts what was proven in the preceding paragraph. Hence $M$ is of dimension one as asserted. (a) and (b) follow immediately. □

Two comments are in order. Firstly, the unit vector $x$ in Proposition 2.1(b) can be easily seen to be the normalized vector of $(y_n)$, where $y_n = 1$, $y_1 = 2w(A)/w_0$ and $y_n = (2w(A)y_{n-1} - w_{n-2}y_{n-2})/w_{n-1}$ for $n \geq 2$. Properties of such vectors, even for the more general Jacobi matrices, were investigated in [12, Section X.4]. Secondly, analogous results as (a) and (b) also
hold for the finite weighted shifts $A$. These can be proven either as above or by invoking the Perron–Frobenius theorem [8, Theorem 1, p. 536] since $R(A)$ is nonnegative irreducible.

Recall that the essential numerical range $W_e(A)$ of an operator $A$ on a separable infinite-dimensional space $H$ is the intersection set $\bigcap \{ \overline{W(A + K)} : K$ compact on $H \}$. Properties of the essential numerical ranges can be found in [7,4]. In particular, it is known that if $W(A)$ is open, then $W_e(A) = W(A)$ (cf. [7, Corollary 2]). The essential numerical radius $w_e(A)$ of $A$ is $\max \{ |z| : z \in W_e(A) \}$.

The next proposition gives information on the (essential) numerical range of a weighted shift with convergent weights.

**Proposition 2.2.** Let $A$ be a unilateral (resp., bilateral) weighted shift with weights $w_n$, $n \geq 0$ (resp., $-\infty < n < \infty$), satisfying $\lim_{n \to \infty} |w_n| = a$ (resp., $\lim_{n \to \pm \infty} |w_n| = b$ and $\lim_{n \to -\infty} |w_n| = c$). Let $a = \max\{b, c\}$ in the bilateral case. Then

(a) $W_e(A) = \{ z \in \mathbb{C} : |z| \leq a \}$,
(b) $W(A) \supseteq a$,
(c) $W(A)$ is closed if and only if $a$ is in $W(A)$, and
(d) $W(A) = \{ z \in \mathbb{C} : |z| < a \}$ if $W(A)$ is open.

**Proof.** We may assume that $w_n \geq 0$ for all $n$.

(a) If $B$ is the unilateral (resp., bilateral) weighted shift with weights $a, a, \ldots$ (resp., ..., $c, c, 0, b, b, \ldots$), then $A - B$ is compact. Hence $W_e(A) = W_e(B) = \{ z \in \mathbb{C} : |z| \leq a \}$, where the last equality follows from the fact that $W_e(S) = \overline{W(S)} = \overline{\mathbb{D}}$, $S$ being the simple unilateral shift (cf. [7, Corollary 2]).

(b) Since $W_e(A)$ contains $W_e(B)$, the assertion follows from (a).

(c) This is a consequence of (a) and the fact that $W(A)$ is closed if and only if $W(A)$ is contained in $W(A)$ (cf. [7, Corollary 1]).

(d) If $W(A)$ is open, then $W(A) = W_e(A)$ by [7, Corollary 2]. Thus $W(A) = \{ z \in \mathbb{C} : |z| < a \}$ from (a).

We remark that, under the conditions of Proposition 2.2(c), $W(A)$ may be bigger than $\{ z \in \mathbb{C} : |z| \leq a \}$. One example is the $A$ with weights $1, 1, 1, \ldots$ (resp., ..., $1, w, 1, 1, \ldots$), where $w > \sqrt{2}$ (resp., $w > 1$). In this case, $W(A) = \{ z \in \mathbb{C} : |z| \leq w^2/(2\sqrt{w^2 - 1}) \}$ (resp., $\{ z \in \mathbb{C} : |z| \leq (w^2 + 1)/(2w) \}$), which is bigger than $\overline{\mathbb{D}}$ (cf. [1, pp. 1053–1054], [13, p. 500], [14, Propositions 2 and 3] or Corollary 4.7 and Theorem 4.9 later). Note also that the converse of Proposition 2.2(d) is false, that is, $w(A) = a$ does not guarantee the openness of $W(A)$.

**Proposition 2.3.** Let $A$ be a unilateral (resp., bilateral) weighted shift with nonzero weights $w_n$, $n \geq 0$ (resp., $-\infty < n < \infty$), satisfying $\lim_{n \to \infty} |w_n| = a > 0$ (resp., $\lim_{n \to \pm \infty} |w_n| = a > 0$) and $|w_n| > a$ for all $n \geq n_0$, where $n_0$ is some fixed nonnegative integer (resp., some fixed integer). Then $W(A)$ is open if and only if $w(A) = a$.

**Proof.** We only prove the unilateral case. In view of Proposition 2.2(d), we need only show that $w(A) = a$ implies the openness of $W(A)$. Assume that $w_n \geq 0$ for all $n$ and $W(A)$ is closed. Proposition 2.1(b) gives a unit vector $x = (x_n)$ in $\ell^2$ with $x_n > 0$ for all $n$ such that $(\text{Re} A)x = ax$. Then $w_n x_1 = 2ax_0$ and $w_{n-1} x_{n-1} + w_n x_{n+1} = 2ax_n$ for all $n \geq 1$. Together with our assumption of $w_n \geq a$ for all $n \geq n_0$, this yields

$$a(x_n - x_{n-1}) \leq w_{n-1} x_{n-1} - ax_n = ax_n - w_n x_{n+1} \leq a(x_n - x_{n+1})$$

for $n \geq n_0 + 1$. Hence the sequence $\{x_n - x_{n+1}\}_{n=n_0+1}^{\infty}$ is increasing. Since $\lim_{n \to n_0} (x_n - x_{n+1}) = a - a = 0$, this implies that $x_{n_0} \leq x_{n_0+1} \leq \cdots = 0$, which contradicts our assumption. Hence $W(A)$ is open as asserted.

The preceding proposition generalizes [2, Example 2], where it is assumed that the unilateral $|w_n|$’s decrease to $a$. Another condition for $W(A)$ to be equal to $\{ z \in \mathbb{C} : |z| < a \}$ is $\lim_{n \to \infty} |w_n| = a > 0$ (resp., $\lim_{n \to \infty} |w_n| = a > 0$ or $\lim_{n \to -\infty} |w_n| = a > 0$) and $|w_n| \leq a$ for all $n$ (cf. [14, Theorem 1]).

The next proposition relates the numerical range $W(A)$ of a weighted shift $A$ to those of its compressions for open $W(A)$.

**Proposition 2.4.** Let $A$ be a unilateral (resp., bilateral) weighted shift with nonzero weights $w_n$, $n \geq 0$ (resp., $-\infty < n < \infty$), and, for each $m \geq 1$, let $A_m$ be the weighted shift with weights $w_m, w_{m+1}, \ldots$ (resp., ..., $w_{-m-1}, w_{-m}, w_m, w_{m+1}, \ldots$). If $W(A)$ is open, then $W(A) = W(A_m)$ for all $m \geq 1$.

**Proof.** We only prove the unilateral case. Since $A_m$ is a compression of $A$, we obviously have $W(A_m) \subseteq W(A)$ for all $m \geq 1$. On the other hand, if $B_m$ denotes the unilateral weighted shift with weights $0, \ldots, 0, w_m, w_{m+1}, \ldots$, then

$$W(A) = W_e(A) \subseteq W(B_m) = W(A_m).$$
where the first equality is the consequence of the openness of $W(A)$ (cf. [7, Corollary 2]). Thus $\overline{W(A)} = \overline{W(A_m)}$. Together with $W(A_m) \subseteq W(A)$ and the openness of $W(A)$, this yields $W(A) = W(A_m)$ for $m \geq 1$. \hfill $\square$

Here the openness of $W(A)$ is essential. For example, if $A$ is the unilateral (resp., bilateral) weighted shift with weights $w, 1, 1, \ldots$ (resp., $w, 1, 1, \ldots$, where $w > \sqrt{2}$ (resp., $w > 1$), then $W(A)$ is a closed circular disc centered at the origin, which is bigger that $W(A_1) = D$ (cf. the remarks after Proposition 2.2). Examples of $A$ with $W(A)$ closed and $W(A) = W(A_m)$ for all $m \geq 1$ will be given in Section 3 as applications of the parametric representation in Theorem 3.1.

We now compare the numerical ranges of two weighted shifts with the moduli of the weights of one less than or equal to those of the other. In this case, their numerical ranges have the same containment relation.

