Minimal Linear Combinations of the Inertia Parameters of a Manipulator

Shir-Kuan Lin

Abstract—This paper deals with the problem of identifying the inertia parameters of a manipulator. We begin by introducing the terminology of minimal linear combinations of the inertia parameters (MLC's) that are linearly independent of one another and determine the manipulator dynamics while keeping the number of linear combinations of the inertia parameters to a minimum. The problem is then to find an identification procedure for estimating the MLC's and to use the MLC's in the inverse dynamics formulation. The resulting formulation is almost as efficient as the most efficient formulation in the literature. This formulation also provides a starting point from which to derive a recursive identification procedure. The identification procedure is simple and efficient, since it does not require symbolic closed-form equations and it has a recursive structure. The three themes concerning the dynamic modeling of a manipulator—the MLC's, the inverse dynamics in terms of the MLC's, and the identification procedure—are treated in sequence in this paper.

I. INTRODUCTION

The dynamic model of a manipulator is highly nonlinear and requires knowledge of the kinematic parameters (relations between two adjacent links) and the inertia parameters (mass, center of mass and inertia tensor of each link). The kinematic parameters are usually provided by the manufacturer or can be calibrated precisely. However, the inertia parameters of industrial robots are almost all unavailable from manufacturers, because these values are not needed for commercial controllers. Yet, the values of the inertia parameters are required for most modern control schemes of manipulators that incorporate the inverse dynamics. To evaluate the inertia parameters of the manipulator dynamics, Armstrong et al. [1] disassembled a PUMA 560 robot and used a mechanical method to measure the parameters. This approach is tedious and does not yield precise results. Fortunately, Atkeson et al. [3] found that the actuator forces of a manipulator are linear functions of the inertia parameters (i.e., the dynamics of a manipulator can be expressed as linear equations with respect to the inertia parameters), provided that friction can be neglected or considered separately. Previous attempts to identify the inertia parameters have tried to formulate the linear equations either explicitly [6], [9], [11], [15], [17] or implicitly [3], [29], [30], [31].

Identifying the inertia parameters is still difficult, however, for not all parameters can be estimated. Some parameters affect the manipulator dynamics jointly, not independently. Khosla and Kanade [17] intuitively regrouped the closed-form dynamic equations, and other researchers [9], [11], [15] developed regrouping rules to minimize the number of inertia parameters appearing in the linear equations. These approaches are not practical for a manipulator with six or more joints since the closed-form dynamic equations of a six-joint manipulator are difficult to analyze. Some authors [3], [7], [34] have presented numerical approaches such as singular value decomposition and the QR method. Because we lack knowledge of the physical meaning of the identified parameters, these parameters cannot be used effectively in computing the inverse dynamics.

Gautier et al. [8], [10], [12] developed a regrouping rule to eliminate redundant inertia parameters and to symbolically form a set of the minimal parameters needed to determine the dynamic model. Mayeda et al. [25]–[28] found the minimal parameters in closed form. Although the results of Gautier et al. and Mayeda et al. are substantially the same [12], the approach of Gautier et al. requires a regrouping process for each type of manipulator.

In this paper, we first show that the manipulator dynamics are uniquely determined by a set of minimal parameters which are linear combinations of the inertia parameters and are linearly independent. These parameters are termed the minimal linear combinations (MLC's) of the inertia parameters in this context. Although the notation of the MLC's is equivalent to that of minimal parameters in the literature, we must emphasize that the minimal parameters are in fact linear combinations of the original inertia parameters. A set of MLC's found by another approach in the present author's earlier work [23] will be used to interpret the concept of MLC's.

Finding the MLC's of the inertia parameters does not provide a complete solution for the dynamic modeling of a manipulator. The central problem is to find an efficient identification procedure for the set of MLC's. Application of the identified MLC's to the inverse or forward dynamics is also essential for manipulator control and simulation. These three problems have seldom been addressed together in the context of a single paper. This paper attempts to solve the three problems in sequence.
A new version of the recursive Newton-Euler formulation in terms of the set of MLC's is derived in this paper. From the new formulation, we deduce an identification procedure for estimating the MLC's of the inertia parameters. The identification procedure is recursive from link $n$ to link 1 and does not require symbolic closed-form dynamic equations. The identified inertia constants of the composite bodies $i + 1$ to $n$ are used to numerically form the linear equations for the actuator force of joint $i$, so that the identifiable inertia constants (i.e., MLC's) of the composite body $i$ can be estimated by the linear least squares method. This procedure is distinct from the one used in an earlier work by the author [23]. The latter strictly requires that only one joint move at a time, so it is an off-line procedure, although the same MLC's are estimated.

However, the identification procedure proposed here is limited in that friction must be treated separately from the dynamic model of the manipulator. The dominant dynamics of direct drive robots such as MIT DDArm [3] and CMU DDArm II [17] can be obtained from the standard Newton-Euler formulation [19], so the present identification method is valid for estimating the MLC's of these manipulators. For a manipulator with high-ratio gear trains, such as the PUMA arm, the present method is valid only when the viscous and static friction and the inertia of the motor actuators can be identified a priori (techniques for doing so can be found in the work of Leahy and Saridis [19]). In any case, this paper provides a starting point for future investigation of the identification problem for manipulators with high-ratio gear trains.

This paper is organized as follows. Section II describes the concept of minimal linear combinations (MLC's) of the inertia parameters. The new version of the recursive Newton-Euler formulation in terms of the MLC's is derived in Section III. Section IV presents the identification procedure.

II. MINIMAL LINEAR COMBINATIONS OF INERTIA PARAMETERS

Knowing that the dynamics of a manipulator (neglecting the effects of friction) can be formulated as linear equations with respect to the inertia parameters [3], we consider dynamic system with the linear deterministic form of

$$y = A(\theta)x$$

(1)

where $y \in \mathbb{R}^n$, $\theta \in \mathbb{R}^m$ are observable signals, $x \in \mathbb{R}^p$ consists of the system parameters, $p > n$, and $A(\theta): \mathbb{R}^m \rightarrow \mathbb{R}^{n \times p}$.

A set of columns $a_i(\theta): \mathbb{R}^m \rightarrow \mathbb{R}^n$ is said to be linearly dependent over $\mathbb{R}^m$ if there exist constants $\alpha_i, i = 1, \ldots, n$, not all zero such that

$$\sum_{i=1}^{n} \alpha_i a_i(\theta) = 0, \quad \forall \theta \in \mathbb{R}^m.$$  

(2)

If $\alpha_i, i = 1, \ldots, n$, are all zero, the set is said to be linearly independent over $\mathbb{R}^m$. By this definition we obtain the following lemma [22].

**Lemma 1:** The number of linearly independent columns of $A(\theta)$ in (1) over $\mathbb{R}^m$ is $k \leq p$ if and only if there exist $\tilde{A}(\theta): \mathbb{R}^m \rightarrow \mathbb{R}^{n \times k}$ whose columns are linearly independent over $\mathbb{R}^m$, and $w(x): \mathbb{R}^p \rightarrow \mathbb{R}^k$ whose elements are linear combinations of $x$ and are linearly independent over $\mathbb{R}^p$, such that

$$A(\theta)x = \tilde{A}(\theta)w(x), \quad \forall \theta \in \mathbb{R}^m \text{ and } x \in \mathbb{R}^p.$$  

(3)

According to the least squares theory [18], not all system parameters of the system (1) can be identified if the columns of $A(\theta)$ are linearly dependent over $\mathbb{R}^m$, i.e., $A$ is rank-deficient. Conversely, if $A$ is of full rank, all system parameters $x$ are identifiable. Lemma 1 then states that a set of linear combinations of the system parameters, $w(x)$, is identifiable since $A(\theta)x$ fully determines $y$, as does $A(\theta)w(x)$. Hence knowledge of $w(x)$ is sufficient to determine $y$. The parameter identification problem of the deterministic system (1) turns out to be to find and identify $w(x)$. To make use of this fact, we introduce the following definition.

**Definition 1:** A set $w(x)$ is a set of minimal linear combinations (MLC's) of the system parameters for the system (1) if the set is linearly independent over the domain of $w$ and there exists $A(\theta)$ whose columns are linearly independent over the domain of $A$ such that (3) holds.

