On positive solutions of some nonlinear differential equations – A probabilistic approach

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Abstract

By using connections between superdiffusions and partial differential equations (established recently by Dynkin, 1991), we study the structure of the set of all positive (bounded or unbounded) solutions for a class of nonlinear elliptic equations. We obtain a complete classification of all bounded solutions. Under more restrictive assumptions, we prove the uniqueness property of unbounded solutions, which was observed earlier by Cheng and Ni (1992).

Keywords: Branching particle systems; Measure-valued processes; Nonlinear elliptic equation; Range; Superdiffusions.

1. Introduction

Throughout this paper we consider positive solutions of the following nonlinear differential equation:

$$Lu(x) = k(x)u^\alpha(x), \quad x \in \mathbb{R}^d,$$

where \(1 < \alpha \leq 2\), \(k\) is a bounded strictly positive continuous function on \(\mathbb{R}^d\) satisfying condition (6) below, and \(L\) is a differential operator in \(\mathbb{R}^d\) of the form

$$L = \sum_{i,j=1}^{d} a_{ij} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} + \sum_{i} b_i \frac{\partial}{\partial x_i}$$

such that it satisfies the following:

1.a) The functions \(a_{ij} = a_{ji}\) and \(b_i\) are bounded smooth functions in \(\mathbb{R}^d\).

1.b) There exists a constant \(c > 0\) such that

$$\sum_{i,j=1}^{d} a_{ij}(x) u_i u_j \geq c \sum u_i^2$$

for all \(x \in \mathbb{R}^n\) and all \(u_1, u_2, ..., u_d\).

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If \( k = 0 \), then Eq. (1) becomes a linear equation and it can be studied probabilistically by using paths of a diffusion \( \zeta = (\xi_t, \Pi_t) \) with the generator \( L \). By using the superdiffusion \( X = (X_t, \Gamma_t, P_\mu) \) with parameters \( (L, \psi) \), where \( \psi(x, z) = k(x)z^2 \), we shall investigate the structure of the set of all positive solutions of Eq. (1).

If \( L \) corresponds to a recurrent diffusion, then there is no nontrivial bounded position solution to (1). (The following argument is provided by an anonymous referee. Assume \( u \) is such a solution and choose \( x_0 \in \mathbb{R}^d \) and a ball \( B \) such that \( u(x_0) > \sup_{y \in B} u(y) \). By Itô's formula, \( \Pi_{x_0}u(\zeta_{t \wedge \tau_B}) \geq u(x_0) \), where \( \tau_B = \inf\{t \geq 0, \ \xi_t \in B\} \). By recurrence, \( \Pi_{x_0}[\tau_B < \infty] = 1 \). Thus letting \( t \to \infty \) gives \( u(x_0) > \Pi_{x_0}u(\zeta_{\tau_B}) \geq u(x_0) \), a contradiction.) Therefore, we assume further that \( L \) corresponds to a transient diffusion.

The superdiffusion \( X = (X_t, \Gamma_t, P_\mu) \) is a branching measure-valued Markov process describing the evolution of a random cloud. It can be obtained as a limit of branching particle systems by speeding up the branching rate, decreasing the mass of particles, and increasing the number of particles. For every \( t > 0 \), the random measure \( X_t \) is a limit of mass distribution of branching particle systems \( X^\beta \) at time \( t \), as \( \beta \to 0 \). For every \( \tau \), the first exit time of \( \zeta \) from a domain \( D \subset \mathbb{R}^d \), the corresponding random measure \( X_{\tau} \) can be obtained as a limit, as \( \beta \to 0 \), of the mass distribution of the particle systems \( X^\beta \) at the first exit time from \( D \) (see Section 2 for more detail). Let \( \tau_n \) be the first exit time of \( \zeta \) from the Euclidean ball of radius \( n \), centered at 0 and let \( H \) stand for the set of all bounded positive functions \( h \) on \( \mathbb{R}^d \). Denote by \( \langle f, \mu \rangle \) the integral of \( f \) with respect to \( \mu \). We write \( P_Y \) for the expected value of a random variable \( Y \) on a probability space \( (\Omega, \mathcal{F}, \mathcal{P}) \). We obtain the following in Section 3.