**Proposition 2.5.** Let $A$ and $B$ be unilateral (resp., bilateral) weighted shifts with weights $w_n$ and $u_n$, $n \geq 0$ (resp., $-\infty < n < \infty$), respectively. If $|w_n| \leq |u_n|$ for all $n$, then the following hold:

(a) $W(A) \subseteq W(B)$ and $W_e(A) \subseteq W_e(B)$.
(b) Assume further that the $w_n$’s are all nonzero. Then $\overline{W(A)} = \overline{W(B)}$ if and only if either $|w_n| = |u_n|$ for all $n$ (when $W(A)$ is closed) or $W(A) = \text{Int } W(B)$ (when $W(A)$ is open). In the latter case, we have $\overline{W(A)} = W_e(A) = W_e(B) = \overline{W(B)}$.

For the proof, we need the following lemma relating the numerical range of a general infinite matrix to those of its compressions. If $A = [a_{ij}]_{i,j=0}^\infty$ (resp., $[a_{ij}]_{i,j=-\infty}^\infty$) and $0 \leq m \leq n \leq \infty$ (resp., $-\infty \leq m \leq n \leq \infty$), we let $A[m,n]$ denote the matrix $[a_{ij}]_{i=m,j=m}^n$. For a subset $\Delta$ of $C$, $\Delta^\circ$ denotes its convex hull, that is, $\Delta^\circ$ is the smallest convex set which contains $\Delta$.

**Lemma 2.6.** If $A = [a_{ij}]_{i,j=0}^\infty$ (resp., $[a_{ij}]_{i,j=-\infty}^\infty$) is an operator on $\ell^2$ (resp., $\ell^2(\mathbb{Z})$), then

(a) $\overline{W(A)} = \bigcap_{n=0}^{\infty} W(A[0,n])$ (resp., $\bigcap_{n=0}^{\infty} W(A[-n,n])$), and
(b) $W_e(A) = \bigcap_{n=0}^{\infty} W_e(A[n,\infty])$ (resp., $\bigcup_{n=0}^{\infty} W_e(A[-\infty,n]) \cup \bigcap_{n=0}^{\infty} W_e(A[n,\infty])$).

**Proof.** We leave the easy proof of (a) and prove only (b). Assume first that $A = [a_{ij}]_{i,j=0}^\infty$. For each $n \geq 0$, let $z_n$ be a point in $W(A(n,\infty])$ and let $A_n = z_n I_n \oplus A[\infty,\infty) \cup \{0\}$. Since $A_n$ is a finite-rank perturbation of $A$, we have

$$W_e(A) \subseteq W(A_n) = \left(\{z_n\} \cup W(A[n,\infty])\right)^\circ = W(A[n,\infty])$$

for $n \geq 0$. Hence $W_e(A) \subseteq \bigcap_{n=0}^{\infty} W(A[n,\infty])$. For the converse containment, let $z$ be any point in $\bigcap_{n=0}^{\infty} W(A[n,\infty])$. Then, for each $n \geq 0$, there is a unit vector $x_n$ in $\ell^2$ such that $\|A[\infty,\infty)|x_n\| < |z| < 1/(n+1)$. Letting $y_n = 0_n \oplus x_n$, where $0_n$ denotes the vector with $n$ zero components, we have $\|A[n,\infty]y_n\| \leq 1/(n+1)$. Since the $x_n$’s are unit vectors which converge to 0 in the weak topology, we infer that $z \in W_e(A)$ (cf. [4, corollary of Theorem 5.1]). This proves that $W_e(A) = \bigcap_{n=0}^{\infty} W(A[n,\infty])$.

Next assume that $A = [a_{ij}]_{i,j=-\infty}^\infty$. Let $z_0$ be any point in $W_e(A(-\infty,-1])$ and let $B = A(-\infty,-1] \oplus \{0\} \oplus A[1,\infty)$. Since $A - B$ is of finite rank, we have

$$W_e(A) = W_e(B) = \left(W_e(A(-\infty,-1]) \cup W_e(A[1,\infty])\right)^\circ = \left(\bigcap_{n=0}^{\infty} W(A(-\infty,n])\right) \cup \left(\bigcap_{n=1}^{\infty} W(A[n,\infty])\right)^\circ,$$

where the last equality is an easy consequence of what was just proven for $W_e([a_{ij}]_{i,j=0}^\infty)$. Our assertion for $W_e(A)$ follows immediately. \hfill $\square$

**Proof of Proposition 2.5.** We only prove the unilateral case and assume that $w_n, u_n \geq 0$ for all $n$. Let $z = (Ax,x)$ be any point in $W(A)$, where $x = (x_n)$ is a unit vector in $\ell^2$. Then

$$|z| = |\sum_{n=0}^{\infty} w_n x_n x_{n+1}| \leq \sum_{n=0}^{\infty} w_n |x_n||x_{n+1}| \leq \sum_{n=0}^{\infty} u_n |x_n||x_{n+1}| = |B|x,|x||,$$

where $|x| = (|x_n|)$ is also a unit vector. Since $W(B)$ is a circular disc centered at the origin, this implies that $|z|$, and hence $z$, is in $W(B)$. Therefore, $W(A) \subseteq W(B)$.

For the essential numerical range, let $A_n$ and $B_n$ be the unilateral weighted shifts with weights $w_n, w_{n+1}, \ldots$ and $u_n, u_{n+1}, \ldots$, respectively, for each $n \geq 0$. Then
\[ W_e(A) = \bigcap_n W(A_n) \subseteq \bigcap_n W(B_n) = W_e(B) \]

by Lemma 2.6(b).

(b) Assume that \( \overline{W(A)} = \overline{W(B)} \), \( W(A) \) is closed, and \( w_{n_0} < u_{n_0} \) for some \( n_0 \geq 0 \). Proposition 2.1(b) says that \( w(A) = \langle Ax, x \rangle \) for some unit vector \( x = (x_n) \) with \( x_n > 0 \) for all \( n \). Then

\[ w(A) = \sum_{n=0}^{\infty} w_n x_n x_{n+1} < \sum_{n=0}^{\infty} u_n x_n x_{n+1} = \langle Bx, x \rangle \subseteq W(B), \]

which is a contradiction. Hence in this case we must have \( w_n = u_n \) for all \( n \). On the other hand, if \( \overline{W(A)} = \overline{W(B)} \) and \( W(A) \) is open, then \( W(A) = \text{Int} W(B) \) and \( W(A) = W_e(A) \) by [7, Corollary 2]. Hence

\[ \overline{W(A)} = W_e(A) \subseteq \overline{W(B)} = W(A) \]

and thus the equalities hold throughout.

The converse implication is trivial. \( \square \)

Note that in Proposition 2.5(b) the assumption of nonzero \( w_n \)'s is essential. For example, the unilateral weighted shifts \( A \) and \( B \) with weights \( 2, 0, 1, 1, \ldots \) and \( 2, 0, \sqrt{2}, 1, 1, \ldots \), respectively, are such that \( W(A) = W(B) = \mathbb{D} \) (cf. [14, Proposition 2]).

3. Parametric representations

We now consider an additional tool for the study of numerical ranges of the weighted shifts. The following theorem gives a refinement of the parametric representation for numerical contractions among weighted shifts. It is due to Eckstein and Rácz [3, Theorem 2.5]. Since the proof in [3] depends on a result from an unpublished article, we present here a more detailed operator-theoretic proof for completeness.

**Theorem 3.1.**

(a) Let \( A \) be a unilateral (resp., bilateral) weighted shift with weights \( w_n, n \geq 0 \) (resp., \( -\infty < n < \infty \)). Then \( w(A) \leq 1 \) if and only if there is a sequence \( \{a_n\}_{n=0}^{\infty} \) (resp., \( \{a_n\}_{n=-\infty}^{\infty} \)) in \( [-1, 1] \) such that \( |w_n|^2 = (1 - a_n)(1 + a_{n+1}) \) for all \( n \).

Moreover, in this case, if \( w_n \neq 0 \) for all \( n \), then the set \( \alpha_A = \{a_0; |w_n|^2 = (1 - a_n)(1 + a_{n+1}) \} \) for some \( \{a_n\}_{n=0}^{\infty} \) (resp., \( \{a_n\}_{n=-\infty}^{\infty} \)) in \( [-1, 1] \) equals \([-1, a]\) for some \( a \) in \([-1, 1]\) (resp., \([a, b]\) for some \( a \leq b \) in \((-1, 1)\)), and for each \( a_0 \in \alpha_A \) such \( a_0 \)'s are uniquely determined.