For a manipulator, we are concerned with the MLC's of the inertia parameters. Ha et al. [15] showed that the dynamic model of a manipulator can be formulated in a form like (3) by using intuitive regrouping rules. Lemma 1 gives a necessary condition for the number of linearly independent columns of $A(\theta)$ and rigorously interprets the relation between the system parameters and the MLC's of the system parameters for the deterministic system (1). Since there are numerous methods for selecting $A(\theta)$ from $A(\theta)$, the set of MLC's is not unique. The notations of base parameters [25]--[28] and minimum inertia parameters [8], [12] in the literature refer to the same basic idea as the MLC's of inertia parameters. The notation of MLC's, however, provides direct insight into the parameter identification of a manipulator. In the following, we use the minimal parameters found in [23] to elucidate the concept of MLC's.

We consider a manipulator with $n$ low-pair joints, which are labeled joints 1 to $n$ outward from the base. Assign a body-fixed frame on each joint (i.e., frame $E_i$) is fixed on joint $i$ in accord with the normal driving-axis coordinate system [20] (known also as modified Denavit-Hartenberg notation [5]). The distance from the origin of $E_i$ to that of $E_j$ is designated $s_{ij}$ and that to the center of mass of link $i$ is designated $c_i$.

In the normal driving-axis coordinate system (see Fig. 1), the $z$-axis of a body-fixed frame is the driving axis of the corresponding link, i.e., the unit vector along joint $i$ is $u(\theta)^{i} = [0, 0, 1]^T$, where the superscript "(i)" denotes the representation of a vector with respect to frame $E_i$. The distance from the origin of frame $E_{i-1}$ to frame $E_i$ is shown to be

$$i^{-1} s^{(i-1)} = \begin{bmatrix} b_i \\ -d_iSb_i \\ d_iC\beta_i \end{bmatrix}, \quad \text{or} \quad i^{-1} s^{(i)} = \begin{bmatrix} b_iC\beta_i \\ -b_iSb_i \\ d_i \end{bmatrix}.$$  

(4)
where \( k, m, c !, e, U = I^{(n)} - m_n [c_n^m] \times [c_n^m] \times \), \( V_n = 0 \) and
\[
k_i = m_i c_i^{(i)} + K_{i+1}^* \begin{bmatrix} \dot{m}_{i+1} + i^{(i)} x_i^{(i)} + i^{(i)} R \dot{c}_{i+1}^{(i)} \
\end{bmatrix} \] (10)
\[
\ell_i = K_{i+1}^* i^{(i)} R \begin{bmatrix} \dot{c}_{i+1}^{(i)} \
\end{bmatrix} + K_{i+1} i^{(i)} R k_{i+1} \] (11)

If joint \( i + 1 \) is a rotational joint, then
\[
U_i = [1 - m_i c_i^{(i)} x_i^{(i)}] + i^{(i)} R U_{i+1} i^{(i)} R T - \dot{m}_{i+1} \times [b_{i+1}^{(i)} x_i^{(i)}] - [b_{i+1}^{(i)} x_i^{(i)}] \times [i^{(i)} R k_{i+1} x_i^{(i)}] \] (12)
\[
\dot{V}_i = i^{(i)} R (V_{i+1} - \dot{m}_{i+1} [d_{i+1}^{(i)} x_i^{(i)}]) + [d_{i+1}^{(i)} x_i^{(i)}] \] (13)

whereas for translational joint \( i + 1 \),
\[
U_i = [1 - m_i c_i^{(i)} x_i^{(i)}] + i^{(i)} R U_{i+1} i^{(i)} R T - \dot{m}_{i+1} \times [b_{i+1}^{(i)} x_i^{(i)}] - [b_{i+1}^{(i)} x_i^{(i)}] \times [i^{(i)} R k_{i+1} x_i^{(i)}] \] (14)
\[
\dot{V}_i = i^{(i)} R (V_{i+1} - \dot{m}_{i+1} [d_{i+1}^{(i)} x_i^{(i)}]) + [d_{i+1}^{(i)} x_i^{(i)}] \] (15)

Note that \( i^{(i)} R k_{i+1} \) is the third column of \( i^{(i)} R \) (i.e., \( i^{(i)} R k_{i+1} = [0, -S \beta_{i+1}, C \beta_{i+1}]^T \), \( b_{i+1}^{(i)} = [b_{i+1}, 0, 0]^T \) and \( d_{i+1}^{(i)} = [0, 0, d_{i+1}]^T \) (i.e., \( i^{(i)} R b_{i+1} = b_{i+1}^{(i)} + d_{i+1}^{(i)} \)).
\( \dot{m}_i \), the vectors \( k_i \) in (10) and the matrices \( U_i \) in (12) and (14) are invariant to manipulator motion; we shall refer to these as inertia constants of composite bodies. It should be remarked that these constants are different from Renaud's
inertia constants (i.e., the first moment and the inertia tensor of an augmented body when the composite body contains only rotational joints [32], [33]). The main difference is that the varying terms in \( c_i^{(0)} \) and \( J_i^{(0)} \) can be calculated with only some (not all) of the inertia constants of composite bodies [23]. This property allows us to set forth the following theorem.

**Theorem 2:** For a manipulator with \( n \) low-pair joints, in which joint \( r \) is the first rotational joint counting from the base and joint \( r \) is the nearest rotational joint not parallel to joint \( r \), a set of MLC's for determining the actuator forces \( \tau \) is the set \( S \) consisting of all nonzero elements of

1) \( K_i^r(U_j)_1 \), \( \delta_j K_i^r(k_j)_x \), \( \delta_j K_i^r(k_j)_y \) for \( r \leq j < s \),
2) \( K_i^r(U_j)_1 - (U_j)_2 \), \( K_i^r(U_j)_3 \), \( K_i^r(U_j)_{12} \), \( K_i^r(U_j)_{13} \), \( K_i^r(U_j)_{23} \), \( K_i^r(k_j)_x \), \( K_i^r(k_j)_y \) for \( s \leq j \leq n \),
3) \( K_i \bar{m}_i \) for \( i = 1, \ldots, n \),
4) \( K_i(K_i)_x \), \( K_i(k_i)_y \), \( K_i(k_i)_z \) for \( s \leq i < s \),
5) \( \sigma_i K_i \left[-(u_i)_x^2 + (u_i)_y^2 + (u_i)_z^2 \right] (k_i)_x + (u_i)_y (k_i)_y \) + \( [1 - (u_i)_x^2] (k_i)_x \), \( \sigma_i K_i \left[-(u_i)_y^2 + (u_i)_z^2 \right] (k_i)_y + (u_i)_x (k_i)_x \) for \( r < i < s \),

where \( \delta_j = 0 \) for the case where \( u_j, u_k, u_i, \forall k < j \leq s \), and \( m_r \) (when \( r > j \)) is zero or parallel to \( u_k \) for every rotational joint \( m_r, r \leq m < j \), otherwise \( \delta_j = 1 \); and \( \sigma_i = 0 \) for the case of \( u_i, u_j, \forall i < s \), otherwise \( \sigma_i = 1 \).

**Remark 1:** In [23], it was shown that knowledge of the set \( S \) in Theorem 2 is sufficient to determine the actuator forces of a manipulator and that all elements of \( S \) are identifiable. According to Lemma 1, we can say that the set \( S \) is a set of the MLC's. However, a direct and rigorous method should show that the dynamic equations of a manipulator can be reformulated as (3) in Lemma 1 with the elements of \( S \) as w. Such a method can be found in [22]. The advantage of this method is that it provides a systematic way of finding the MLC's.

**Remark 2:** The set of the MLC's of the inertia parameters in Theorem 2 is different from the results of Gautier et al. [8], [10], [12] and Mayeda et al. [25]-[28] only in some minor terms. In particular, the present result and theirs are almost the same (the \( U_i \) are slightly different) when a manipulator has only rotational joints. Suppose that joint \( r \) is a translational joint and joints \( i + 1, \ldots, n \) are rotational joints; then \( \bar{m}_{i-1} \) and \( k_{i-1} \) in this paper can be compared with their counterparts \( m_{R_{i-1}} \) and \( m_{X_{i-1}}, m_{Y_{i-1}}, m_{Z_{i-1}} \) \( i \) in [8], [10], [12] as shown in Table I. It is apparent that the present set of MLC's is not identical to that of Gautier et al. for a manipulator with translational joints. Note that \( U_{i-1} \) is also different since it contains \( (k_i)_x \). The merit of Theorem 2 is that it clearly describes the set of MLC's by introducing joints \( r \) and \( s \).

