Numerical ranges of weighted shift matrices with periodic weights

Ming Cheng Tsai

Department of Applied Mathematics, National Chiao Tung University, Hsinchu 30010, Taiwan

ARTICLE INFO

Article history:
Received 25 February 2011
Accepted 14 April 2011
Available online 14 May 2011
Submitted by M. Tsatsomeros

AMS classification:
15A60

Keywords:
Numerical range
Weighted shift matrix
Periodic weights
Degree-$n$ homogeneous polynomial
Reducible matrix

ABSTRACT

Let $A$ be an $n$-by-$n$ ($n \geq 2$) matrix of the form

$$
\begin{bmatrix}
0 & a_1 \\
0 & \ddots \\
\vdots & \ddots & \ddots \\
0 & \ddots & \ddots & a_{n-1} \\
0 & \ddots & \ddots & \ddots & a_n
\end{bmatrix}
$$

We show that if the $a_j$’s are nonzero and their moduli are periodic, then the boundary of its numerical range contains a line segment. We also prove that $\partial W(A)$ contains a noncircular elliptic arc if and only if the $a_j$’s are nonzero, $n$ is even, $|a_1| = |a_3| = \cdots = |a_{n-1}|$, $|a_2| = |a_4| = \cdots = |a_n|$ and $|a_1| \neq |a_2|$. Finally, we give a criterion for $A$ to be reducible and completely characterize the numerical ranges of such matrices.

© 2011 Elsevier Inc. All rights reserved.

An $n$-by-$n$ ($n \geq 2$) weighted shift matrix $A$ is one of the form

$$
\begin{bmatrix}
0 & a_1 \\
0 & \ddots \\
\vdots & \ddots & \ddots \\
0 & \ddots & \ddots & \ddots & a_{n-1} \\
0 & \ddots & \ddots & \ddots & \ddots & a_n
\end{bmatrix}
$$

The contents of this paper from part of the author’s Ph.D. dissertation supervised by Pei Yuan Wu.

E-mail address: mctsai2@gmail.com

0024-3795/$-$ see front matter © 2011 Elsevier Inc. All rights reserved.
where the \( a_j \)'s, called the weights of \( A \), are complex numbers. The purpose of this paper is to study the numerical ranges of such matrices with periodic weights.

Recall that the \textit{numerical range} \( W(A) \) of an \( n \)-by-\( n \) complex matrix \( A \) is by definition the subset \( \{ \langle Ax, x \rangle : x \in \mathbb{C}^n, |x| = 1 \} \) of the complex plane, where \( \langle \cdot, \cdot \rangle \) and \( \| \cdot \| \) denote the standard inner product and Euclidean norm in \( \mathbb{C}^n \). It is known that \( W(A) \) is a nonempty compact convex subset of \( \mathbb{C} \). For other properties, the reader may consult \cite[Chapter 1]{4} or \cite{2}.

The study of the numerical ranges of the weighted shift matrices was started in \cite{6}. It was proven in \cite[Theorem 1]{6} that the boundary of the numerical range of such a matrix \( A \) has a line segment if and only if the weights are all nonzero and all its \((n-1)\)-by-\((n-1)\) principal submatrices have identical numerical ranges (which are necessarily circular discs centered at the origin). In such discussions, we may assume that the weights are all nonnegative. In Theorem 1 below, we show that if the weights \( a_1, \ldots, a_n \) of an \( n \)-by-\( n \) weighted shift matrix \( A \) are nonzero and periodic with period \( k \), then \( W(A) = W(B) \), where \( B = \sum_{j=0}^{(n/k)-1} e^{i\theta j} C \) and \( C \) is the \( k \)-by-\( k \) weighted shift matrix with weights \( a_1, \ldots, a_{k-1}, \alpha a_k, \alpha = (a_1 \cdots a_n)^{1/k} \) and \( \theta = 2\pi/k \). In this case, the boundary of \( W(A) \) has a line segment. In Theorem 3, we give a necessary and sufficient condition for the boundary of \( W(A) \) to have a noncircular elliptic arc. More specifically, it is shown that this is the case if and only if the \( a_j \)'s are nonzero, \( n \) is even, \( |a_1| = |a_2| = \cdots = |a_{n-1}|, |a_2| = |a_4| = \cdots = |a_n| \) and \( |a_1| \neq |a_2| \). For \( n = 4 \), this essentially generalizes \cite[Proposition 12]{6}.