**Theorem I.** (a) If \( h \in H \), then \( Z_h = \lim_{n \to \infty} \langle h, X_{\tau_n} \rangle \) exists a.s. (which means \( P_\mu \)-a.s. for all \( \mu \)) and the function

\[
v_h(x) = -\log P_{x_0} \exp\{-Z_h\}
\]

is the unique positive solution of (1) with \( u = h \) at \( \infty \). (If \( u \) and \( v \) are two functions on \( \mathbb{R}^d \), we write \( u = v \) at \( \infty \) if \( u(x) - v(x) \to 0 \) as \( ||x|| \to \infty \).

(b) If \( u \) is a bounded solution of (1), then \( u = v_h \) for some \( h \in H \).

Therefore we characterize all bounded solutions of (1). (A similar result was established earlier by Cheng and Ni (1992) under more restrictive assumptions.)

Denote by \( E \) the set of all unbounded solutions \( u(x) \) of (1) with \( u(x) \to \infty \) as \( ||x|| \to \infty \). The following theorem implies that \( E \) is not empty.

**Theorem II.** (a) The function

\[
I(x) = -\log P_{x_0} [X_{\tau_n} = 0 \text{ for } n \text{ sufficiently large }], \quad x \in \mathbb{R}^d
\]

is the largest element of \( E \).

(b) The function

\[
J(x) = -\log P_{x_0} [X_{\tau_n} \to 0 \text{ as } n \to \infty ], \quad x \in \mathbb{R}^d
\]

is the smallest element of \( E \).
We will prove Theorem II in Section 4, and give equivalence conditions in Theorem 4.2, for the uniqueness of unbounded solutions $u$ of (1) with $u(x) \to \infty$ as $\|x\| \to \infty$. As an application of Theorem 4.2, we give an alternative probabilistic proof of Cheng and Ni's result (cf. Cheng and Ni, 1992, Theorem II):

**Theorem III.** If $L = \Delta$, where $\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$, and $k(x) \sim \|x\|^{-l}$, $l > 2$, as $\|x\| \to \infty$, then there is only one unbounded solution $u$ of (1) with $u(x) \to \infty$, as $\|x\| \to \infty$.

(Writing $u \sim v$ as $\|x\| \to \infty$ means there exist two constants $c_1, c_2 > 0$ such that $c_1 v(x) \ll u(x) \ll c_2 v(x)$ for $x$ sufficiently large.)

In Section 5, as an application of Theorem III, we evaluate the probability for the range $\mathcal{R}$ of $X$ to be compact.

In this paper $c$ always denotes a constant and it may have different values in different lines. The notation $B_n$ stands for the open ball with radius $n$, centered at $0$.

2. Preliminaries

2.1 Let $L$ be a differential operator in $\mathbb{R}^d$ of the form (2) satisfying conditions (1.a) and (1.b). Then there exists a Markov process $\xi = (\xi_t, \Pi_x)$ in $\mathbb{R}^d$ with continuous paths such that for every bounded continuous function $f$ on $\mathbb{R}^d$,

$$u_t(x) = \Pi_x f(\xi_t)$$

is the unique solution of the equation

$$\frac{\partial u}{\partial t} = Lu$$

with the property $u_t(x) \to f(x)$ as $t \downarrow 0$ (see, e.g. Stroock and Varadhan, 1979).

We call $\xi$ the diffusion with the generator $L$.

2.2 Denote by $\mathcal{B}$ the Borel $\sigma$-algebra in $\mathbb{R}^d$ and by $M$ the set of all finite measures on $\mathcal{B}$. Write $\mathcal{M}$ for the $\sigma$-algebra in $M$ generated by the functions $f_B(\mu) = \mu(B)$, $B \in \mathcal{B}$. For every positive bounded Borel function $k(x)$ in $\mathbb{R}^d$ and $1 < \alpha \leq 2$, there exists a Markov process $X = (X_t, P_\mu)$ in $(M, \mathcal{M})$ such that the following conditions are satisfied.