### III. INVERSE DYNAMICS

Khalil and Kleinfinger [16] modified the recursive Newton-Euler formulation [24] by using the first moments \( (m_i c_i^{(0)}) \) and the inertia tensors about the origin on the driving joint \( (J_i^{(0)}) \) instead of the centers of mass \( (c_i^{(0)}) \) and the inertia tensors about the center of mass \( (J_i^{(0)}) \). Their set of MLC's [8], [10], [12] can then be used in this modified formulation by replacing the first moment and the inertia tensor of each link with their counterparts in the MLC's (if the counterparts are not redundant) or with zero (if they are redundant, i.e., not in the MLC's). This approach draws on the work of Atkeson et al. [3], who suggested that the value of each linear combination in the MLC's be kept the same while one original inertia parameter in the linear combination is assigned the same value as the linear combination and the other inertia parameters are set to zero. Since the dynamic model is linear with respect to the MLC's, the same values for the MLC's determine the same dynamic model. However, this property does not hold for all sets of MLC's.

Consider a manipulator with rotational joint \( i - 1 \) and translational joint \( i, i > s \). The set of MLC's in Theorem 2 for the manipulator contains the \( x \) - and \( y \)-components of \( k_{i-1} \) and all three components of \( k_i \). Since joint \( i \) is a translational joint, \( k_{i-1} \) has the contribution of \( k_i \) (see Table I). \( m_{i-1} c_{i-1}^{(0)} \) should not be assigned the same value as \( k_{i-1} \), otherwise \( k_i \) must be zero, which contradicts the principle that the values of MLCs should be preserved while distributing their values to the original inertia parameters. Hence the modified formulation [16] cannot be applied to the general case, where the values of the MLC's are not preserved while using...
Atkeson’s technique, and so it cannot be applied to the present set of MLC’s.

Thus it is necessary to derive a new version of the recursive Newton-Euler formulation in terms of the present set of MLC’s. The derivation process presented in this section could also be used for other sets of MLC’s. At the end of this section, it will be seen that the result is identical to that of Khalil and Kleinfinger [16] when the manipulator has only rotational joints. This identity is because the two sets of MLC’s are almost the same for a manipulator with only rotational joints. This also verifies the result of Khalil and Kleinfinger. Since these two formulations both use minimal parameters, it will be found that they are about equally efficient for a manipulator with one or more translational joints.

On the other hand, the formulation of the forward dynamics in terms of the present MLC’s is easy to derive [23]. Renaud’s formulation of the forward dynamics [32], [33] uses the masses, first moments, and inertia tensors of composite bodies ($\tilde{m}_i$, $\tilde{c}_i$, and $\tilde{J}_i$), which are directly related to the set of MLC’s in the form of (8)-(15). Since all possible constant terms of $\tilde{c}_i$ and $\tilde{J}_i$ are concatenated to be $k_i$ and $U_i$, respectively, the computation of $\tilde{c}_i$ and $\tilde{J}_i$ in terms of the present set of MLC’s is more efficient than that in terms of the set of Gautier et al. when joint $i + 1$ is a translational joint and joint $i$ is a rotational joint, the present set requires $8M + 7A$ and $50M + 40A$ for computing $\tilde{c}_i$ and $\tilde{J}_i$, respectively (see (11) in this paper and (36) in [23]), while the set of Gautier et al. requires $11M + 9A$ and $55M + 41A$ (see (A7) and (A8) in [21]). Note that $M$ denotes multiplications and $A$ additions/subtractions.

In the following, we first derive a formulation in terms of the inertia constants of composite bodies, and then reduce this formulation so that it is expressed in terms of the present set of MLC’s.

**A. Formulation**

We start with the recursive Newton-Euler formulation [24]. Let $\dot{\omega}_i$ and $\omega_i$ be the angular acceleration and velocity of link $i$, $a_i$, the acceleration of the origin of frame $E_i$, $g$ the gravitational acceleration, $q_i$ the joint displacement of joint $i$, $f_i$ and $t_i$ the inertia force and torque of link $i$, $f_i$ and $t_i$ the force and torque exerted on link $i$ by joint $i$, and $\tau_i$ the actuator force (or torque) of joint $i$. The Newton-Euler formulation based on the normal driving-axis coordinate system is [5], [20]:

$$\omega_i = i^{-1} R_{\omega_{i-1}} + K_i u_i \dot{q}_i$$

$$\dot{\omega}_i = i^{-1} R_{\dot{\omega}_{i-1}} + K_i (\dot{u}_i \dot{q}_i + a_i)$$

$$a_i = i^{-1} R [a_{i-1} + \dot{\omega}_{i-1} \times \omega_{i-1} + \omega_{i-1} \times (\omega_{i-1} \times i^{-1} s_{i-1})] + K_i (u_i \dot{q}_i + 2 \omega_i \times u_i \dot{q}_i)$$

Note that $f_i = -m_i (a_i + \omega_i \times \omega_i)$, $t_i = m_i (a_i + \omega_i \times \omega_i)$, and $(n+1)g_i = E_i g_i$, where $f_i$ and $t_i$ are the external force and torque acting on the gripper point of the end-effector, respectively, and $E_i g_i$ is the distance from the origin of frame $E_i$ to the gripper point of the end-effector.

We define the following notation:

$$\Omega_i \equiv \begin{bmatrix} \omega_i \times \omega_i \times \omega_i \times \end{bmatrix}$$

$$\psi(A) \equiv \begin{bmatrix} A_{11} \\ A_{22} \\ A_{33} \\ A_{12} \\ A_{13} \\ A_{23} \end{bmatrix}$$

where $A \in \mathbb{R}^{3x3}$ is a symmetric matrix and $\alpha_x$ and $\omega_x$ are the $x$-components of $\alpha$ and $\omega$, respectively. The purpose of this notation is to rewrite (19) and (20) in matrix form,

$$f_i = -m_i (a_i + \Omega_i c_i)$$

$$t_i = -\psi(\omega_i \times \omega) \psi(I_i)$$

The following three equalities are relevant to the derivation of the new formulation. The proof of them can be found in the Appendix.

$$\begin{bmatrix} a \times (\Omega_i b) + b \times (\Omega_i a) = \psi(\omega_i \times \omega) \psi(A) \\
\Omega_i \begin{bmatrix} 0 \\ 0 \\ k \end{bmatrix} = \Omega_i \begin{bmatrix} 0 \\ 0 \\ k \end{bmatrix} \end{bmatrix}$$

$$\begin{bmatrix} a \times (\Omega_i b) + b \times (\Omega_i a) = \psi(\omega_i \times \omega) \psi(A) \\
\Omega_i \begin{bmatrix} 0 \\ 0 \\ k \end{bmatrix} = \Omega_i \begin{bmatrix} 0 \\ 0 \\ k \end{bmatrix} \end{bmatrix}$$

$$\begin{bmatrix} a \times (\Omega_i b) + b \times (\Omega_i a) = \psi(\omega_i \times \omega) \psi(A) \\
\Omega_i \begin{bmatrix} 0 \\ 0 \\ k \end{bmatrix} = \Omega_i \begin{bmatrix} 0 \\ 0 \\ k \end{bmatrix} \end{bmatrix}$$

where $A = \begin{bmatrix} a R \times b \times a \times c \times a \times b \times c \end{bmatrix}$ denotes a diagonal matrix with entries of $a$, $b$, and $c$. 