Finally, we give a criterion for \( A \) to be reducible and characterize their numerical ranges in Theorem 4. In particular, it says that, for \( n = 4 \), \( A \) is reducible if and only if either (a) \( a_i = a_j = 0 \) for some \( i \) and \( j, 1 \leq i < j \leq n \), or (b) all the \( a_j \)'s have the same absolute values.

For an \( n \)-by-\( n \) matrix \( A \), let \( A^T \) denote its transpose, \( A^* \) its adjoint, \( Re \) its real part \((A + A^*)/2 \) and \( Im \) its imaginary part \((A - A^*)/2i \). Let \( A \) its imaginary part \((A - A^*)/2i \). For \( 1 \leq i_1 < \cdots < l_m \leq n \), let \( A[i_1, \ldots, l_m] \) denote the \((n-m)\)-by-\((n-m)\) principal submatrix of \( A \) obtained by deleting its rows and columns indexed by \( i_1, \ldots, l_m \). The numerical radius \( w(A) \) and generalized Crawford number \( w_0(A) \) of \( A \) are, by definition, \( \max \{ |z| : z \in W(A) \} \) and \( \min \{ |z| : z \in \partial W(A) \} \), respectively. A diagonal matrix with diagonals \( a_1, \ldots, a_n \) is denoted by diag\((a_1, \ldots, a_n)\). Our basic reference for properties of matrices is \cite{3}.

For an \( n \)-by-\( n \) matrix \( A \), consider the degree-\( n \) homogeneous polynomial \( p_A(x, y, z) = \det(xRe A + yIm A + zI_n) \). A result of Kippenhahn \cite[p. 199]{5} says that the numerical range \( W(A) \) is the convex hull of the real points in the dual of the curve \( p_A(x, y, z) = 0 \), that is, \( W(A) = \{ a + ib \in \mathbb{C} : a, b \ real, ax + by + z = 0 \} \). Here, for any subset \( \Delta \) of \( \mathbb{C} \), \( \Delta^\circ \) denotes its convex hull, that is, \( \Delta^\circ \) is the smallest convex set containing \( \Delta \).

For any nonzero complex number \( z = x + iy \) (\( x \) and \( y \) real), \( arg z \) is the angle \( \theta \), \( 0 \leq \theta < 2\pi \), from the positive \( x \)-axis to the vector \((x, y)\). If \( z = 0 \), then \( arg z \) can be an arbitrary real number. In the following, let \( \omega_n = e^{2\pi i/n} \) for \( n \geq 1 \).

The main result of this paper is the following.

\textbf{Theorem 1.} \textit{Let} \( A \) \textit{be an} \( n \)-by-\( n \) (\( n \geq 3 \)) \textit{weighted shift matrix with nonzero weights} \( a_1, \ldots, a_n \). \textit{Assume that} \( |a_j| = |a_{k+j}| = \cdots = |a_{(m-1)k+j}| \) for all \( 1 \leq j \leq k \), \textit{where} \( n = km \) \textit{for some} \( k \) \textit{and} \( m \), \( k, m \geq 2 \). \textit{Then}

\begin{enumerate}
\item \( A \) \textit{is reducible and} \( W(A) = W(B) \), \textit{where} \( B = C \oplus (e^{i\theta} C) \oplus \cdots \oplus (e^{i(k-1)\theta} C) \) \textit{and} \( C \) \textit{is the} \( k \)-by-\( k \) \textit{weighted shift matrix with weights} \( a_1, \ldots, a_{k-1}, \alpha a_k, \alpha = (a_1 \cdots a_n)^{1/m} / (a_1 \cdots a_k) \) \textit{and} \( \theta = 2\pi/k \).
\item \( \partial W(A) \) \textit{has a line segment} \( L \) \textit{and} \( dist(0, L) = w_0(A) = w(A[i]) \) \textit{is maximum zero of} \( \det(\lambda I_{n-1} - Re A[i]) \) \textit{for every} \( i \), \( 1 \leq i \leq n \).
\end{enumerate}

\textit{Note that} \( \partial W(A) \) \textit{has a line segment for} \( k = 1 \) \textit{by} \cite[Proposition 4]{6}.