(b) Let \( A \) be the \( n \)-by-\( n \) weighted shift

\[
\begin{pmatrix}
0 & \cdots & w_{n-1} & w_n \\
-1 & \cdots & 0 & \vdots \\
1 & \cdots & -1 & 0 \\
\vdots & \ddots & \vdots & \ddots \\
\end{pmatrix}
\]

with \( w_j \neq 0 \) for all \( j \). Then \( w(A) < 1 \) (resp., \( w(A) = 1 \)) if and only if \( |w_j|^2 = (1 - a_j)(1 + a_{j+1}) \) for some \( a_j \) with \( a_1 = -1 \) and \(-1 < a_j < 1 \) for \( 1 < j \leq n \) (resp., \( a_1 = -1, -1 < a_j < 1 \) for \( 1 < j < n \) and \( a_n = 1 \)).

**Proof.** (a) Assume first that \( A \) is a unilateral weighted shift with nonnegative weights \( w_n \). Since \( A \) is the direct sum of finite weighted shifts and a unilateral shift with nonzero weights (either may be absent), we may consider these latter two types separately.

Assume that \( w_n \neq 0 \) for all \( n \) and \( w(A) \leq 1 \). Since the latter is equivalent to \( \text{Re} A \leq I \), we may apply the Gram–Schmidt process to the columns of \( (I - \text{Re} A)^{1/2} \) to obtain its QR-decomposition: \((I - \text{Re} A)^{1/2} = Q R\), where \( Q \) is an isometry and \( R \) is an upper triangular. Note that since \( I - \text{Re} A \) is a real matrix, so are \((I - \text{Re} A)^{1/2}, Q \) and \( R \) (cf. [6, Problem 121]). Then \( I - \text{Re} A = R^*Q^*QR = R R^* \) is a Cholesky decomposition of \( I - \text{Re} A\):

\[
\begin{pmatrix}
1 & -w_{0}/2 & \cdots & -w_{n}/2 \\
-w_{0}/2 & 1 & \cdots & -w_{n}/2 \\
\cdots & \cdots & \cdots & \cdots \\
-w_{n}/2 & \cdots & 1 & -w_{0}/2 \\
\end{pmatrix}
= \begin{pmatrix}
0 & \cdots & 0 & \cdots & 0 \\
r_{00} & \cdots & 0 & \cdots & 0 \\
r_{01} & \cdots & r_{11} & \cdots & 0 \\
r_{02} & \cdots & r_{12} & \cdots & r_{22} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
r_{n0} & \cdots & r_{n1} & \cdots & r_{n2} \\
r_{n1} & \cdots & r_{n2} & \cdots & \cdots \\
r_{n2} & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix}
\]

In particular, we have \( r_{00}^2 = 1 \). If, for some \( n_0 \geq 1 \), \( r_{n0}/n_0 = 0 \) and \( r_{n0} \neq 0 \) for \( 0 \leq n < n_0 \), then, carrying out the matrix multiplication in (1), we deduce that \( r_{nm} = 0 \) for \( 0 \leq m < n_0 \) and \( n \geq m + 2 \), and \(-w_{n0}/2 = \sum_{m=0}^{n_0} r_{nm}/(n_0 + 1) = \cdots \).
\[ r_{n_0, n_0 + 1} r_{n_0} = 0, \text{ which contradicts our assumption. Hence we must have } r_{m,n} \neq 0 \text{ for all } n \text{ and thus } r_{m,n} = 0 \text{ for } 0 \leq m \leq n - 2, \]
\[ n_n^{n_1 + 1} r_n = 1 \text{ and } r_{n+1,n} = -w_n/2 \text{ for } n \geq 0. \]

If \( a_0 = 1 - 2r^2 \) for \( n \geq 0 \), then \( a_0 = -1, -1 < a_n < 1 \) for \( n \geq 1 \) and
\[ w_n^2 = 4r^{2}_{n,n+1} + 1 = 4(1 - r^{2}_{n+1,n+1}) = (a_n - 1) + (a_{n+1}), \quad n \geq 0. \]

Conversely, if the \( a_n \)'s in \([-1, 1]\) satisfy \( w_n^2 = (1 - a_0)(1 + a_{n+1}) \) for all \( n \), then, defining \( b_n \)'s inductively by \( b_0 = -1 \) and \( (1 - b_n)(1 + b_{n+1}) = w_n^2 \) for \( n \geq 0 \), we have \( -1 < b_n < a_n < 1 \) for all \( n \). If
\[ r_{m,n} = \begin{cases} \sqrt{(1-b_n)/2} & \text{for } m = n \geq 0, \\ -\sqrt{(1+b_n)/2} & \text{for } m = n - 1 \geq 0, \\ 0 & \text{for } 0 \leq m < n - 1, \end{cases} \]
then \( r_{00} = 1, r_{n,n+1} = r_{n+1,n+1} = 1 \) and \( r_{n+1,n+1} w_n = -w_{n+1}/2 \) for \( n \geq 0 \). Hence (1) holds. This shows that \( \text{Re} A \leq 1 \) and thus \( w(A) \leq 1 \).

Now consider the set \( \alpha_A \) under the assumption of \( w_n > 0 \) for all \( n \). We have \( -1 \in \alpha_A \) from above. If the sequence \( \{a^{(m)}_n\}_{m=1}^{\infty} \) is in \( \alpha_A \) with \( \lim_m a^{(m)}_0 = a_0 \) and with the corresponding \( \{a^{(m)}_n\}_{m=0}^{\infty} \) satisfying \( w_n^2 = (1 - a^{(m)}_n)(1 + a^{(m)}_{n+1}) \) for all \( m \) and \( n \), then we easily infer that \( a_0 = \lim_m a^{(m)}_0 \) exists for each \( n \geq 1 \) and \( a_0 \) is in \( \alpha_A \) with the corresponding sequence \( \{a^{(m)}_n\}_{m=0}^{\infty} \) in \([-1, 1]\). This shows that \( \alpha_A \) is a closed subset of \([-1, 1]\). Moreover, if \( a_0 < b_0 \) are both in \( \alpha_A \) with the corresponding \( \{a^{(m)}_n\}_{m=0}^{\infty} \) and \( \{b^{(m)}_n\}_{m=0}^{\infty} \) and if \( a_0 < c_0 < b_0 \), then the sequence \( \{c^{(m)}_n\}_{m=0}^{\infty} \) defined inductively by \( w_n^2 = (1 - c_n)(1 + c_{n+1}) \) obviously satisfies \(-1 \leq a_n < c_n < b_n < 1 \) for all \( n \). We conclude that \( \alpha_A \) is a closed subinterval of \([-1, 1]\) and hence of the form \([-a, 1]\) for some \( a \) in \([-1, 1]\).

We now move to the proof of (b) and will be back to (a) again for the bilateral case.

(b) Assume that \( w_j > 0 \) for all \( j \). If \( w(A) < 1 \), then \( I_n - \text{Re} A \) is positive definite. We may proceed as in (a) to obtain the Cholesky decomposition of \( I_n - \text{Re} A \):
\[ \left( \begin{array}{ccccc} 1 & -w_1/2 & & & \\
-w_1/2 & 1 & & & \\
& & \ddots & & \\
& & & \ddots & \\
& & & & 1 \end{array} \right) = \left( \begin{array}{ccc} R_{11} & & \\
R_{12} & R_{22} & \\
& & \ddots & \\
& & & \ddots & \\
& & & & R_{nn} \end{array} \right)^2 \]

with \( r_{ij} \neq 0 \) for all \( j \). Letting \( a_j = 1 - 2r_{ij} \) for \( 1 \leq j \leq n \), we have \( a_1 = -1, -1 < a_j < 1 \) for \( 1 < j \leq n \) and \( w_j^2 = (1 - a_j)(1 + a_{j+1}) \) for all \( j \) as before. On the other hand, if \( w(A) = 1 \), then \( I_n - \text{Re} A \) is noninvertible and hence, in (2), some \( r_{ij} \) must be equal to 0. If \( r_{j_0,j_0} = 0 \) and \( r_{ij} \neq 0 \) for \( 1 \leq j < j_0 \), then, as in (a), we would have \(-w_{j_0}^2 = \sum_{k=j_0}^{j_0-1} r_{k,j_0} + r_{j_0,j_0} = r_{j_0,j_0} + r_{j_0,j_0} = 0 \), a contradiction. Thus \( r_{ij} \neq 0 \) for all \( j, 1 \leq j < n \), and hence \( r_{j_0,j_0} = 0 \). This gives \( a_1 = -1, -1 < a_j < 1 \) for \( 1 < j < n \), and \( a_n = 1 \).