\begin{align}
\begin{aligned}
\dot{\omega}_i^{(n)} &= -m_i \left[ a_i^{(n)} + \omega_i^{(n)} \times c_i^{(n)} \right] + \omega_i^{(n)} \times \left[ \omega_i^{(n)} \times c_i^{(n)} \right] \\
\tau_i^{(n)} &= -I_i^{(n)} \dot{\omega}_i^{(n)} - \omega_i^{(n)} \times \left( I_i^{(n)} \dot{\omega}_i^{(n)} \right) \\
f_i^{(n)} &= i^{-1} R_{i+1} \dot{E}_i^{(n+1)} - m_i g_i - f_i^{(n)} \\
t_i^{(n)} &= i^{-1} R_{i+1} \dot{E}_i^{(n+1)} - \tau_i^{(n)} - c_i^{(n)} \\
\end{aligned}
\end{align}
Lemma 3: The joint force of joint \( i \) acting on link \( i \) by joint \( i \) is

\[
\mathbf{f}_i^{(i)} = m_i \left( \mathbf{a}_i^{(i)} - \mathbf{g}_i^{(i)} \right) + \Omega_i^{(i)} \mathbf{k}_i + \mu_i^{(i)}
\]

where \( \mu_{n+1} = -\mathbf{f}_E^{(n)} \) and

\[
\mu_i^{(i)} = -\mathbf{f}_E^{(i)} + \sum_{j=1}^n \mathbf{J}_j \mathbf{R}_j \left( \mathbf{k}_j \frac{\partial}{\partial \mathbf{q}_j} + \mathbf{m}_j \mathbf{c}_j^{(i)} \right)
\]

\[
= \mathbf{f}_E^{(i)} + \sum_{j=1}^n \mathbf{J}_j \mathbf{m}_j \left( \mathbf{a}_j^{(i)} - \mathbf{g}_j^{(i)} + \Omega_j^{(i)} \mathbf{c}_j^{(i)} \right)
\]

\[
= \mathbf{f}_E^{(i)} - \sum_{j=i+1}^n \mathbf{J}_j \mathbf{m}_j \left( \mathbf{a}_j^{(i)} - \mathbf{g}_j^{(i)} + \Omega_j^{(i)} \mathbf{c}_j^{(i)} \right) + \mathbf{J}_i \mathbf{m}_i \left( \mathbf{a}_i^{(i)} - \mathbf{g}_i^{(i)} + \Omega_i^{(i)} \mathbf{c}_i^{(i)} \right),
\]

\[
= \mathbf{f}_E^{(i)} + \sum_{j=i+1}^n \mathbf{J}_j \mathbf{m}_j \left( \mathbf{a}_j^{(i)} - \mathbf{g}_j^{(i)} + \Omega_j^{(i)} \mathbf{c}_j^{(i)} \right)
\]

in which the latter follows from (10). Since the angular velocity and acceleration of link \( j \) and those of link \( j + 1 \) are the same for the case where joint \( j + 1 \) is a translational joint, we obtain

\[
K_{j+1} \Omega_j^{(i)} R_k + k_{j+1} = K_{j+1}^{(i+1)} \Omega_{j+1}^{(i+1)} R_{k_{j+1}}
\]

Substituting (30) and (35)–(37) into (34) yields (32).

Proof: According to the recursive Newton-Euler formulation (16)–(23), we have

\[
\mathbf{f}_i^{(i)} = \mathbf{f}_E^{(i)} - \sum_{j=1}^n \left( \mathbf{J}_j \mathbf{R}_j m_j \mathbf{c}_j^{(i)} \right)
\]

where \( \sum_{j=1}^i \) for \( j < i \) is assumed to be zero. It can be shown that

\[
\sum_{j=i+1}^n \sum_{k=i+1}^n \sum_{k=1}^j \sum_{k=1}^n \sum_{k=1}^n
\]

\[
= k_{j+1} \mathbf{R} \left\{ \begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ \vdots \\ 0 \\ 0 \end{array} \right\}
\]

\[
+ K_{j+1} \left( k_{j+1} - m_{j+1} \right) \mathbf{d}_{j+1}
\]

for \( j < n \)
where
\[
\zeta_i^{(i)} = \zeta_i^{(i+1)} + \sum_{k=i}^{i+1} K_i (a_k - g_k) + \Psi(\omega_i^{(i)}, \omega_i^{(i)}) D_i^{(i)}
\]

Note that \(\mu_i^{(n)} = -\zeta_i^{(n)} = -\Psi^{(n)} - \beta_i^{(n)} \times E_i^{(n)}\).

**Proof:** It can be shown (see the Appendix) that

\[
t_i^{(i)} = \sum_{j=i}^{n} \mathbf{R} \Psi(\omega_j^{(i)}, \omega_j^{(i)}) \psi(A_j^{(i)})
\]

Comparing (47) and (38) yields

\[
\sum_{k=i}^{i+1} \mathbf{A}_k^{(i)} = D_i^{(i)} + \text{diag}([U_j^2, U_j^2, 0])
\]

Substituting (51) and (52) into (46) and using (31), we obtain the joint torque of rotational joint \(i\),

\[
t_i^{(i)} = \eta_i^{(i)} + \left[ \begin{array}{c} 0 \\ [k_i] \end{array} \right] \times (a_i^{(i)} - g_i^{(i)})
\]

The rule of the vector product and (25) imply that

\[
\eta_i^{(i)} = \eta_i^{(i+1)} + \left[ \begin{array}{c} 0 \\ [k_i] \end{array} \right] \times (a_i^{(i)} - g_i^{(i)})
\]

Consider the case where joint \(i\) is a rotational joint and joint \(m_i\) is the nearest rotational joint behind joint \(i\). Thus

\[
\dot{\zeta}_i^{(i)} = \dot{\zeta}_i^{(i+1)} = \cdots = \dot{\zeta}_m^{(i+1)}
\]

\[
\dot{\omega}_i^{(i)} = \dot{\omega}_i^{(i+1)} = \cdots = \dot{\omega}_m^{(i+1)}
\]
\[ h_i^{(i)} = K_i \left( a_i^{(i)} \ddot{q}_i + 2 \omega_i^{(i)} \times u_i^{(i)} \right) \]  
(61)

Thus (64) and (65) can be rewritten in more compact form as follows:

\[ \ddot{\mu}_i^{(i)} = K_i^* (h_i^{(i)}) \left( \begin{bmatrix} 0 \\ \frac{b_0}{d_i} \end{bmatrix} \right) + K_i \ddot{m}_i \ddot{h}_i^* \]  
(72)

\[ \ddot{\zeta}_i^{(i)} = K_i^* \left( \begin{bmatrix} 0 \\ \frac{b_0}{d_i} \end{bmatrix} \right) x a_i^{(i)} + \Psi (\omega_i^{(i)}, \omega_i^{(i)}) \psi (D_i^{(i)}) \]  
(73)

Furthermore, a technique can be used to save some computation for \( \zeta_i^{(i)} \). Define

\[ \zeta_i^{(i+1)} = \zeta_i^{(i+1)} + \begin{bmatrix} 0 \\ \frac{b_{i+1}}{d_{i+1}} \end{bmatrix} x \mu_i^{(i+1)} \]  
(74)

Then (67) can be reformulated as

\[ \zeta_i^{(i)} = \zeta_i^{(i+1)} + i+1 R \zeta_i^{(i+1)} + \begin{bmatrix} 0 \\ \frac{b_{i+1}}{d_{i+1}} \end{bmatrix} x \mu_i^{(i+1)} \]  
(75)

**B. Algorithm**

The above formulation is suited for any joint of a manipulator. However, it requires knowledge of some parameters other than the MLC’s for joint \( i \), \( i < s \). In order to replace the inertia constants with the MLC’s, the individual joints must be taken into account. As usual, we set \( \ddot{h}_i^{(i)} = -g(0) \).

In accord with Theorem 2, the joints are classified into three groups.

1) Joints remaining in front of the first rotational joint \( r \), i.e., \( i < r \).

The angular velocity and acceleration of the links remaining in front of joint \( r \) are all zero and \( \zeta_i^{(i)} \) are not required for \( \tau_i \). Therefore, (57)–(61), (65), and (67) are redundant, while (62)–(64) can be replaced by

\[ a_i^{(i)} = i-1 R a_{i-1}^{(i-1)} s^{(i-1)} + K_i \ddot{h}_i^* \]  
(77)

\[ \rho_i = \ddot{m}_i (a_i^{(i)})_s \]  
(78)

\[ \ddot{\mu}_i^{(i)} = m_i (a_i^{(i)})_s \]  
(79)

in which \( \ddot{\mu}_i^{(i)} \) is also redundant (see (67)). As a result, only the members of the MLC’s (i.e., \( \ddot{m}_i \)) are required in the algorithm for \( i < r \).

2) Joints remaining between joint \( r - 1 \) and joint \( s \), i.e., \( r \leq i < s \).