\textit{An easy consequence of the preceding theorem is the following:}

\textbf{Corollary 2.} \textit{Let} \( A \) \textit{be an} \( n \)-by-\( n \) (\( n \geq 3 \)) \textit{weighted shift matrix with weights} \( a_1, \ldots, a_n \). \textit{Suppose that} \( n - 2 \) \textit{of the} \( a_j \)'s \textit{have equal absolute value and the remaining two terms are} \( a_k \) and \( a_l \). \textit{Then} \( \partial W(A) \) \textit{has a line segment if and only if all the} \( a_j \)'s \textit{are nonzero and either}

\begin{enumerate}
\item \( n \) \textit{is even,} \( |k - l| = n/2 \), \( |a_k| = |a_l| \), \textit{or}
\item \textit{all the} \( a_j \)'s \textit{have the same absolute values.} \end{enumerate}
Proof. The sufficiency follows easily from Theorem 1. Now we prove the necessity. By [6, Theorem 1], we have that the \( a_i \)'s are nonzero and \( W(A[1]) = \cdots = W(A[n]) \). If \( n \) is even and \( |k - l| = n/2 \), then we may assume that \( k = n/2, l = n, a_i = a_j > 0 \) for \( 1 \leq i < j \leq n - 1, i, j \neq n/2 \) and \( a_{n/2} \), \( a_n > 0 \) by [6, Lemma 2(1) and (2)]. Also, \( W(A[n/2]) = W(A[n]) \) implies that \( |a_{n/2}| = |a_n| \) by [6, Lemma 5]. Otherwise, we may assume that \( 1 \leq k < n/2, l = n, a_i = a_j > 0 \) for \( 1 \leq i < j \leq n - 1, i, j \neq k \) and \( a_k, a_n > 0 \) by [6, Lemma 2(1) and (2)]. Let \( a \equiv a_i \), where \( i \neq k, n \). Note that we have \( W(A[k]) = W(A[2k]) = W(A[n - k]) = W(A[n]) \) and \( A[n] \) is the \((n - 1)\)-by-\((n - 1)\) matrix \( W_i \) of \( |a_i| \), where \( s_{i,i+1} = a \) for \( 1 \leq i < n - 2, i \neq k, n \), \( s_{k,k+1} = a_k \) and \( s_{i,j} = 0 \) otherwise. By [6, Lemma 2(1)], we may assume that \( A[k] \) is the \((n - 1)\)-by-\((n - 1)\) matrix \( [t_{ij}]_{i,j=1}^{n-1} \), where \( t_{i,i+1} = a \) for \( 1 \leq i < n - 2, i \neq k, n - k, n - k + 1 = a_n \) and \( t_{i,j} = 0 \) otherwise. For the orders of \( a_n, a_k, a \), consider the following cases:

1. \( a_n \geq a \geq a_k \) or \( a_n \leq a \leq a_k \). Since \( W(A[n]) = W(A[k]) \), by [6, Lemma 5(2)], we infer that
   \[
   a_n = a = a_k.
   \]

2. \( a_n \geq a_k \geq a \) or \( a_n \leq a_k \leq a \). By [6, Lemma 2(1)], we may assume that \( A[n - k] \) is the \((n - 1)\)-by-\((n - 1)\) matrix \( [u_{ij}]_{i,j=1}^{n-1} \), where \( u_{i,i+1} = a \) for \( 1 \leq i < n - 2, i \neq k, 2k, u_{k,k+1} = a_n, u_{2k,2k+1} = a_k \) and \( u_{i,j} = 0 \) otherwise. Since \( W(A[n]) = W(A[n - k]) \), by [6, Lemma 5(2)], we also infer that \( a_n = a = a_k \).

3. \( a_k \geq a_n \geq a \) or \( a_k \leq a_n \leq a \). By [6, Lemma 2(1)], we may assume that \( A[2k] \) is the \((n - 1)\)-by-\((n - 1)\) matrix \( [v_{ij}]_{i,j=1}^{n-1} \), where \( v_{i,i+1} = a \) for \( 1 \leq i < n - 2, i \neq n - k, n - 2k, v_{n-2k,n-2k+1} = a_n, v_{n-k,n-k+1} = a_k \) and \( v_{i,j} = 0 \) otherwise. Since \( W(A[k]) = W(A[2k]) \), by [6, Lemma 5(2)], we obtain that \( a_n = a = a_k \) and complete the proof. \( \square \)

We are now ready to prove Theorem 1.