Conversely, if the \( a_j \)'s satisfying \( w_j^2 = (1 - a_j)(1 + a_{j+1}) \) for all \( j \) exist, then letting
\[ r_{ij} = \begin{cases} \sqrt{(1-a_j)/2} & \text{for } i = j, \\ -\sqrt{(1+a_j)/2} & \text{for } i = j - 1, \\ 0 & \text{for } i < j - 1, \end{cases} \]
we obtain the equality in (2). Hence \( I_n - \text{Re} A \) is either positive definite or positive semidefinite with nonzero kernel depending on whether \( r_{mn} > 0 \) or \( = 0 \), and thus \( w(A) < 1 \) or \( = 1 \) accordingly.

Finally, we come back to (a).

(a) Consider a bilateral weighted shift \( A \) with weights \( w_n > 0 \). For each integer \( m \), let \( A_m \) be the unilateral weighted shift with weights \( w_m, w_{m+1}, \ldots \). If \( w(A) \leq 1 \), then \( w(A_m) \leq 1 \) for all \( m \). By what was proven before, for each \( m \) there is a sequence \( \{a_{n}^{(m)}\}_{n=0}^{\infty} \) in \([-1, 1]\) such that \( w_{n_{m+1}}^2 = (1 - a^{(m)}_0)(1 + a^{(m)}_{n+1}) \) for all \( m \) and \( n \), and \( a_0^{(m)} \) is such that
\[ [-1, a_0^{(m)}] \text{ for all integers } m \text{ and all } n \geq 0, \text{ we have } a_0^{(m)} > a_{m-1}^{(m-1)} \text{ and } a_{m-1}^{(m-1)} > a^{(m-2)}_{m-1} \text{. From } \\
(1 - a_0^{(m-1)})(1 + a_{m-1}^{(m-1)}) = w_{m-1}^2 = (1 - a_{m-1}^{(m-2)})(1 + a_{m}^{(m-2)}), \]
we deduce that \( a_{m-1}^{(m-2)} > a^{(m-2)}_{m-1} \). Similarly, we obtain, by induction, that the sequence \( \{a_{j}^{(m-j)}\}_{j=0}^{\infty} \) is decreasing. If \( a_m = \lim_m a_j^{(m-j)} \) for each \( m \), then \( a_0 \) is in \([-1, 1]\) and satisfies \( w_{n_{m+1}}^2 = (1 - a_0)(1 + a_{n+1}) \) for all \( m \). Conversely, the existence of such \( a_0 \)'s would imply, by the proven unilateral case, that \( w(A_m) \leq 1 \) for all \( m \). Hence \( w(A) = \lim_{m \to -\infty} w(A_m) \leq 1 \) by Lemma 2.6(a). Finally, if \( w_0 \neq 0 \) for all \( n \), then \( \alpha_A = [a, b] \) for some \( a \leq b \) in \((-1, 1)\) can be proven as in the unilateral case. \( \Box \)
The next lemma will be needed in Section 4. It implies, in particular, that if $A$ is the simple unilateral (resp., bilateral) shift, then $\alpha_A = [-1, 0]$ (resp., $\alpha_A = [0])$.

**Lemma 3.2.** Let $A$ be a unilateral (resp., bilateral) shift with unit vectors $w, w, \ldots$ (resp., $w, w, w, \ldots$), where $0 < |w| \leq 1$, then $\alpha_A = [-1, 1, \sqrt{1-|w|^2}]$ (resp., $\alpha_A = [\sqrt{1-|w|^2}]$).

**Proof.** We first consider the unilateral case. Let $(a_n)_{n=0}^\infty$ be such that $-1 \leq a_n < 1$ and $|w|^2 = (1 - a_0)(1 + a_{n+1})$ for all $n$. If $a_0^2 > 1 - |w|^2$, then we derive from $1 + a_0 = |w|^2/(1 - a_0)$ that $1 + a_0 > 1 + a_0$ and hence $a_0^2 > 1 - |w|^2$. Repeating this argument, we obtain the increasingness of the $a_n$'s. If $a = \lim_n a_n$, then

$$|w|^2 = \lim_n (1 - a_n)(1 + a_{n+1}) = (1 - a)(1 + a) = 1 - a^2,$$

which yields that $a^2 = 1 - |w|^2 < a_0^2$, a contradiction. Hence we must have $a_0^2 \leq 1 - |w|^2$ or $\alpha_A \subseteq [-1, 1, \sqrt{1-|w|^2}]$. On the other hand, letting $a_n = \sqrt{1 - |w|^2}$ for all $n$ gives $|w|^2 = (1 - a_0)(1 + a_{n+1})$, which shows that $\sqrt{1 - |w|^2}$ is in $\alpha_A$. Hence $\alpha_A = [-1, 1, \sqrt{1-|w|^2}]$ as asserted.

Similarly, for the bilateral case, if $(a_n)_{n=-\infty}^\infty$ is such that $-1 < a_n < 1$ and $|w|^2 = (1 - a_0)(1 + a_{n+1})$ for all $n$, and if $a_0 < \sqrt{1 - |w|^2}$, then, as above, the sequence $(a_n)_{n=-\infty}^\infty$ is decreasing and we would obtain the contradictory $\lim_{n \to -\infty} a_n = 1 - |w|^2$. Therefore $\alpha_A = [\sqrt{1-|w|^2}]$ as required. \hfill $\square$

As was noted in Section 1, if $A$ is a weighted shift, then $W(A)$ is either open or closed. In terms of the parameters in Theorem 3.1, we can now give a complete characterization of those A's with $W(A)$ open (or closed).

**Theorem 3.3.** Let $A$ be a unilateral (resp., bilateral) weighted shift with nonzero weights $w_n, n \geq 0$ (resp., $-\infty < n < \infty$). Assume that $w_0 = 1$ and let $a_n, n \geq 0$ (resp., $-\infty < n < \infty$), be such that $-1 \leq a_n < 1$ (resp., $-1 < a_n < 1$) and $|w_0|^2 = (1 - a_0)(1 + a_{n+1})$ for all $n$. Then $W(A)$ is closed if and only if $a_0 = -1$ and $\sum_{n=0}^\infty \prod_{k=0}^{n} (1 - a_k)(1 + a_{n+1}) < \infty$ (resp., $\sum_{n=0}^\infty (\prod_{k=0}^{n} (1 - a_k)/\prod_{k=0}^{n} (1 + a_{n+1}))/1 - a_{n+1}) < \infty$).

**Proof.** We only prove the unilateral case and assume that $w_n > 0$ for all $n$. Then $-1 \leq a_n < 1$ and $-1 < a_n < 1$ for all $n \geq 1$.

Assume first that $W(A)$ is closed, that is, $W(A) = \overline{B}$, and $a_0 > -1$. Let

$$u_n = \begin{cases} \sqrt{(1 - a_n)(1 + a_{n+1})} & \text{for } n \geq 0, \\ \sqrt{1 - a_0} & \text{for } n = -1, \\ 1 & \text{for } n \leq -2, \end{cases}$$

and let $B$ be the bilateral weighted shift with weights $u_n > 0$, $-\infty < n < \infty$. Then, obviously, $W(A) \subseteq W(B)$, and $W(B) \subseteq \overline{B}$ by Theorem 3.1(a). Hence we have $W(B) = \overline{B}$. By Proposition 2.1(b), there is a unit vector $x = (x_n)$ in $l^2(\mathbb{Z})$ with $x_n > 0$ for all $n$ such that $(Bx, x) = w(B) = 1$. In particular, we have $(Bx, x) = x$. A simple calculation yields that $x_{n+1} - x_n = x_{n+1} - x_n = d$ for all $n \leq -2$. Hence $x_n = x_{n+1} + (n + 1)d$ for $n \leq -2$. Since $\prod_{n=0}^\infty x_n = 1$, this implies that $d = 0$ and thus $x_{n+1} = x_{n+1} = \cdots = 0$, which contradicts our assumption. Therefore, we must have $a_0 = -1$. Moreover, by Theorem 3.1(a), there is a unit vector $y = (y_n)$ in $l^2$ with $y_n > 0$ for all $n$ such that $(Ay, y) = w(A) = 1$. This yields that

$$0 = 1 - \sum_{n=0}^\infty w_n y_n y_{n+1}$$

$$= \sum_{n=0}^\infty y_n^2 - \sum_{n=0}^\infty w_n y_n y_{n+1}$$

$$= \sum_{n=0}^\infty \left[ \frac{1}{2} y_n^2 + \frac{1}{2} (1 + a_{n+1}) y_{n+1}^2 \right] - \sum_{n=0}^\infty \sqrt{1 - a_n} \sqrt{1 + a_{n+1}} y_n y_{n+1}$$

$$= \sum_{n=0}^\infty \left( \sqrt{1 - a_n} y_n - \frac{1}{2} (1 + a_{n+1}) y_{n+1} \right)^2. \quad (3)$$

Hence $y_n = \frac{1}{\sqrt{1 - a_n}}(1 + a_{n+1})^\infty$ for all $n \geq 0$ and thus $y_{n+1} = y_0 \prod_{k=0}^{n-1} \sqrt{(1 - a_k)/(1 + a_{k+1})}$, $n \geq 0$. Thus $\sum_{n=0}^\infty \prod_{k=0}^{n-1} (1 - a_k)/(1 + a_{k+1}) < \infty$ as asserted.