The rotational joints remaining in front of joint \( s \) are parallel to one another, so that \( \bar{\zeta}_i^{(i)} = \pm \bar{\zeta}_i^{(i)} \) for rotational joint \( i \) (and then only its \( z \)-component must be computed) and

\[ \omega_i^{(i)} = u_i^{(i)} (\omega_i^{(i)})_z \]  
(80)

\[ \bar{\omega}_i^{(i)} = u_i^{(i)} (\bar{\omega}_i^{(i)})_z \]  
(81)

which replace (57) and (58). Note that \( \bar{\mu}_i^{(i)} \) is redundant if \( r = 1 \).
As was mentioned above, (62) is replaced by (69)-(71). Applying (80) and (81), we modify (61) and (63)-(65) as follows:

\[
\dot{h}_i^{(r)} = K_i \begin{bmatrix} 0 \\ 0 \\ \omega_i^{(r)}(x) \end{bmatrix} + \kappa_{1i}(\omega_i^{(r)}(x)) \dot{q}_i \\
\rho_i = K_i \begin{bmatrix} 0 \\ 0 \\ \omega_i^{(r)}(x) \end{bmatrix} - \kappa_{2i}(\omega_i^{(r)}(x)) \dot{q}_i \\
\mu_i^{(r)} = \delta_i \mathbf{k}_i \mathbf{O}_i^{(r)} \begin{bmatrix} (k_i)x \\ (k_i)y \end{bmatrix} + K_i \dot{m}_i \dot{h}_i^* \\
(\zeta_i^{(r)})^\top_x = K_i \begin{bmatrix} (a_i^{(r)})(x) \\ (a_i^{(r)})(y) \end{bmatrix} \cdot \begin{bmatrix} (k_i)x \\ (k_i)y \end{bmatrix} \\
+ \omega_i^{(r)}(x) \cdot \begin{bmatrix} (D_i^{(r)})(x) \\ (D_i^{(r)})(y) \end{bmatrix} + \kappa_{1i}(\omega_i^{(r)}(x)) \dot{m}_i \dot{d}_i \\
+ \kappa_{2i}(\omega_i^{(r)}(x)) \dot{q}_i \\
\]

where the constants are
\[
\kappa_{1i} \equiv -(\mathbf{u}_i^{(r)})(x) \cdot (k_i)x + (\mathbf{u}_i^{(r)})(x) \cdot (k_i)y \\
\kappa_{2i} \equiv -(\mathbf{u}_i^{(r)})(x) \cdot (\mathbf{u}_i^{(r)})(x) \cdot (k_i)x + (\mathbf{u}_i^{(r)})(x) \cdot (k_i)y \\
+ \begin{bmatrix} 1 - (\mathbf{u}_i^{(r)})(x)^2 \end{bmatrix} \cdot (k_i)x \\
\]

which are the elements of the set of MLC's. Note that \(\mathbf{u}_i^{(r)}, i < s\), are constant vectors. The reason for (84) is that the components of \(\mathbf{k}_i\) are not MLC's and thus are redundant for determining the actuator forces if \(\delta_i = 0\) (see Theorem 2). Furthermore, (60) is redundant since (85) does not require it to compute \((\zeta_i^{(r)})^\top_x\).

In the backward recursion, only (67) needs to be modified since only the z-component of \(\zeta_i^{(r)}\) must be computed. According to (74) and (75), (67) is replaced by

\[
(\zeta_i^{(r)})^\top_z = \mathbf{u}_i^{(r-1)} \cdot \mathbf{R}_i \left( \mathbf{z}_i^{(r)} + \begin{bmatrix} 0 \\ 0 \\ \mu_i \end{bmatrix} \right) \\
\]

for \(i = s\) (87)

\[
(\zeta_i^{(r)})^\top_z = (\zeta_i^{(r)})^\top_x + \mathbf{u}_i^{(r+1)} \cdot \mathbf{R}_i \left( \begin{bmatrix} 0 \\ 0 \\ \mu_i \end{bmatrix} \right) \\
\]

for \(i < s\) (89)

An examination of (83)-(85) indicates that all inertia constants in the formulation have been replaced by the MLC's described in Theorem 2.

3) Joints remaining behind joint \(s - 1\), i.e., \(i \geq s\):
None of (57)-(68) needs to be further modified since joints in this group fall under the general case. However, (62), (64), (65), and (67) are replaced by (69)-(75), respectively.

In addition, \(\mathbf{D}_i^{(r)}\) in (38) varies and must be calculated for each rotational joint each recursion. To save computation, (38) can be rewritten in the following more compact form:

\[
\mathbf{D}_i^{(r)} = \mathbf{U}_i^* - \mathbf{diag}[\mathbf{U}_i^{(22)}(\mathbf{U}_i^{(22)})^T, 0] \\
+ \sum_{k=i+1}^{m_i-1} \mathbf{d}_k \mathbf{W}_k^{(r)} \\
\]

Table II

<table>
<thead>
<tr>
<th>(i &lt; r)</th>
<th>(i = r)</th>
<th>(r &lt; i &lt; s)</th>
<th>(s &lt; i \leq n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\omega_i^{(r)})</td>
<td>0</td>
<td>1A</td>
<td>3M</td>
</tr>
<tr>
<td>(\omega_i^{(r)})</td>
<td>0</td>
<td>1A</td>
<td>3M</td>
</tr>
<tr>
<td>(\Omega_i^{(r)})</td>
<td>1M</td>
<td>1M</td>
<td>6M9A</td>
</tr>
<tr>
<td>(\psi)</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>(\rho_i)</td>
<td>1M</td>
<td>-</td>
<td>4M2A</td>
</tr>
<tr>
<td>(\rho_i)</td>
<td>1M</td>
<td>0</td>
<td>4M2A</td>
</tr>
<tr>
<td>(\zeta_i^{(r)})</td>
<td>-</td>
<td>3M2A</td>
<td>3M2A</td>
</tr>
<tr>
<td>(\zeta_i^{(r)})</td>
<td>-</td>
<td>8M4A</td>
<td>8M7A</td>
</tr>
<tr>
<td>(\mu_i^{(r)})</td>
<td>-</td>
<td>3M2A</td>
<td>3M2A</td>
</tr>
<tr>
<td>(\mu_i^{(r)})</td>
<td>-</td>
<td>8M4A</td>
<td>8M7A</td>
</tr>
<tr>
<td>(\tau_i)</td>
<td>1A</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

† It additionally requires 3M2A for rotational joint s-1 or 11M6A for translational joint s-1.
TABLE III

<table>
<thead>
<tr>
<th>Method</th>
<th>General Robot with n Rotational Joints</th>
<th>Stanford Arm</th>
</tr>
</thead>
<tbody>
<tr>
<td>Khalil and</td>
<td>M: 101n - 129</td>
<td>240</td>
</tr>
<tr>
<td>Kleinfinger [16]</td>
<td>A: 90n - 118</td>
<td>227</td>
</tr>
<tr>
<td>Balafoutis et</td>
<td>M: 93n - 69</td>
<td>-</td>
</tr>
<tr>
<td>al. [4]</td>
<td>A: 81n - 66</td>
<td>-</td>
</tr>
<tr>
<td>Gautier and</td>
<td>M: 92n - 127</td>
<td>232</td>
</tr>
<tr>
<td>Khalil [10]</td>
<td>A: 81n - 117</td>
<td>218</td>
</tr>
<tr>
<td>Present</td>
<td>M: 89n - 82</td>
<td>231</td>
</tr>
<tr>
<td></td>
<td>A: 77n - 71</td>
<td>219</td>
</tr>
</tbody>
</table>

where the first two terms on the right-hand side are calculated by using (42) and

\[
W_k^{(i)} = \begin{bmatrix} 2(k_k)_x & 0 & -(k_k)_y \\ 0 & 2(k_k)_y & -(k_k)_y \\ -(k_k)_y & -(k_k)_y & 0 \end{bmatrix} R^T \quad (92)
\]

which is a constant symmetric matrix since \( k_k \), \( i < k < m_i \), is constant. Note that \( U_i \) is also constant. For a rotational joint in front of joint \( s \), only the \( (3, 3) \)th entry of \( D_{i}^{(i)} \) is required, which is

\[
(D_{i}^{(i)})_{33} = (U_i^*)_{33} + \sum_{k=i+1}^{m_i-1} d_k k_{2k} d_k \quad \text{for } i < s \quad (93)
\]

This algorithm has been verified by a FORTRAN program. The number of operations for each variable in the algorithm is listed in Table II. If we consider a manipulator with \( n \) rotational joints whose second joint is not parallel to the first joint, the total number of computations of the present algorithm is \((89n - 82)M + (77n - 71)A\), where “M” and “A” denote multiplication and addition/subtraction, respectively. The number of computations for the coordinate transformation matrices \( R_{i} \) and the distance between two frames \( s^{(i)} \) has not been taken into account; these total \( 4nM \) and \( n \) pairs of sin and cos for \( n \) rotational joints. In most industrial robots, adjacent joints are either parallel or perpendicular to each other, which reduces the number of computations for the product of a coordinate transformation and a vector to \( 4M + 2A \). The total number of computations for an industrial robot is then \( 5(n - 1)(4M + 2A) + 2M + 1A \) less than the number for a general manipulator since only the \( z \)-component of \( \zeta_{i}^{(i)} \) needs to be computed.