Proof of Theorem 1

(a) Let \( B = C \oplus (e^{i\theta} C) \oplus \cdots \oplus (e^{i(m-1)\theta}) C \), where \( C \) is the \( k \)-by-\( k \) weighted shift matrix with weights \( a_1, \ldots, a_{k-1}, \alpha \alpha_k, \alpha = (a_1 \cdots a_n)^{1/m} / (a_1 \cdots a_k) \) and \( \theta = 2\pi/n \). Since \( |a_j| = |a_{j+k}| = \cdots = |a_{(m-1)k+j}| \neq 0 \) for \( 1 \leq j \leq k \) and \( \arg (a_1 \cdots a_n)/((a_1 \cdots a_k)m^m a^m) = 0 \), we may assume that \( A \) is the \( n \)-by-\( n \) weighted shift matrix with periodic weights \( a_1, \ldots, a_{k-1}, \alpha \alpha_k, \ldots, a_1, \ldots, a_{k-1}, \alpha \alpha_k \)

by [6, Lemma 2(2)]. Let the matrix \( xRe A + yIm A + zI_n \) be partitioned as

\[
\begin{bmatrix}
C_{11} & \cdots & C_{1m} \\
\vdots & & \vdots \\
C_{m1} & \cdots & C_{mm}
\end{bmatrix}
\]

sizes \( k \)-by-\( k \) for all \( i, j, 1 \leq i, j \leq n \). Since \( C_{ij} + \cdots + C_{mj} = xRe C + yIm C + zI_k \), for all \( j, 1 \leq j \leq m \), we have

\[
p_A(x, y, z) = \det(xRe A + yIm A + zI_n)
\]

\[
= \det \begin{bmatrix}
C_{11} + \cdots + C_{1m} & C_{12} + \cdots + C_{2m} & \cdots & C_{1m} + \cdots + C_{mm} \\
C_{21} & C_{22} & \cdots & C_{2m} \\
\vdots & \vdots & & \vdots \\
C_{m1} & C_{m2} & \cdots & C_{mm}
\end{bmatrix}
\]

\[
= \det \begin{bmatrix}
xRe C + yIm C + zI_k & 0 & \cdots & 0 \\
* & * & * & \cdots \\
* & * & * & \cdots \\
* & * & * & \cdots 
\end{bmatrix}.
\]
Hence $p_C|p_A$. Since $A$ and $\omega_j^1 A$ are unitarily equivalent for all integer $j$, then $p_C|p_{\omega_j^1 A}$ for all integer $j$, consequently, $p_{\omega_j^1 C}|p_A$ for all $j = 0, 1, \ldots, m - 1$. Note that the real foci of the curve $p_{\omega_j^1 C} = 0$ are eigenvalues of $\omega_j^1 C$ for each $j$. Since $\sigma(C) = \{\lambda, \omega_k \lambda, \ldots, \omega_k^{-1} \lambda\}$, where $\lambda = (a_1 \cdots a_k)^{1/k}$, it follows that $\sigma(\omega_j^1 C) \cap \sigma(\omega_j C) = 0$ for any $0 \leq i < j \leq m - 1$, thus the homogeneous polynomials $p_{\omega_j^1 C}$ and $p_{\omega_j C}$ have no common factor for any $0 \leq i < j \leq m - 1$. Therefore, we deduce that $p_A = \prod_{j=0}^{m-1} p_{\omega_j^1 C}$ or $W(A) = W(B)$. This completes the proof.

(by [6, Proposition 3(6)] and its proof, we need only prove that $\partial W(A)$ has a line segment. Since the $a_j$'s are nonzero, from [6, Lemma 2(2)], we may assume that $a_j > 0$ for all $j$. Then $C$ is a $k$-by-$k$ weighted shift matrix with positive weights, thus $W(C) \in W(C)$. Let $D = \{x \in \mathbb{C} : |x| < \omega(C)\}$ and $B = C \oplus B'$, where $B' = \omega_n C \oplus \omega_n^2 C \oplus \cdots \oplus \omega_n^{m-1} C$. For each $j = 0, 1, \ldots, m - 1$, by [6, Proposition 3(4)], we have $W(\omega_j^1 C) \subseteq D$ and $W(\omega_j^2 C) \cap \partial D = \{\omega_j^1 \omega_k^1 w(C) : i = 0, 1, \ldots, k - 1\}$. Since each point on $\partial D$ is an extreme point of $D$, thus $W(B') \subseteq D$ and $W(B') \cap \partial D = \{\omega_j^1 w(C) : j = 0, 1, \ldots, n - 1\} \setminus \{\omega_j^1 w(C) : j = 0, 1, \ldots, k - 1\}$. It follows that $W(C) \in W(C) \setminus W(B')$ and $\omega_n w(C) \notin W(B') \setminus W(C)$, that is, $W(C) \cap W(B') = \{\omega_j^1 \omega_k w(C) : j = 0, 1, \ldots, n - 1\} \setminus \{\omega_j^1 w(C) : j = 0, 1, \ldots, k - 1\}$.