Conversely, if $a_0 = -1$ and $\alpha = \sum_{n=0}^\infty \prod_{k=0}^{n-1} (1 - a_k)/(1 + a_{k+1}) < \infty$, then, letting $y_0 = 1/\alpha$ and $y_n = \sqrt{(1 - a_n)/(1 + a_{n+1})}$ for $n \geq 0$, we obtain that $\sum_{n=0}^\infty y_n^2 = 1$ and $\sum_{n=0}^\infty w_n y_n y_{n+1} = 1$ by (3). Hence $y = (y_n)$ is a unit vector in $l^2$ with $(Ay, y) = 1$. This shows that $W(A) = \overline{B}$ is closed. \hfill $\square$
We now use this theorem to give an example of a unilateral weighted shift $A$ with positive weights $w_n$, $n \geq 0$, such that $W(A) = W_e(A) = W(A_m)$ for all $m \geq 1$, where $A_m$ is the weighted shift with weights $w_m, w_{m+1}, \ldots$ (cf. [13, Note (4), p. 502]). Indeed, let $a_0 = -1$, $a_n = 1/(n+1)$ for $n \geq 1$, $w_n = \sqrt{1-a_n}(1+a_{n+1})$ for $n \geq 0$, and $A$ be the weighted shift with weights $w_n$, $n \geq 0$. Then $w_0 = \sqrt{2}$, $0 < w_n = \sqrt{n/(n+1)(n+2)}$ for $n \geq 1$, $\lim_h w_n = 1$, and $w(A) = w(A_m) = 1$ for all $m \geq 1$ (by Proposition 2.2(b) and Theorem 3.1(a)). Moreover, since

$$
\sum_{n=m}^{\infty} \prod_{k=m}^{n} \frac{1-a_k}{1+a_k+1} \leq \sum_{n=m}^{\infty} \prod_{k=m}^{n} \frac{1}{(1+a_k+1)^2} = (m+2)^2 \sum_{n=m}^{\infty} \frac{1}{(n+3)^2} < \infty
$$

for any $m \geq 0$, Theorem 3.3 says that $W(A)$ and $W(A_m)$ are closed for all $m \geq 1$. Hence $W(A) = W(A_m) = \mathbb{D} = W_e(A)$ by Proposition 2.2(a).

Similarly, it can be shown that if

$$a_n = \begin{cases} 1/(n+1) & \text{for } n \geq 1, \\ 0 & \text{for } n = 0, \\ 1/(n-1) & \text{for } n \leq -1, \end{cases}
$$

and $w_n = \sqrt{(1-a_n)(1+a_{n+1})}$ for all $n$, then the bilateral weighted shift $A$ with weights $w_n$ is such that $W(A) = W_e(A) = W(A_m) = \mathbb{D}$ for all integers $m$, where $A_m$ is the unilateral weighted shift with weights $w_m, w_{m+1}, \ldots$.

4. Periodic weights

If $A$ is a (unilateral or bilateral) weighted shift with positive periodic weights, then it was shown in [13, Proposition 6] that $W(A)$ is open (cf. also [14, Theorem 4]). In the following, we prove this via the openness criterion in Section 3. We start by showing that if the weights $w_n$ are periodic, then the parameters $a_n$ can be chosen to be periodic too.

**Lemma 4.1.** Let $A$ be a unilateral (resp., bilateral) weighted shift with weights $w_n$, $n \geq 0$ (resp., $-\infty < n < \infty$). Assume that $w(A) \leq 1$. Then the $\{w_n\}$’s are periodic with period $p \geq 1$ if and only if there is a periodic sequence $\{a_n\}_{n=0}^\infty$ (resp., $\{a_n\}_{n=-\infty}^\infty$) with period $p$ in $[-1, 1]$ such that $|w_n|^2 = (1-a_n)(1+a_{n+1})$ for all $n$. In this case, if the $w_n$’s are all nonzero, then the $a_n$’s are in $(-1, 1)$.

**Proof.** We only prove the unilateral case and assume that $w_{n+p} = w_n \geq 0$ for all $n \geq 0$. Let $\{b_n\}_{n=0}^\infty$ be a sequence in $[-1, 1]$ satisfying $w_n^2 = (1-b_n)(1+b_{n+1})$ for all $n$. If $b_0 \leq b_p$, then we deduce from

$$(1-b_p)(1+b_{p+1}) = w_p^2 = w_0^2 = (1-b_0)(1+b_1)$$

that $b_1 \leq b_{p+1}$. Inductively, we obtain $b_n \leq b_{n+p}$ for all $n$ or, in other words, $b_l \leq b_{l+p} \leq \cdots \leq b_{l+mp} \leq \cdots \leq 1$ for all $l, 0 \leq l < p$, and all $n \geq 0$. If $c_l = \lim_{n \to \infty} b_{l+np}$ for $0 \leq l < p$ and $c_p = c_0$, then

$$w_p^2 = \lim_{n \to \infty} w_n^2 = \lim_{n \to \infty} \lim_{m \to \infty} (1-b_{1+mp})(1+b_{1+mp+1}) = (1-c_1)(1+c_{l+1}).$$

We define the sequence $\{a_n\}_{n=0}^\infty$ by $a_{l+mp} = c_l$ for $0 \leq l < p$ and $m \geq 0$. It is easily seen that $\{a_n\}_{n=0}^\infty$ is periodic with period $p$ and satisfies $-1 \leq a_n \leq 1$ and $w_n^2 = (1-a_n)(1+a_{n+1})$ for all $n$. Similarly, if $b_0 > b_p$, then the sequence $\{b_{l+mp}\}_{m=0}^\infty$ is decreasing and we can also obtain the required periodic $\{a_n\}_{n=0}^\infty$ as above.

That the periodicity of the $a_n$’s implies that of the $w_n$’s is trivial. So is our assertion for nonzero $w_n$’s. □

The next proposition was proven in [13, Proposition 6] and [14, Theorem 4].

**Proposition 4.2.** If $A$ is a unilateral (resp., bilateral) weighted shift with nonzero periodic weights, then $W(A)$ is open.

**Proof.** We only prove the unilateral case. Assume that $A$ has weights $w_n$, $n \geq 0$, with $w_{n+p} = w_n > 0$ ($p \geq 1$) for all $n$, and $w(A) = 1$. By Lemma 4.1, there are $a_n$’s in $(-1, 1)$ such that $a_{n+p} = a_n$ and $w_n^2 = (1-a_n)(1+a_{n+1})$ for all $n$. Then

$$\sum_{n=0}^{\infty} \prod_{k=0}^{n} \frac{1-a_k}{1+a_k+1} \geq \sum_{m=1}^{mp-1} \prod_{k=0}^{m} \frac{1-a_k}{1+a_k+1} = \sum_{m=1}^{m} \prod_{l=0}^{p-1} \frac{1-a_l}{1+a_l+1} = \infty.$$
Theorem 4.3.

(a) Let $A$ be a unilateral weighted shift with nonzero weights $w_n$, $n \geq 0$, such that, for some $a_0, \ldots, a_{p-1} > 0$ and some integer $n_0 \geq 0$, we have $|w_{n_0+kp+j}| \geq a_j$ for all $k \geq 0$ and all $j \neq 0$, $0 \leq j < p$, and $\lim_{k \to \infty} |w_{n_0+kp+j}| = a_j$ for all $j$. Then $W(A)$ is open if and only if $w(A) = w(C(a_0, \ldots, a_{p-1}))$.