The efficiency of the present algorithm is compared with that of the other algorithms [4], [10], [16] in Table III. The algorithm of Gautier and Khalil [10] is a reformulation of Khalil and Kleinfinger [16] using only the MLC's. As was mentioned above, the present formulation is identical to that in [16] for a general manipulator with only rotational joints, so these two formulations should have same efficiency in this case. The difference in their efficiency shown in Table III may come from different programming techniques or different assumptions. For instance, if joint 1 is parallel to the direction of gravity, the operation count of the present formulation listed in Table III can be further reduced. Nevertheless, Table III shows that the present formulation is almost as efficient as the most efficient formulation in the literature. This is demonstrated by the comparison for the Stanford arm, which has one translational joint.

IV. IDENTIFICATION

Our goal is to formulate the linear equations in a recursive form, so that the procedure for identifying the inertia constants of the composite body \( i \) can be executed on the basis of knowledge of the inertia constants of the composite bodies \( i + 1 \) to \( n \).

Substituting (63)-(67) into (68), we obtain

\[
\begin{align*}
\tau_i = & \ K_i \begin{bmatrix} v_{Ti}^T w_{Ti} + \left( \mu_{i+1}^{(i)} \right)_x \\ v_{Ti}^T w_{Ri} + \left( \psi \omega_i^{(o)}, \omega_i^{(o)} \right) \psi (D_i^{(i)}) \\ + i+1 \dot{s}^{(i)} \times \mu_{i+1}^{(i)} + \zeta_{i+1}^{(i)} \end{bmatrix} \\
& + \ K_i^T \begin{bmatrix} v_{Ti}^T w_{Ti} + \left( \mu_{i+1}^{(i)} \right)_x \\ v_{Ti}^T w_{Ri} + \left( \psi \omega_i^{(o)}, \omega_i^{(o)} \right) \psi (D_i^{(i)}) \\ + i+1 \dot{s}^{(i)} \times \mu_{i+1}^{(i)} + \zeta_{i+1}^{(i)} \end{bmatrix}
\end{align*}
\quad (94)
\]

where

\[
w_{Ti} = \begin{bmatrix} \dot{m}_i \\ k_i \end{bmatrix}
\quad (95)
\]

\[
w_{Ri} = \begin{bmatrix} (k_i)_x \\ (k_i)_y \\ (\Omega_i)_{11} - (\Omega_i)_{22} \\ (\Omega_i)_{33} \\ (\Omega_i)_{12} \\ (\Omega_i)_{13} \\ (\Omega_i)_{23} \\ (\Omega_i)_{31} \\ (\Omega_i)_{32} \\ (\Omega_i)_{33} \end{bmatrix}
\quad (96)
\]

\[
v_{Ti} = \begin{bmatrix} (\dot{\omega}_i^{(o)})_x - (\dot{\omega}_i^{(o)})_x \\ -(\dot{\omega}_i^{(o)})_y + (\dot{\omega}_i^{(o)})_x (\omega_i^{(o)})_x \\ (\dot{\omega}_i^{(o)})_x + (\dot{\omega}_i^{(o)})_y (\omega_i^{(o)})_x \\ - (\dot{\omega}_i^{(o)})_x^2 - (\dot{\omega}_i^{(o)})_y^2 \end{bmatrix}
\quad (97)
\]
\[
\begin{align*}
\mathbf{v}_{Ri} &= \begin{bmatrix}
(a_i^{(t)} - g_i^{(t)})_y \\
-(a_i^{(t)} - g_i^{(t)})_x \\
-(\omega_i^{(t)})_y \\
(\omega_i^{(t)})_x \\
(\omega_i^{(t)})_z x (a_i^{(t)} - g_i^{(t)})_x \\
(\omega_i^{(t)})_z \end{bmatrix}, \\
\mathbf{v}_{Ri} &= \begin{bmatrix}
-(a_i^{(t)} - g_i^{(t)})_y \\
(\omega_i^{(t)} + \omega_i^{(t)})_x \\
(\omega_i^{(t)} + \omega_i^{(t)})_y \\
(\omega_i^{(t)} + \omega_i^{(t)})_z \\
(\omega_i^{(t)} + \omega_i^{(t)})_z x (a_i^{(t)} - g_i^{(t)})_x \\
(\omega_i^{(t)} + \omega_i^{(t)})_z \end{bmatrix}, \\
\mathbf{A}_i &= \begin{bmatrix}
(a_i^{(t)} - g_i^{(t)})_y \\
-(a_i^{(t)} - g_i^{(t)})_x \\
(\omega_i^{(t)} + \omega_i^{(t)})_x \\
(\omega_i^{(t)} + \omega_i^{(t)})_y \\
(\omega_i^{(t)} + \omega_i^{(t)})_z \\
(\omega_i^{(t)} + \omega_i^{(t)})_z x (a_i^{(t)} - g_i^{(t)})_x \\
(\omega_i^{(t)} + \omega_i^{(t)})_z \end{bmatrix} \\
\gamma_i &= \begin{bmatrix}
\gamma_i^{(d)} \\
\gamma_i^{(d)} \\
\gamma_i^{(d)} \\
\gamma_i^{(d)} \\
\gamma_i^{(d)} \\
\gamma_i^{(d)} \\
\gamma_i^{(d)} \\
\gamma_i^{(d)} \\
\gamma_i^{(d)} \\
\gamma_i^{(d)} \end{bmatrix} \\
\end{align*}
\]
where
\[
\mathbf{v}_i = \begin{cases}
\mathbf{v}_{Ti} & \text{for translational joint } i, i \geq s, \\
\mathbf{v}_{Ri} & \text{for rotational joint } i, i \geq s, \\
\mathbf{v}_{Ti} & \text{for translational joint } i, r \leq i < s, \\
\mathbf{v}_{Ri} & \text{for rotational joint } i, r \leq i < s,
\end{cases}
\]

\[
\mathbf{A}_i = \begin{bmatrix}
(a_i^{(t)} - g_i^{(t)})_y \\
-(a_i^{(t)} - g_i^{(t)})_x \\
(\omega_i^{(t)} + \omega_i^{(t)})_x \\
(\omega_i^{(t)} + \omega_i^{(t)})_y \\
(\omega_i^{(t)} + \omega_i^{(t)})_z \\
(\omega_i^{(t)} + \omega_i^{(t)})_z x (a_i^{(t)} - g_i^{(t)})_x \\
(\omega_i^{(t)} + \omega_i^{(t)})_z \end{bmatrix}
\]
(107)

\[
\gamma_i = \begin{bmatrix}
\gamma_i^{(d)} \\
\gamma_i^{(d)} \\
\gamma_i^{(d)} \\
\gamma_i^{(d)} \\
\gamma_i^{(d)} \\
\gamma_i^{(d)} \\
\gamma_i^{(d)} \\
\gamma_i^{(d)} \\
\gamma_i^{(d)} \\
\gamma_i^{(d)} \end{bmatrix}
\]
(108)

\[
\mathbf{D}_i^{(t)} = \mathbf{D}_i^{(r)} - \sum_{k=1}^{m_i-1} d_k \mathbf{W}_k^{(i)} + \mathbf{G}_{m_i}^{(i)} - \text{diag}[(\mathbf{G}_{m_i}^{(i), 22}), (\mathbf{G}_{m_i}^{(i), 22}), 0] + \sum_{k=i+1}^{m_i-1} d_k \mathbf{W}_k^{(i)}
\]
(99)

provided joint \( m_i \) is the nearest rotational joint behind joint \( i \).