The next theorem gives a necessary and sufficient condition for an $n$-by-$n$ weighted shift matrix $A$ to have a noncircular elliptic arc in $\partial W(A)$. Moreover, in this case, $\partial W(A)$ also has a line segment.

**Theorem 3.** Let $A$ be an $n$-by-$n$ $(n \geq 3)$ weighted shift matrix with weights $a_1, \ldots, a_n$. Then $\partial W(A)$ has a noncircular elliptic arc if and only if the $a_j$'s are nonzero, $n$ is even, $|a_1| = |a_2| = \cdots = |a_{n-1}|$, $|a_2| = |a_4| = \cdots = |a_n|$ and $|a_1| \neq |a_2|$. In this case, $W(A) = W(B)$, where $B = C \oplus (e^{i\theta} C) \oplus \cdots \oplus (e^{(n/2-1)i\theta} C)$. Let $C = \begin{bmatrix} 0 & a_1 \\ a_2 & 0 \end{bmatrix}$, $\alpha = (a_1 \cdots a_n)^{2/n} / (a_1 a_2)$ and $\theta = 2\pi / n$, and $\partial W(A)$ has a line segment.

**Proof.** The sufficiency follows easily from Theorem 1(a) and the fact that $W\left(\begin{bmatrix} 0 & a_1 \\ a_2 & 0 \end{bmatrix}\right)$ is a noncircular elliptic disc as $|a_1| \neq |a_2|$ and both are nonzero, where $\alpha = (a_1 \cdots a_n)^{2/n} / (a_1 a_2)$.

To prove the necessity, by [6, Lemma 2(2)], we have that $A$ is unitarily equivalent to $e^{i\phi} A'$, where $\phi = (\sum_{j=1}^{n} \arg a_j) / n$ and $A'$ is the $n$-by-$n$ weighted shift matrix with weights $|a_1|, \ldots, |a_n|$. Then $\sigma(A) = \{|a_1 \cdots a_n|^{1/n} \omega_j^1 : j = 0, 1, \ldots, n - 1\}$. Since $\partial W(A)$ has a noncircular elliptic arc, by [1, Theorem], there is a 2-by-2 matrix $C_1$ such that $p_{C_1}|p_{A'}$ and $\sigma(C_1) \subseteq \sigma(A')$, say, $\sigma(C_1) = \{\beta, \gamma\}$. From [6, Proposition 3(1)], we infer that $p_{\omega_j^1 C_1}|p_{A'}$ and $\sigma(\omega_j^1 C_1) \subseteq \sigma(A')$ for all $j = 0, 1, \ldots, n - 1$. Therefore, $\sigma(A') \supseteq \{\omega_j^1 \beta : j = 0, \ldots, n - 1\} \cup \{\omega_j^1 \gamma : j = 0, \ldots, n - 1\}$. Since these sets consist of $n$ distinct elements, we deduce that $\sigma(A') = \{\omega_j^1 \beta : j = 0, \ldots, n - 1\} = \{\omega_j^1 \gamma : j = 0, \ldots, n - 1\}$. Therefore, we may assume that $\beta = |a_1 \cdots a_n|^{1/n}$ and $\gamma = \omega_j^1 \beta$ for some $j_0$. Now, if $\omega_j^1 \beta \neq -1$ or $n$ is odd, then these irreducible homogeneous polynomials $p_{C_1}, p_{\omega_1 C_1}, \ldots, p_{\omega_n^{(n/2)-1} C_1}$ are distinct, it follows that $p_{A'}$ can be divided by the homogeneous polynomial $\prod_{j=0}^{n/2} p_{\omega_j^1 C_1}$ of degree $2(n/2 + 1) > n$. This contradicts the fact that $p_{A'}$ is of degree $n$. Therefore, we deduce that $\omega_j^1 \beta = -1$ and $n$ is even.
Moreover, \( p_{A'} = \prod_{j=0}^{(n/2)-1} p_{\alpha j} c_1 \). On the other hand, since \( C_1 \) is 2-by-2 with eigenvalues \( \pm|a_1 \cdots a_n|^{1/n} \), by unitarily equivalence, we may assume that \( C_1 = \begin{bmatrix} 0 & b_1 \\ b_2 & 0 \end{bmatrix} \), where \( b_1, b_2 > 0 \), \( b_1 \neq b_2 \) and \( b_1b_2 = |a_1 \cdots a_n|^{2/n} \). Let \( B' = C_1 \oplus \omega_n C_1 \oplus \cdots \oplus \omega_n^{(n/2)-1} C_1 \) and \( B_1 \) be the \( n \)-by-\( n \) weighted shift matrix with periodic weights \( b_1, b_2, b_1, b_2, \ldots, b_1, b_2 \). By Theorem 1(a), we have \( p_{B_1} = p_{B'} = p_{A'} \). Compute now the coefficients of \( (x^2 + y^2)^{n/2} \) and \( y^n \) of \( p_{A'} \) and \( p_{B_1} \). Since \( p_{A'} = p_{B_1} \), we have \( \sum_{j=1}^{n} |a_j|^2 = (b_1^2 + b_2^2)n/2 \) and \((\prod_{j=1}^{n} |a_{2j-1}|) - (\prod_{j=1}^{n} |a_{2j}|)^2 = (b_1^{n/2} - b_2^{n/2})^2 \). Hence we may assume that \( b_1^{n/2} - b_2^{n/2} = \prod_{j=1}^{n} |a_{2j-1}| - \prod_{j=1}^{n} |a_{2j}| \). In addition, \( b_1b_2 = |a_1 \cdots a_n|^{2/n} \) implies that \( b_1^{n/2} = \prod_{j=1}^{n} |a_{2j-1}| \) and \( b_2^{n/2} = \prod_{j=1}^{n} |a_{2j}| \). We also have