(2.2.a) If $f$ is a bounded continuous function, then $\langle f, X_t \rangle$ is right continuous in $t$ on $\mathbb{R}^+$. (2.2.b) For every $\mu \in M$ and for every positive bounded Borel function $f$,

$$P_\mu \exp\{-\langle f, X_t \rangle\} = \exp\{-\langle v_t, \mu \rangle\},$$

where $v_t$ is the unique solution of the integral equation

$$v_t(x) + \Pi_x \left[ \int_0^t k(\xi_s) v_{s-}^2(\xi_s) \, ds \right] = \Pi_x f(\xi_t).$$
Moreover, to every $D \in \mathcal{B}$, there corresponds a random measure $X_\tau$ on $(\mathbb{R}^d, \mathcal{B})$ associated with the first exit time $\tau = \inf\{t, \xi_t \notin D\}$ from $D$ by the formula

$$P_\mu \exp\{-\langle f, X_\tau \rangle\} = \exp\{-\langle v, \mu \rangle\}$$

where $v$ satisfies the integral equation

$$v(x) + \Pi_x \left[ \int_0^t k(\xi_s)v^\alpha(\xi_s) \, ds \right] = \Pi_x f(\xi_t).$$

(See Dawson (1993) or Dynkin (1994).) Note that for every $\mu \in M$ with $\text{supp}(\mu) \subset D$, $X_\tau$ concentrates on $\partial D, P_\mu$-a.s.

We call $X = (X_t, X_\tau, P_\mu)$ the superdiffusion with parameters $(L, \psi)$, where $\psi(x, z) = k(x)z^\alpha$. We explain the heuristic meaning of $X_t$ and $X_\tau$ in terms of branching particle systems. Consider a system of particles which undergo random motion and branching on $\mathbb{R}^d$ according to the following rules.

1. Particles are distributed at time 0 according to the Poisson point process with intensity $\mu \in M$.
2. Each particle survives with probability $\exp\{-\int_0^t k(\xi_s) \, ds\}$ at time $t$.
3. At the end of its lifetime, a dying particle gives birth to $n$ offsprings at its own site, with probability $p_n$, where if $1 < \alpha < 2$,
   $$p_n = \begin{cases} 0 & \text{if } n = 1, \\ \frac{1}{\alpha} (-1)^n \left(\frac{\alpha}{n}\right) & \text{if } n \neq 1, \end{cases}$$
4. During its lifetime, the motion of each particle is governed by the process $\xi$.
5. All particle lifetimes, motions, and branching are independent of one another.

The historical path $H_t^a$ of a particle consists of its own trajectory and the trajectories of all its ancestors. If each particle has mass $\beta$, then

$$X_t^\beta(B) = \beta \sum_a 1_B(H_t^a)$$

is the mass distribution at time $t$. (The sum is taken over all particles which are alive at time $t$.) Set

$$X_\tau^\beta(B) = \beta \sum_a 1_B(H_\tau^a),$$

where $\tau_a = \inf\{t, H_t^a \notin D\}$. (Here we identify particles $a$ and $b$ if $\tau_a = \tau_b$ and $H_t^a = H_t^b$ for all $s \leq \tau_a$.) If $k_\beta = \frac{k}{\beta^{d-1}}$ and $\mu_\beta = \frac{\mu}{\beta}$, then $X_t^\beta$ and $X_\tau^\beta$ converge weakly to $X_t$ and $X_\tau$ as $\beta \to 0$. Let $\tau$ be the first exit time of $\xi$ from a domain $D \subset \mathbb{R}^d$. A boundary point $y$ of $D$ is called regular if $\Pi_y[\tau = 0] = 1$. We quote two theorems from Dynkin (1991).

**Theorem 2.1.** Let $D$ be a bounded domain in $\mathbb{R}^d$. For every positive bounded Borel function $f$ on $\partial D$, the function

$$v(x) = -\log P_\delta, \exp\{-\langle f, X_\tau \rangle\}$$
satisfies the equation

\[ L v(x) = k(x) v^2(x) \quad (5) \]

for \( x \in D \). Moreover if \( D \) is regular and if \( f \) is continuous, then \( v = f \) on \( \partial D \). (We write \( v = f \) on \( K \subset \partial D \) if for every \( y \in K \), \( \lim_{x \to y, x \to D} v(x) = f(y) \).)

**Theorem 2.2.** Let \( D \) be an arbitrary domain in \( \mathbb{R}^d \). Choose a sequence of bounded regular domains \( \{D_n\}_n \) with \( D_n \uparrow D \) and let \( \tau_n \) be the first exit time of \( \xi \) from \( D_n \). If \( u \) is a solution of (5) in \( D \), then \( Z_u = \lim_{n \to \infty} (u, X_{\tau_n}) \) exists a.s. and \( u(x) = -\log P_{\delta_x} \exp\{-Z_u\} \) for all \( x \in D \).