It should be remarked that the parameters in the linear equations (94) and (100) (i.e., \( \mathbf{W}_{Ti}, \mathbf{W}_{Ri}, \mathbf{A}_i, \mathbf{D}_i \)) are MLCS.

For the case of \( r \leq i < s \), the angular velocity and acceleration of link \( i \) are in alignment with the direction of joint \( r \) (see (80) and (81)). Equation (94) can be reduced to

\[
\begin{align*}
\mathbf{T} &= \begin{bmatrix} \mathbf{v}_{Ti} & \mathbf{v}_{Ri} \end{bmatrix}
\end{align*}
\]
(100)

where

\[
\mathbf{w}_{Ti} = \begin{bmatrix} \mathbf{w}_{Ti} \end{bmatrix}
\]
(101)

\[
\mathbf{w}_{Ri} = \begin{bmatrix} \mathbf{w}_{Ri} \end{bmatrix}
\]
(102)

\[
\mathbf{v}_{Ti} = \begin{bmatrix} \mathbf{v}_{Ti} \end{bmatrix}
\]
(103)

\[
\mathbf{v}_{Ri} = \begin{bmatrix} \mathbf{v}_{Ri} \end{bmatrix}
\]
(104)

\[
\mathbf{D}_i^{(t)} = (\mathbf{G}_{m_i}^{(i), 22}) + \sum_{k=i+1}^{m_i-1} 2k \mathbf{d}_k
\]
(105)

It should be remarked that the parameters in the linear equations (94) and (100) (i.e., \( \mathbf{w}_{Ti}, \mathbf{w}_{Ri}, \mathbf{A}_i, \mathbf{D}_i \)) are MLCS.

For the case of \( i < r \), (94) leads to

\[
\begin{align*}
\mathbf{T} &= \begin{bmatrix} \mathbf{v}_{Ti} & \mathbf{v}_{Ri} \end{bmatrix}
\end{align*}
\]
(106)

since the angular velocity and acceleration of link \( i \) are both zero. The fact that \( (a_i^{(t)} - g_i^{(t)})_x, i < r \), is not zero implies that \( m_i \) can be identified by means of (106).
Step 2: Use the values of $\omega_\alpha^{(n)}$, $\omega_\beta^{(n)}$, $\bar{h}_n^{(n)}$, $a_n^{(n)}$ to calculate $v_n$ by (97), (98), (103) or (104) for each sampling point from point 1 to point $N$, and then form $A_n$. Use the least squares method to solve $w_n$ from

$$A_n w_n = \begin{bmatrix} \tau_{n}^\alpha \\ \tau_{n}^\beta \\ \vdots \\ \tau_{n}^\gamma \end{bmatrix}$$  \hspace{1cm} (113)

Step 3: If $r > n - 1$, go to Step 4; otherwise, set $D_{1}^{(n)} = 0$ and do the following substeps for joint $i$ recursively from $i = n - 1$ to $i = r$.

3.1 For joint $i$, do the following substeps for each sampling point from point 1 to point $N$.

3.1.1 If joint $i+1$ is a rotational joint, compute $D_{i+1}^{(i+1)}$ for $i + 1 \geq s$ by

$$D_{i+1}^{(i+1)} = \begin{bmatrix} (U_{i+1})_{11} & (U_{i+1})_{12} & (U_{i+1})_{13} \\ (U_{i+1})_{12} & 0 & (U_{i+1})_{23} \\ (U_{i+1})_{13} & (U_{i+1})_{23} & (U_{i+1})_{33} \end{bmatrix}$$

or compute $(D_{i+1}^{(i+1)})_{33}$ for $i + 1 < s$ by

$$(D_{i+1}^{(i+1)})_{33} = (\bar{D}_{i+1}^{(i+1)})_{33} + (U_{i+1})_{33}$$  \hspace{1cm} (115)

3.1.2 Use the values of $\dot{\omega}_{i+1}^{(i+1)}$, $\ddot{\omega}_{i+1}^{(i+1)}$, $\dot{h}_{i+1}^{(i+1)}$, $a_{i+1}^{(i+1)}$, $w_{i+1}^{(i+1)}$ (i.e., parts of $\dot{m}_{i+1}$, $k_{i+1}$, and $U_{i+1}$), and $D_{i+1}^{(i+1)}$ to calculate $\mu_{i+1}^{(i+1)}$ by (59), (60), (64) and (65). Then compute $\mu_{i+1}^{(i+1)}$ and $v_{i+1}^{(i+1)}$ by (66) and (67) and transform them to $\mu_{i+1}$ and $v_{i+1}$.

3.1.3 If joint $i$ is a rotational joint, compute $D_{i}^{(i)}$ for $i \geq s$ by (99) or $(\bar{D}_{i}^{(i)})_{33}$ for $i < s$ by (105).

3.1.4 Compute $\phi_i$ by (110) and form $v_i$ by (97), (98), (103) or (104).

3.2 Compute $\gamma_i$ by (108) and then form (111). Finally, use the least squares method to solve $w_i$ from (111).

Step 4: If $r > 1$, do the following substeps for joint $i$ recursively from $i = r - 1$ to $i = 1$.

4.1 For joint $i$, do the following substep for each sampling point from point 1 to point $N$.

4.1.1 Use the values of $\dot{\omega}_{i+1}^{(i+1)}$, $\ddot{\omega}_{i+1}^{(i+1)}$, $\dot{h}_{i+1}^{(i+1)}$, $a_{i+1}^{(i+1)}$, $w_{i+1}^{(i+1)}$ to calculate $\mu_{i+1}^{(i+1)}$ by (59) and (64). Then compute $\mu_{i+1}^{(i+1)}$ by (66) and transform it to $\mu_{i+1}^{(i)}$.

4.2 Form

$$\begin{bmatrix} \dot{\theta}_{i}^{(i)} - \theta_{i}^{(i)} \\ \dot{\gamma}_{i}^{(i)} \end{bmatrix} = \begin{bmatrix} \alpha_{i}^{(i)} - (\mu_{i+1}^{(i+1)})_{z} \\ \gamma_{i}^{(i)} - (\mu_{i+1}^{(i+1)})_{z} \end{bmatrix}$$

and use the least squares method to solve $\dot{m}_{i}$ from (116).

As just stated, the identification procedure requires a persistently exciting trajectory along which all modes of the system should be excited [35]. Since the actuator forces of a manipulator are bound, we can describe the persistently exciting trajectory in the following mathematical form.

Definition 2: The sampling signals $\{q, \dot{q}, \ddot{q}\}$ of a trajectory are said to be persistently exciting for the least squares estimation of the system (111) if

$$M_i = \sum_{k=1}^{N} \dot{v}_k Z(v_k)T$$  \hspace{1cm} (117)

is positive definite, where $N$ is the number of sampling points in the trajectory.

It is apparent that $M_i = A_i^T A_i$. If matrix $A_i$ is of full rank, $A_i^T A_i$ is symmetric and positive definite [14]. Examining (97), (98), (103), and (104), we find that $A_i$ is of full rank for most trajectories. This is why the trajectories arbitrarily selected in the literature [3], [15] are all persistently exciting. We performed computer simulations of the identification procedure on the Stanford arm for several persistently exciting trajectories, and the identified results matched the true values very closely. A detailed report on these simulations can be found in [22].

In practical identification, there are measurement errors, which were not taken into account in the computer simulation. The measured values, especially the joint velocities and accelerations, are perturbed within a certain error bound. In regression theory, the width of the prediction interval of the estimated values is proportional to the standard deviation of the residuals (i.e., the square root of the error mean square). Least squares theory [14], [18] indicates that the upper bound of the relative error, and thus that of the residuals, is about proportional to the condition number of the excitation matrix $A_i$ in (111). As a result, the accuracy of the least squares estimation depends on the condition number of the excitation matrix. An arbitrary, persistently exciting trajectory cannot ensure small condition numbers for $A_i$, $i = 1, \ldots, n$, so it is necessary to search for an optimal exciting trajectory. Two good references [2], [13] discuss this optimization problem.