\[
\sum_{j=1}^{n} |a_j|^2 = \sum_{j=1}^{n} |a_{2j-1}|^2 + \sum_{j=1}^{n} |a_{2j}|^2 \\
\geq n \left( \prod_{j=1}^{n} |a_{2j-1}|^2 \right)^{\frac{2}{n}} + n \left( \prod_{j=1}^{n} |a_{2j}|^2 \right)^{\frac{2}{n}} = \frac{n}{2} \left( b_1^2 + b_2^2 \right).
\]

Therefore, the equality holds if and only if \( b_1 = |a_{2j-1}| \neq 0 \), \( b_2 = |a_{2j}| \neq 0 \) for all \( j = 1 \leq j \leq n/2 \) and \( b_1 \neq b_2 \). Let \( C = e^{i\phi} C_1 \) and \( B = e^{i\phi} B_1 \). Then \( C \) is unitarily equivalent to \( \begin{bmatrix} 0 & a_1 \\ \alpha a_2 & 0 \end{bmatrix} \), where \( \alpha = e^{i(2\phi - \arg a_1 - \arg a_2)} = (a_1 \cdots a_n)^{2/n} / (a_1 a_2) \) and \( W(A) = e^{i\phi} W(A') = e^{i\phi} W(B_1) = W(B) \). This proves our assertion. In particular, it follows from Theorem 1(b) that \( \partial W(A) \) has a line segment. \( \square \)

Note that the weighted shift matrix \( A \) in the above theorem is a special case of the ones considered in Theorem 1. The next theorem is another special case. Recall that a matrix \( \mathbf{A} \) is said to be reducible if it is unitarily equivalent to the direct sum of two matrices; otherwise, \( \mathbf{A} \) is irreducible. We characterize those \( n \)-by-\( n \) weighted shift matrices \( \mathbf{A} \) which are reducible in the following theorem.

**Theorem 4.** Let \( A \) be an \( n \)-by-\( n \) \( (n \geq 2) \) weighted shift matrix with weights \( a_1, \ldots, a_n \). Then \( A \) is reducible if and only if one of the following cases holds:

1. \( a_i = a_j = 0 \) for some \( 1 \leq i < j \leq n \).
2. \( n \) is odd, \( |a_i| = |a_j| \neq 0 \) for all \( 1 \leq i < j \leq n \).
3. \( n \) is even, \( |a_i| = |a_{i+(n/2)}| \neq 0 \) for all \( 1 \leq i \leq n/2 \).