(b) Let $B$ be a bilateral weighted shift with nonzero weights $w_n$, $-\infty < n < \infty$, such that for some $a_0, \ldots, a_{p-1}, b_0, \ldots, b_{q-1} > 0$ and some integer $n_0 \geq 0$, we have $|w_{n_0+kp+j}| \geq a_j$ for all $k \geq 0$ and all $j \neq 0$, $0 \leq j < p$, and $\lim_{k \to \infty} |w_{n_0+kp+j}| = a_j$ for all $j$, $|w_{qk-j-1}| \geq b_j$ for $k \leq 0$ and all $j$, $0 \leq j < q$, and $\lim_{k \to \infty} |w_{qk-j-1}| = b_j$ for all $j$. Then $W(B)$ is open if and only if $w(A) = \max\{w(C(a_0, \ldots, a_{p-1})), w(C(b_0, \ldots, b_{q-1}))\}$.

For the proof, we need the following lemmas.

Lemma 4.4. Let $A$ and $B$ be unilateral (resp., bilateral) weighted shifts with nonzero periodic weights $w_n$ and $u_n$, $n \geq 0$, respectively, of period $p$. Then

(a) $w(A) = w(C(w_0, \ldots, w_{p-1}))$,
(b) $w(A) \leq w(B)$ if $|w_n| \leq |u_n|$ for all $n$ and
(c) $w(A) < w(B)$ if $|w_n| \leq |u_n|$ for all $n$ and $|w_{n_0}| < |u_{n_0}|$ for some $n_0$.

Proof. (a) was proven in [10, Theorem 1]. (b) and (c) follow from (a) and [9, Corollary 3.6]. (Note that (b) also follows from Proposition 2.5(a)).

Lemma 4.5. Let $A$ be a unilateral (resp., bilateral) weighted shift with nonzero periodic weights $w_n$, $n \geq 0$ (resp., $-\infty < n < \infty$), of period $p$.

(a) Assume that $w(A) \leq 1$ and $(a_n)_{n=0}^\infty$ (resp., $(a_n)_{n=-\infty}^\infty$) is a periodic sequence of period $p$ with $-1 < a_n < 1$ and $|w_n|^2 = (1-a_n)/(1+a_{n+1})$ for all $n$. Then $w(A) = 1$ if and only if $\prod_{n=0}^{p-1}(1-a_n)/(1+a_{n+1}) = 1$.

(b) If $w(A) = 1$, $P = \{(\alpha_n)_{n=0}^{p-1} : -1 < \alpha_n < 1$ and $|w_n|^2 = (1-\alpha_n)(1+\alpha_{n+1})$ for $0 \leq n < p$, and $\alpha_p \equiv \alpha_0$, and $M_j = \sup|\alpha_j| : (\alpha_n)_{n=0}^{p-1} \in P\}$ for $0 \leq j < p$, then $\{M_n\}_{n=0}^{p-1}$ is in $P$ and, for any sequence $(b_n)_{n=0}^\infty$ (resp., $(b_n)_{n=-\infty}^\infty$) satisfying $-1 < b_n < 1$ and $|w_n|^2 = (1-b_n)/(1+b_{n+1})$ for all $n$, we have $b_n \leq M_n/(\prod_{n=0}^{p-1}(1-a_n)/(1+a_{n+1}))$ for all $n$.

Proof. We assume that $A$ is a unilateral weighted shift with periodic weights $w_n > 0$.

(a) Suppose that $w(A) = 1$ and $\prod_{n=0}^{p-1}(1-a_n)/(1+a_{n+1}) < 1$. Let $\epsilon > 0$ be such that $\prod_{n=0}^{p-1}[(1-a_n)/(1+a_{n+1})] + \epsilon < 1$, $b_0, \ldots, b_{p-1}$ be such that $b_n > (1-a_n)/(1+a_{n+1}) + \epsilon$ for $0 \leq n < p$ and $\prod_{n=0}^{p-1}b_n = 1$, and $\epsilon_0$ be such that $0 < \epsilon_0 < \min(1+a_n, (1+a_{n+1})\epsilon)$ and $\epsilon_n \equiv \epsilon_0/(b_0 \cdots b_{n-1}) < \min(1+a_n, (1+a_{n+1})\epsilon)$ for $1 \leq n < p$. Then

$$\frac{\epsilon_n}{\epsilon_{n+1}} > b_n > \frac{1-a_n}{1+a_{n+1}} + \epsilon > \frac{1-a_n + \epsilon_n}{1+a_{n+1}}$$

or

$$\epsilon_n^2 > (1-a_n + \epsilon_n)(1+a_{n+1} - \epsilon_{n+1}) > (1-a_n)(1+a_{n+1}) = w_n^2.$$
for \(0 \leq n < p\) \((\epsilon_p \equiv \epsilon_0)\). Let \(B = A(u_0, \ldots, u_{p-1})\). Since \(u_n > w_n\) for all \(n\), \(0 \leq n < p\), we have \(w(A) < w(B)\) by Lemma 4.4(c). Note that \(w(B) \leq 1\) by Theorem 3.1(a) because its weights \(u_n\) are associated with the sequence \(\{a_0 - \epsilon_0, \ldots, a_{p-1} - \epsilon_1, a_0 - \epsilon_0, \ldots, a_{p-1} - \epsilon_1, \ldots\}\) in \((-1, 1)\). These two together yield \(w(A) < 1\) in contradiction to our assumption.

In a similar fashion, if \(\prod_{n=0}^{p-1} (1 - a_n)/(1 + a_{n+1}) > 1\), then we can choose \(\epsilon_p' > 0\), \(0 \leq n < p\), such that \(a_n + \epsilon_p'\) is in \((-1, 1)\) and \(\epsilon_p'/\epsilon_{n+1} < (1 - a_n - \epsilon_p')/(1 + a_{n+1})\) for all \(n\), which would lead to \(w(A) < 1\) as above, a contradiction again. We thus conclude that \(\prod_{n=0}^{p-1} (1 - a_n)/(1 + a_{n+1}) = 1\).

To prove the converse, suppose that \(w(A) < 1\). Since \(w(A/w(A)) = 1\), there is, by Lemma 4.1, a periodic sequence \(\{b_n\}_{n=0}^{\infty}\) of period \(p\) in \((-1, 1)\) such that \((w_n/w(A))^2 = (1 - b_n)(1 + b_{n+1})\) for all \(n\). Assume first that \(a_0 \leq b_0\). Then

\[1 + a_1 = \frac{w_0^2}{1 - a_0} < \frac{w_0^2}{w(A)^2} \frac{1}{1 - b_0} = 1 + b_1,\]

which implies that \(a_1 < b_1\). Inductively, we obtain \(a_n < b_n\) for all \(n\). Let \(\delta_n = b_n - a_n > 0\). Then

\[(1 - a_n - \delta_n)(1 + a_{n+1} + \delta_{n+1}) = (1 - b_n)(1 + b_{n+1}) = \frac{w_n^2}{w(A)^2},\]

and therefore

\[\prod_{n=0}^{p-1} \frac{1 - a_n}{1 + a_{n+1}} > \prod_{n=0}^{p-1} \frac{1 - a_n - \delta_n}{1 + a_{n+1} + \delta_{n+1}} > \prod_{n=0}^{p-1} \frac{\delta_n}{\delta_{n+1}} = 1\]

since \(\delta_p = b_p - a_p = b_p - a_0 = \delta_0\). Similarly, if \(a_0 > b_0\), then we obtain \(a_n > b_n\) for all \(n\) and hence \(\prod_{n=0}^{p-1} (1 - a_n)/(1 + a_{n+1}) < 1\) as above. These show that if \(\prod_{n=0}^{p-1} (1 - a_n)/(1 + a_{n+1}) = 1\), then \(w(A) = 1\).

(b) To prove that \(\{M_n\}_{n=0}^{p-1}\) is in \(P\), let \(\{a_n^{(m)}\}_{n=0}^{p-1}\), \(m \geq 1\), be in \(P\) such that \(M_0 = \lim_m a_0^{(m)}\). From \(w_n = (1 - a_n^{(m)})/(1 + a_{n+1}^{(m)})\) for \(0 \leq n < p\) and \(m \geq 1\), we infer that \(M_n = \lim_m a_n^{(m)}\) exists for all \(n\), \(1 \leq n < p\), \(N_p = M_0\) and \(\{M_0, M_1, \ldots, M_{p-1}\}\) is in \(P\). In particular, we have \(N_n \leq M_n\) for \(1 \leq n < p\). If \(N_1 < M_1\), then there exists some \(b_n^{(p-1)}\) in \(P\) such that \(N_1 < b_1^{(p-1)} \leq M_1\). We infer inductively from

\[w_n^2 = (1 - N_n)(1 + N_{n+1}) = (1 - b_n^{(p-1)})(1 + b_{n+1}),\]

\((\beta_p = \beta_0)\) that \(N_n < b_n^{(p-1)}\) for all \(n\), \(1 \leq n < p\), and thus, in particular, \(M_0 = N_p < \beta_p = \beta_0\), which is a contradiction. Therefore, we must have \(N_1 = M_1\). Similarly, we can prove inductively that \(N_n = M_n\) for all \(n\), \(2 \leq n < p\), and hence \(\{M_n\}_{n=0}^{p-1} = \{M_0, M_1, \ldots, M_{p-1}\}\) is in \(P\) as asserted.