V. CONCLUSION

This paper addresses the minimal linear combinations (MLC's) of the inertia parameters of a manipulator, the inverse dynamics in terms the MLC's, and an identification procedure for estimating the MLC's. These three themes are closely related to one another in the sense of the dynamic modeling of a manipulator, and so should be treated together. Knowledge of a set of MLC's facilitates parameter identification. The purpose of identifying the parameters is to use them in the inverse dynamics to control a manipulator. This paper formulates the inverse dynamics in terms the MLC's, and an identification procedure for estimating the MLC's; the proposed formulation is almost as efficient as the most efficient formulation of the inverse dynamics [10], [16]. It is interesting that the identification procedure is derived from the formulation of the inverse dynamics. The identification procedure is simple and efficient, since it
does not require symbolic closed-form equations and it has a recursive structure.

In the literature [3], [17], there have been successful examples of experimental identification of the parameters of a manipulator whose dominant dynamics can be described by the standard Newton-Euler formulation, e.g., a direct drive robot. It is reasonable to believe that the present identification procedure is valid in practice for direct drive robots since the procedure is also based on the Newton-Euler formulation.

The effects of friction from high-ratio gear trains on the manipulator dynamics will invalidate the proposed identification procedure for manipulators with high-ratio gear trains. Further investigation will be required to extract the friction terms from the dynamic model and to estimate them separately, so that the present procedure can be applied.

APPENDIX

Proof of (29)–(31): It is easy to show that

\[
a \times (\omega \times (\omega \times b)) + b \times (\omega \times (\omega \times a)) = -\omega \times (a \times (b \times \omega)) + b \times (a \times (\omega \times a)) \tag{A1}
\]

Expanding the left-hand side of (29) and applying (A1) and the equivalence property of (20) and (28), we then obtain (29).

Equation (30) is due to the assignment of the normal driving-axis coordinate system. Substituting (16) and (17) into (30) and expanding both sides while noting that \(u_i^{(0)} = [0, 0, 0]^T\), we find that (30) is true.

It is easy to show that for any constant matrix \(A \in \mathbb{R}^{3 \times 3}\),

\[
(i-1)^{\mathbf{R}}(A \alpha + \omega \times (A \omega)) = (i-1)^{\mathbf{R}}, \alpha (i-1)^{\mathbf{R}} \omega + (i-1)^{\mathbf{R}}(\omega \times (i-1)^{\mathbf{R}}) \omega - (i-1)^{\mathbf{R}} (A \omega) \times \omega \quad (A2)
\]

Since \(\text{diag}[a, a, 0] \in \mathbb{R}^{3 \times 3}\) and \(i-1)^{\mathbf{R}}(\omega \times (i-1)^{\mathbf{R}}) \omega = \text{diag}[a, a, 0] \times \omega \), the second equality in (31) is true. Substituting (16) and (17) into (28) and replacing \(A\) with \(\text{diag}[a, a, 0]\), we have the first equality of (31).

Proof of (46): According to the recursive Newton-Euler formulation (16)–(23), we have

\[
t_i^{(0)} = -\sum_{j=i}^{n} (\mathbf{R}_{j-1} (i_j^{(0)} + \frac{\mathbf{b}_j^{(0)}}{\eta_j^{(0)}} + \mathbf{c}_j^{(0)} \times (\mathbf{f}_j^{(0)} + \eta_j^{(0)} \mathbf{g}_j^{(0)})))
\]

\[
= -t_i^{(0)} - (\mathbf{f}_E^{(i)} + \frac{\mathbf{b}_i^{(0)}}{\eta_i^{(0)}} + \mathbf{c}_i^{(0)} \times (\mathbf{f}_E^{(i)})) \tag{A3}
\]

It follows from (27)–(29) that

\[
t_i^{(0)} + \mathbf{c}_i^{(0)} \times (\mathbf{f}_E^{(i)} + \eta_i^{(0)} \mathbf{g}_i^{(0)})
\]

\[
= -\mathbf{R} (\mathbf{J}_i^{(0)} - m_j \mathbf{c}_j^{(0)} \times (\mathbf{e}_j^{(0)} \times \mathbf{c}_j^{(0)})) \tag{A4}
\]

where \(\mathbf{J}_i^{(0)} = \mathbf{I}_i^{(0)} - m_j \mathbf{c}_j^{(0)} \times (\mathbf{e}_j^{(0)} \times \mathbf{c}_j^{(0)})\). It is easy to show that

\[
\sum_{j=i}^{n} (\mathbf{R} (\mathbf{I}_j^{(0)} - m_j \mathbf{c}_j^{(0)} \times (\mathbf{e}_j^{(0)} \times \mathbf{c}_j^{(0)})) - \mathbf{I}_i^{(0)} - \frac{\mathbf{b}_i^{(0)}}{\eta_i^{(0)}} + \mathbf{c}_i^{(0)} \times (\mathbf{f}_E^{(i)}))
\]

\[
= \sum_{j=i}^{n} (\mathbf{R} (\mathbf{I}_j^{(0)} - m_j \mathbf{c}_j^{(0)} \times (\mathbf{e}_j^{(0)} \times \mathbf{c}_j^{(0)}))) \tag{A5}
\]

We substitute (A4) and (A5) into (A3) to obtain

\[
t_i^{(0)} + \mathbf{f}_E^{(i)} + \frac{\mathbf{b}_i^{(0)}}{\eta_i^{(0)}} + \mathbf{c}_i^{(0)} \times (\mathbf{f}_E^{(i)} + \eta_i^{(0)} \mathbf{g}_i^{(0)})
\]

\[
= \mathbf{k}_i^{(0)} \times (\mathbf{a}_i^{(0)} - \mathbf{g}_i^{(0)}) + \mathbf{u}_i^{(0)}
\]

\[
+ \sum_{j=i}^{n} (\mathbf{R} \mathbf{J}_j^{(0)} - \mathbf{R} \mathbf{J}_j^{(0)} \mathbf{J}_i^{(0)} - \mathbf{R} \mathbf{J}_j^{(0)} \mathbf{J}_i^{(0)} - \mathbf{R} \mathbf{J}_j^{(0)} \mathbf{J}_i^{(0)}) \tag{A6}
\]

by using (30), (32), (36), (37) and

\[
a_i^{(j+1)} = a_i^{(j)} + \mathbf{J}_j^{(0)} - \mathbf{R} (\mathbf{J}_j^{(0)} - \mathbf{R} \mathbf{J}_j^{(0)} - \mathbf{R} \mathbf{J}_j^{(0)} - \mathbf{R} \mathbf{J}_j^{(0)}) \tag{A7}
\]

Note that \(\eta_i^{(j)}\) is defined in (48). By (29) and

\[
[i_j^{(0)} \times] = [i_j^{(0)} \times] + [i_j^{(0)} \times] + [i_j^{(0)} \times] + [i_j^{(0)} \times] \tag{A8}
\]

(A6) turns out to be (46) with

\[
A_j^{(0)} = \mathbf{I}_j^{(0)} - m_j \mathbf{c}_j^{(0)} \times (\mathbf{e}_j^{(0)} \times \mathbf{c}_j^{(0)}) - \mathbf{R} \mathbf{J}_j^{(0)} - \mathbf{J}_j^{(0)} \times [i_j^{(0)} \times] = \mathbf{R} [i_j^{(0)} \times] + [i_j^{(0)} \times] + [i_j^{(0)} \times] + [i_j^{(0)} \times] \tag{A9}
\]

which is identical to (47) according to (12) and (14).

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Shih-Kuan Lin received the B.S. degree in aeronautical engineering from National Cheng Kung University, Taiwan, the M.S. degree in power mechanical engineering from National Tsing Hua University, Taiwan, and the Dr.-Ing. degree in manufacturing automation from Universität Erlangen-Nürnberg, Germany, in 1979, 1983, and 1988, respectively. From 1984 to 1988, he was a recipient of the DAAD fellowship by the Deutschen Akademischen Austauschdienst. Since February 1989, he has been an Associate Professor in the Department of Control Engineering at the National Chiao Tung University, Taiwan. His major research interests include parameter identification, robotic control, multitask and multiprocessor system, servo control, and manufacturing automation.