In case (1), \( A \) is unitarily equivalent to \( B_1 \oplus B_2 \), where \( B_1 \) and \( B_2 \) are the weighted shift matrices with weights \( a_{j+1}, \ldots, a_{i-1}, 0 \) and \( a_{i+1}, \ldots, a_{j-1}, 0 \), respectively (\( a_r = a_{n+r} \) for \( 1 \leq r \leq n \), \( B_1 \equiv [0] \) if \( i = 1, j = n \) and \( B_2 \equiv [0] \) if \( i = j - 1 \)). Hence \( W(A) \) is a circular disc centered at the origin. In case (2), \( A \) is unitarily equivalent to \( \text{diag}(\alpha, \alpha \omega_1, \ldots, \alpha \omega_1^{n-1}) \), where \( \omega_n = e^{2\pi i / n} \) and \( \alpha = (a_1 \cdots a_n)^{1/n} \). Hence \( W(A) \) is a closed regular \( n \)-gonal region centered at the origin and the distance from the origin to its vertices equals \( |a_1 \cdots a_n|^{1/n} \). In case (3), \( A \) is unitarily equivalent to \( A_1 \oplus e^{i\theta} A_1 \), where \( \theta = 2\pi / n \) and \( A_1 \) is an \( (n/2) \)-by-\( (n/2) \) weighted shift matrix with weights \( a_1, \ldots, a_{(n/2)-1}, \alpha a_{n/2}, \alpha = (a_1 \cdots a_n)^{1/2} / (a_1 \cdots a_{n/2}) \). In particular, \( \partial W(A) \) has a line segment.

**Proof**

1. Let \( a_i = a_j = 0 \) for some \( i, j = 1 \leq i < j \leq n \). Also, by [6, Lemma 2(1)], we may assume that \( j = n \). Then \( A = B_1 \oplus B_2 \), where \( B_1 \) and \( B_2 \) are the weighted shift matrices with weights \( a_{j+1}, \ldots, a_n, a_1, \ldots, a_{i-1}, 0 \) and \( a_{i+1}, \ldots, a_{j-1}, 0 \), respectively (\( a_r = a_{n+r} \) for \( 1 \leq r \leq n \), \( B_1 \equiv [0] \) for \( 1 \leq i < j \).
if \( i = 1, j = n \) and \( B_2 \equiv [0] \) if \( i = j - 1 \). Hence \( W(A) \) is a circular disc centered at the origin. Let \( a_i = 0 \) for some \( i, 1 \leq i \leq n \), and \( a_j \neq 0 \) for all \( j \neq i \). Again, by [6, Lemma 2(1)], we may assume that \( i = n \). Then for any orthogonal projection \( P = [p_{ij}]_{i,j=1}^n \) such that \( AP = PA \), we have 
\[
a_i (p_{1,i} - p_{i,1}) = a_{i+1} (p_{i+1,i+1} - p_{i+2,i+2}) = 0 \quad 0 \leq i \leq n - 1.\]

In addition, \( AP = PA \) also implies that \( a_i p_{i+1,i+1} = 0 \) for \( 0 \leq i \leq n - 1 \). We substitute \( p_{i+1,1} = 0 \) in these equalities for \( AP = PA \). Then \( a_i p_{i+1,2} = a_i p_{i,1} = 0 \) for \( 2 \leq i \leq n - 1 \). Proceeding successively with the remaining equalities for \( AP = PA \), we have \( p_{ij} = 0 \) for \( i > j \). Hence the assumption \( P = P^* = P^2 \) implies that \( P = P = P_i \). Therefore, \( A \) is irreducible.

(2) If \( n \) is odd and \( a_i \neq 0 \) for all \( 1 \leq i \leq n \), then we may assume that \( a_i > 0 \) by [6, Lemma 2(2)]. For any orthogonal projection \( P = [p_{ij}]_{i,j=1}^n \) such that \( AP = PA \), we have \( a_i (p_{1,i} - p_{i,1}) = a_2 (p_{2,2} - p_{3,3}) = \cdots = a_n (p_{n,n} - p_{1,1}) = 0 \). Thus \( p_{1,1} = p_{2,2} = \cdots = p_{n,n} \). In addition, \( AP = PA \) also implies that 
\[
a_i p_{i+1,i+1} = a_{i+1} (p_{i+1,i+1} - p_{i+2,i+2}) = 0 \quad 1 \leq i \leq n - 1.\]

Since \( P = P^* \), we have 
\[
a_{i+1} p_{i+1,i+2} = a(p_{i+1,i+1}) \quad 1 \leq i \leq n - 1.\]

Thus \( a_i > 0 \) for all \( 1 \leq i \leq n \). Since \( n - 1 \) is even, by the same process, we have \( p_{ij} = 0 \) for all \( i < j \) and \( a_i = \cdots = a_n \). Thus \( P = P^* = P^2 \) implies that \( P \) equals \( 0 \) or \( I_n \), or \( a_i = \cdots = a_n \). That is, \( A \) is reducible if and only if \( |a_i| = |a_n| = 0 \). Hence the assertion on \( W(A) \) follows from [6, Proposition 4].