Let \(\{b_n\}_{n=0}^{\infty}\) be a sequence in \((-1, 1)\) satisfying \(w_n^2 = (1 - b_n)(1 + b_{n+1})\) for all \(n\). We need check that \(b_{k+1} < M_j\) for \(k \geq 0\) and \(0 \leq j < p\). If \(b_0 > b_p\), then let \(u_n = \begin{cases} \sqrt{(1 - b_n)(1 + b_{n+1})} & \text{if } 0 \leq n < p - 1, \\ \sqrt{(1 - b_{p-1})(1 + b_{p+1})} & \text{if } n = p - 1, \end{cases}\)

and let \(B = A(u_0, \ldots, u_{p-1})\). Since \(w_n = u_n\) for all \(n\), \(0 \leq n < p - 1\), and \(w_{p-1} < u_{p-1}\), we infer from Lemma 4.4(c) that \(w(A) < w(B)\). On the other hand, we also have \(w(B) \leq 1\) by Theorem 3.1(a). These result in the contradictory \(w(A) < 1\). Hence we must have \(b_0 \leq b_p\). It follows from \(w_n^2 = (1 - b_n)(1 + b_{n+1})\) for all \(n\) and the periodicity of the \(w_n\)’s that \(\{b_{n+1}\} < \{b_{(k+1)p+j}\} + \{b_{k+1}\}\) for all \(k \geq 0\) and \(0 \leq j < p\). Let \(\alpha_j = \lim b_{kp+j}\) for each \(j\), \(0 \leq j < p\), and \(\alpha_0 = \alpha_0\). Then

\[w_j^2 = \lim_k w_k^2_{kp+j} = \lim_k (1 - b_{kp+j})(1 + b_{kp+j}) = (1 - \alpha_j)(1 + \alpha_j+1)\]

for \(0 \leq j < p\). This shows that \(\alpha_j^{p-1}\) is in \(P\) and therefore \(b_{kp+j} \leq \alpha_j \leq M_j\) for all \(k\) and \(j\) as asserted. This completes the proof.

We are now ready to prove Theorem 4.3.

**Proof of Theorem 4.3.** We prove only (a); (b) can be done analogously, which we omit. Assume that \(w_n > 0\) for all \(n\) and \(W(A)\) is open. Since \(A\) and the unilateral weighted shift \(A'\) with weights \(w_0, \ldots, w_{n-1}, a_0, a_1, \ldots, a_{p-1}, a_0, \ldots, a_{p-1}, \ldots\) differ by a compact operator, we have \(W_e(A) = W_e(A')\). Hence

\[w(A) = w_e(A) = w_e(A') = w_e(A(a_0, \ldots, a_{p-1})) \leq w(A(a_0, \ldots, a_{p-1})) = w(C(a_0, \ldots, a_{p-1})),\]
where the first equality is because \( W(A) = W_C(A) \) by the openness of \( W(A) \) (cf. [7, Corollary 2]), and the last equality is by Lemma 4.4(a). On the other hand, we also have \( W(A) \supset W(A') \) by Lemma 4.4(b) and \( W(A') = W(C(a_0, \ldots, a_{p-1})) = W(C(q_0, \ldots, a_{p-1})) \) by Lemma 4.4(a). Thus we conclude that \( W(A) = W(C(a_0, \ldots, a_{p-1})) \) as asserted.

To prove the converse, we may assume that \( W(A) = W(C(a_0, \ldots, a_{p-1})) = W(A(0 \ldots, a_{p-1})) = 1 \). Let \( \{b_n\}_{n=0}^\infty \) be a sequence with \( b_0 = -1, -1 < b_n < 1 \) for \( n \geq 1 \) and \( w_n^2 = (1 - b_n)(1 + b_{n+1}) \) for all \( n \) by Theorem 3.1(a). We need check that

\[
\sum_{n=0}^\infty \sum_{k=0}^n \frac{1 - b_k}{1 + b_{k+1}} = \infty
\]

(4)

holds. The openness of \( W(A) \) will then follow from Theorem 3.3. For this, let \( M_j = \sup \{\alpha_j : \text{there is some } \{\alpha_n\}_{n=0}^{p-1} \text{ in } (-1, 1) \text{ such that } \alpha_n = 1 - \alpha_{n+1} \text{ for } 0 \leq n < p \text{ and } \alpha_0 = \alpha_{p-1} \} \) for \( 0 \leq j < p \). In the following, we show that \( b_{n_0 + p \cdot j} \leq M_j \) for all \( l \geq 0 \) and all \( j \), \( 0 \leq j < p \). Indeed, if this is the case, then

\[
\prod_{j=0}^{p-1} \frac{1 - b_{n_0 + p \cdot j}}{1 + b_{n_0 + p \cdot j + 1}} = \prod_{j=0}^{p-1} \frac{1 - M_j}{1 + M_j} + 1
\]

for all \( l \geq 0 \) by Lemma 4.5(b) and (a), from which we obtain

\[
\sum_{n=0}^\infty \sum_{k=0}^n \frac{1 - b_k}{1 + b_{k+1}} \geq \prod_{l=0}^{p-1} \left( \sum_{k=0}^{n_0-1} \frac{1 - b_k}{1 + b_{k+1}} \right) \left( \prod_{j=0}^{p-1} \frac{1 - b_{n_0 + p \cdot j}}{1 + b_{n_0 + p \cdot j + 1}} \right)
\]

\[
\geq \sum_{l=0}^{n_0-1} \prod_{k=0}^{p-1} \frac{1 - b_k}{1 + b_{k+1}} = \infty,
\]

and thus (4) holds.

For convenience, we only show that \( b_{n_0} \leq M_0 \); that \( b_{n_0 + p \cdot j} \leq M_j \) for general values of \( l \) and \( j \) can be done analogously. Assume otherwise that \( b_{n_0} > M_0 \). Then, in particular, \( n_0 \geq 1 \). Let \( \{c_n\}_{n=0}^\infty \) be a sequence with \( c_0 = b_{n_0} \), \( c_k = (1 - a_m - k)(1 + c_{m+k} + k) \) for all \( m \geq 0 \) and all \( k, 0 \leq k < p \). Then \( -1 < c_0 < 1 \). We infer from

\[
(1 - b_{n_0})(1 + b_{n_0 + 1}) = w_0^2 \geq a_0^2 = (1 - c_0)(1 + c_1) = (1 - b_{n_0})(1 + c_1)
\]

that \( b_{n_0 + 1} \geq c_1 \) and thus \( -1 < c_1 < 1 \). In a similar fashion, we obtain inductively that \( -1 < c_k < 1 \) for all \( n \). Thus \( c_n \) is a sequence in \((-1, 1)\) associated with \( A(a_0, \ldots, a_{p-1}) \) as in Theorem 3.1(a). On the other hand, we also have \( a_k^2 = (1 - M_k)(1 + M_{k+1}) \) for \( 0 \leq k < p \). If \( b_{n_0} > M_0 \), then we obtain \( c_1 > M_1 \) which is in contradiction to the second assertion in Lemma 4.5(b). Hence we have \( b_{n_0} \leq M_0 \) as asserted. This completes the proof of (a).

Note that the preceding theorem generalizes both Propositions 2.3 and 4.2.

We conclude this paper by determining the numerical ranges of the unilateral (resp., bilateral) weighted shifts whose weights are such that all but one have equal moduli.

**Theorem 4.6.** Let \( A \) be the unilateral weighted shift with weights \( 1, \ldots, 1, c, 1, 1, \ldots \), where \( c \) appears in the \( m \)th position \( (m \geq 0) \). Then

(a) \( W(A) = \mathbb{D} \) if and only if \( |c| \leq \sqrt{(m+2)/(m+1)} \), and

(b) if \( |c| > \sqrt{(m+2)/(m+1)} \), then \( W(A) = \{z \in \mathbb{C} : |z| \leq r \} \), where \( r \) satisfies the \( m+1 \) equations \( r^{-2} = 2(1 + a_1), r^{-2} = (1 - a_n)(1 + a_{n+1}) \) for \( 1 \leq n < m \) and \( |c|^2 r^{-2} = (1 - a_m)(1 + \sqrt{1 - r^{-2}}) \) for some \( a_1, \ldots, a_m \in (-1, 1) \) (if \( m = 0 \), this is interpreted as \( r \) satisfies \( |c|^2 r^{-2} = 2(1 + \sqrt{1 - r^{-2}}) \)).