(3) If \( n \) is even and \( a_i \neq 0 \) for all \( 1 \leq i \leq n \), then we may assume that \( a_i > 0 \) by [6, Lemma 2(2)]. For any orthogonal projection \( P = [p_{ij}]_{i,j=1}^n \) such that \( AP = PA \), following a similar argument as in the proof of (2), we obtain 
\[
p_{1,1} = p_{2,2} = \cdots = p_{n,n} \quad \text{and} \quad p_{ij} = 0 \quad \text{all} \quad i \neq j, |j - i| \neq n/2.\]

In addition, we also have 
\[
a_{i+1} (p_{i,i+1}) = a_{n/2} (p_{i,n/2}) + a_i (p_{i,i+1} + p_{i+1,i}) = a_{n/2+i} p_{n/2+i} + a_i (p_{i,i+1}) \quad \text{for every} \quad i, 1 \leq i \leq n/2.\]

Hence \( P = P^* \) implies that \( P \) equals \( 0 \) or \( I_n \), or \( a_i = a_{n/2} + a_{i+1} = a_n/2 \). Therefore, \( A \) is reducible if and only if \( |a_i| = |a_{n/2}+i| \) for all \( i, 1 \leq i \leq n/2 \). Hence \( \partial W(A) \) has a line segment by Theorem 1(b). Moreover, by [6, Lemma 2(2)], \( A \) is unitarily equivalent to \( e^{i\psi} B \), where \( \psi = (\sum_{j=1}^n \arg a_j)/n \) and \( B \) is the \( n \times n \) weighted shift matrix with weights \( |a_1|, \ldots , |a_n/2|, |a_1|, \ldots , |a_n/2| \). Let \( U = \left( 1/(\sqrt{2}) \right) \begin{bmatrix} I_{n/2} & I_{n/2} \\ I_{n/2} & -I_{n/2} \end{bmatrix} \).

Then \( U^*BU = B_1 \oplus e^{i\theta} B_1 \), where \( \theta = 2\pi/\alpha \in B_1 \) is the \((n/2)\)-by-(\(n/2\)) weighted shift matrix with weights \( |a_1|, \ldots , |a_n/2| \). Hence \( A \) is unitarily equivalent to \( e^{i\psi} B_1 \oplus e^{i\theta} (e^{i\psi} B_1) \). Let \( A_1 = e^{i\psi} B_1 \). Then \( A_1 \) is the \((n/2)\)-by-(\(n/2\)) weighted shift matrix with weights \( a_1, \ldots , a_{n/2-1} \), \( |a_{n/2}| \), where \( \alpha = e^{i\phi} \) and \( \phi = (n/2)\theta - (\sum_{j=1}^{n/2} \arg a_j) = (n/2)\sum_{j=1}^{n/2} \arg a_j)/n - (\sum_{j=1}^{n/2} \arg a_j) = (\sum_{j=1}^{n/2} \arg a_{n/2}+j - \sum_{j=1}^{n/2} \arg a_j)/2 \). This proves our assertion. \( \square \)

An immediate corollary of Theorem 4 and [1, Theorem] is the following:

Corollary 5. Let \( A \) be an \( n \)-by-\( n \) (\( n \geq 3 \)) weighted shift matrix with weights \( a_1, \ldots , a_n \) and \( a_i = 0 \) for some \( i, 1 \leq i \leq n \). Then

(1) \( p_A \) is reducible.
(2) \( A \) is reducible if and only if \( a_j = 0 \) for some \( j \neq i, 1 \leq j \leq n \).

Recall that the reducibility of an \( n \)-by-\( n \) matrix \( A \) implies the reducibility of \( p_A \) but the converse is in general not true. We give two examples of weighted shift matrices \( A \) for which \( p_A \) is reducible but \( A \) is irreducible.

Example 6

(1) If \( A = F_n \) (\( n \geq 3 \)), then \( A \) is irreducible, \( p_A \) is reducible and \( \partial W(A) \) has no line segment.
(2) If \( A \) is a \( 6 \)-by-\( 6 \) weighted shift matrix with weights \( 1, 2, 1, 2, 1, 2 \), then \( A \) is irreducible, \( p_A \) is reducible but \( \partial W(A) \) has a line segment.
Proof

(1) From [6, Proposition 3(3)], we obtain that $W(A)$ is a circular disc centered at the origin. Hence the assertion follows directly from [1, Theorem] and Theorem 4.

(2) Follow directly from Theorems 1 and 4. □

Acknowledgements

The author would like to thank Professor Pei Yuan Wu, the thesis advisor of the author, for useful discussion. He is also grateful to the (anonymous) referee whose comments greatly help in improving his presentation.

References