The following two corollaries give the cases of \( m = 0 \) and \( m = 1 \). The former was obtained before in [1, pp. 1053–1054] (cf. also [13, p. 500] and [14, Proposition 2]), whose proof we omit.

**Corollary 4.7.** Let \( A \) be the unilateral weighted shift with weights \( c, 1, 1, \ldots \). Then

(a) \( W(A) = \mathbb{D} \) if and only if \( |c| \leq \sqrt{2}, \) and

(b) if \( |c| > \sqrt{2}, \) then \( W(A) = \{z \in \mathbb{C} : |z| \leq |c|^2/(2\sqrt{|c|^2-1}) \} \).
Corollary 4.8. Let $A$ be the unilateral weighted shift with weights $1, c, 1, \ldots$. Then

(a) $W(A) = \mathbb{D}$ if and only if $|c| \leq \sqrt[3]{2}$. and
(b) if $|c| > \sqrt[3]{2}$ then $W(A) = \{z \in \mathbb{C} : |z| \leq (2|c|^2 + 2|c|^2 - 3 - 2|c|^4 - 2|c|^2 + 4)^{-1/2}\}$.

Proof. We need only prove (b). Assuming that $|c| > \sqrt[3]{2}$, let $s = 1/r$ and $t = \sqrt{1 - s^2}$. We have to solve the equations $s^2 = 2(1 + a_1)$ and $|c|^2 s^2 = (1 - a_1)(1 + \sqrt{1 - s^2})$ for $s$ and $a_1$. These can be written as $1 - t^2 = 2(1 + a_1)$ and $|c|^2(1 - t^2) = (1 - a_1)(1 + t)$. Their solutions are easily seen to be $t = |c|^2 + \sqrt{|c|^4 + 2|c|^2 - 3}$ and $s = (2|c|^2 \sqrt{|c|^4 + 2|c|^2 - 3} - 2|c|^4 - 2|c|^2 + 4)^{1/2}$. Our assertion then follows. □

We now prove Theorem 4.6.

Proof of Theorem 4.6(a). We may assume that $c > 0$. Letting $a_0 = m + 1$ and $p = 1$ in Theorem 4.3(a), we have that $W(A)$ is open if and only if $w(A) = 1$. Thus to complete the proof, we need show that $w(A) = 1$ if and only if $c \leq \sqrt{(m + 2)/(m + 1)}$. Indeed, if $w(A) = 1$, then there is a sequence $\{a_n\}_{n=0}^{\infty}$ with $a_0 = -1, -1 < a_n < 1$ for $n \geq 1$, $1 = (1 - a_n)(1 + a_{n+1})$ for $n \neq m$, and $c^2 = (1 - a_m)(1 + a_{m+1})$ by Theorem 3.1(a). We derive from

\[ 1 = (1 + a_1)(1 - a_1)(1 + a_2) = \cdots = (1 - a_m)(1 + a_{m+1}) \]

that $a_1 = -1/2, a_2 = -1/3, \ldots, a_n = -1/(m + 1)$ and from $c^2 = (1 - a_m)(1 + a_{m+1})$ that $a_{m+1} = [c^2(m + 1) - (m + 2)]/(m + 2)$. Since $1 = (1 - a_n)(1 + a_{n+1})$ for all $n \geq m + 1$, the sequence $\{a_n\}_{n=m+1}^{\infty}$ is associated with the simple unilateral shift as in Theorem 3.1(a). In particular, we have $a_{m+1} \leq 0$ by Lemma 3.2. It follows that $c \leq \sqrt{(m + 2)/(m + 1)}$. Conversely, if $c \leq \sqrt{(m + 2)/(m + 1)}$, then letting

\[ a_n = \begin{cases} -1/(n + 1) & \text{if } 0 \leq n \leq m, \\ (c^2(m + 1) - (m + 2))/(m + 2) & \text{if } n = m + 1, \\ (c^2(n + 1) - (n + 2))/(n + 2) & \text{if } n \geq m + 2, \end{cases} \]

we have $-1 \leq a_n \leq 0$ for all $n$, $1 = (1 - a_n)(1 + a_{n+1})$ for $n \neq m$, and $c^2 = (1 - a_m)(1 + a_{m+1})$. Thus $w(A) \leq 1$ by Theorem 3.1(a). Since the simple unilateral shift is a compression of $A$, we also have $w(A) \geq w(1) = 1$. This shows that $w(A) = 1$ as required.

(b) If $c > \sqrt{(m + 2)/(m + 1)}$, then obviously $W(A) = \{z \in \mathbb{C} : |z| \leq r\}$ for some $r \geq 1$ from (a). We now show that $r$ is of the asserted form. Let $\{a_n\}_{n=0}^{\infty}$ be such that $a_0 = -1, -1 < a_n < 1$ for $n \geq 1, r^2 = (1 - a_n)(1 + a_{n+1})$ for $n \neq m$ and $c^2r^{-2} = (1 - a_m)(1 + a_{m+1})$. By Lemma 3.2, we have $a_{m+1} \leq \sqrt{1 - r^{-2}}$. If $a_{m+1} < \sqrt{1 - r^{-2}}$, then let

\[ b_n = \begin{cases} a_n & \text{if } 0 \leq n \leq m, \\ \sqrt{1 - r^{-2}} & \text{if } n \geq m + 1. \end{cases} \]

Let $c' > c$ be such that $c'r^{-2} = (1 - b_m)(1 + b_{m+1})$ and let $A'$ be the unilateral weighted shift with weights $1, 1, 1, \ldots$. Then there exists $r' = (1 - b_n)(1 + b_{n+1})$ for $n \neq m$ and $c'r^{-2} = (1 - a_m)(1 + a_{m+1})$. This contradicts (b) of Proposition 2.1(b), a unit vector $x$ in $l^2$ with strictly positive components such that $\langle Ax, x \rangle = w(A)$. Thus

\[ r = w(A) = \langle Ax, x \rangle < \langle A'x, x \rangle \leq w(A') \leq r, \]

which is a contradiction. Hence we must have $a_{m+1} = \sqrt{1 - r^{-2}}$ and, therefore, $r$ satisfies the asserted $m + 1$ equations. □

The bilateral case of Theorem 4.6 was essentially obtained in [13, p. 500] and [14, Proposition 3]. We give an alternative proof here based on Proposition 2.3 and Theorem 4.6.

Theorem 4.9. Let $A$ be the bilateral weighted shift with weights $\ldots, 1, c, 1, 1, \ldots$, where $c$ appears in the $m$th position ($-\infty < m < \infty$). Then

(a) $W(A) = \mathbb{D}$ if and only if $|c| \leq 1$, and
(b) if $|c| > 1$, then $W(A) = \{z \in \mathbb{C} : |z| \leq (|c|^2 + 1)/(2|c|)\}$.

Proof. (a) is an easy consequence of Proposition 2.3. To prove (b), assume that $|c| > 1$. Then $W(A) = \{z \in \mathbb{C} : |z| \leq r\}$ for some $r > 1$. Since $w(A/r) = 1$, by Theorem 3.1(a) there is a sequence $\{a_n\}_{n=-\infty}^{\infty}$ in $(-1, 1)$ such that $r^{-2} = (1 - a_n)(1 + a_{n+1})$ for $n \neq m$ and $|c|^2r^{-2} = (1 - a_m)(1 + a_{m+1})$. Then both $\{a_n\}_{n=m+1}^{\infty}$ and $\{-a_n\}_{n=m}^{\infty}$ are associated with the unilateral weighted shift with weights $1/r, 1/r, \ldots$. We infer from Lemma 3.2 that $a_{m+1} = -a_m \leq \sqrt{1 - r^{-2}}$. Then using the closedness of $W(A)$, we may argue as in the proof of Theorem 4.6(b) that $a_{m+1} = -a_m = \sqrt{1 - r^{-2}}$. Thus $|c|^2r^{-2} = (1 + \sqrt{1 - r^{-2}})^2$ and it follows that $r = (|c|^2 + 1)/(2|c|)$. This completes the proof. □
Acknowledgments

The researches of the two authors were partially supported by the National Science Council of the Republic of China under NSC-99-2115-M-009-013-MY2 and NSC-99-2115-M-009-002-MY2, respectively. They were also supported by the MOE-ATU.

References