### 3. Bounded solutions of \( Lu = ku^a \)

Throughout this paper we consider a strictly positive bounded continuous function \( k(x) \) on \( \mathbb{R}^d \) satisfying the condition:

\[ \lim_{r \to \infty} \sup_{x \in \mathbb{R}^d} \int_{\|y\| > r} g(x, y) k(y) \, dy = 0, \quad (6) \]

where \( g(x, y) \) is the Green function of the operator \( L \) on \( \mathbb{R}^d \).

**Example 1.** If \( k(x) \leq h(||x||) \) for \( x \) sufficiently large, and \( \int_0^\infty sh(s) \, ds < \infty \) for some \( a > 0 \), then \( k \) satisfies condition (6) for \( L = \Delta \) and \( d = 3 \).

**Proof.** It follows from Eq. (13.74) in Dynkin (1965) that \( g(x, y) \leq c ||x - y||^{2-d} \) for all \( x, y \in \mathbb{R}^d \). Our conclusion follows from Zhao (1993, Propositions 1 and 2). \[ \square \]

As before, \( \tau_n \) stands for the first exit time of the diffusion \( \xi \) from the domain \( B_n \) and \( X = (X_t, \tau_n, P_x) \) is the superdiffusion with parameters \( (L, \psi) \), where \( \psi(x, z) = k(x) z^a \). Put \( Z_{h,n} = \langle h, X_{\tau_n} \rangle \) if \( h \in H \).

**Lemma 3.1.** For every \( \mu \in M \) and \( h \in H \), \( \mu \)-a.s., \( Z_h = \lim_{n \to \infty} Z_{h,n} \) exists and \( Z_h < \infty \).

**Proof.** Let \( \mathbb{F}_n = \sigma\{X_{\tau_k}, 1 \leq k \leq n\} \). If \( m < n \), then, by the strong Markov property and Eqs. (3) and (4),

\[ P_{\mu}[\exp\{-Z_{h,n}\} | \mathbb{F}_m] = P_{X_{\tau_m}} \exp\{-Z_{h,n}\} = \exp\{-\langle w_n, X_{\tau_n} \rangle\}, \quad (7) \]

where \( w_n \) satisfies the equation

\[ w_n(x) + \Pi_x \left[ \int_0^{\tau_n} w_n^*(\xi_s) k(\xi_s) \, ds \right] = h(x), \quad x \in B_n. \]

Clearly \( w_n(x) \leq h(x) \) and we have, by (7),

\[ P_{\mu}[\exp\{-Z_{h,n}\} | \mathbb{F}_m] \geq \exp\{-Z_{h,m}\}. \]
Therefore, \((\exp\{-Z_{h,n}\}, \mathcal{F}_n, \mathbb{P}_\mu)\) is a bounded submartingale and \(Z_h = \lim_{n \to \infty} Z_{h,n}\) exists \(\mathbb{P}_\mu\)-a.s.

It follows from (3) and (4) that \(\mathbb{P}_\mu Z_{h,n} = \Pi h(\xi_{t_n}) = \langle h, \mu \rangle\). By Fatou's lemma,

\[
\mathbb{P}_\mu Z \leq \liminf_{n \to \infty} \mathbb{P}_\mu Z_{h,n} = \langle h, \mu \rangle < \infty,
\]

which implies that \(Z_h < \infty, \mathbb{P}_\mu\)-a.s. \(\square\)

We will write \(Z\) for \(Z_h\) if \(h(x) = 1\) for all \(x \in \mathbb{R}^d\).

**Theorem 3.2.** For every \(h \in H\), the function

\[
v_h(x) = -\log \mathbb{P}_\delta \exp\{-Z_h\}, \quad x \in \mathbb{R}^d,
\]

is the unique solution of (1) with \(v_h = h\) at \(\infty\). Moreover \(v_h\) satisfies the integral equation

\[
v(x) + \int_{\mathbb{R}^d} g(x, y)k(y)v_h(y)\,dy = h(x)
\]

in \(\mathbb{R}^d\).

**Proof.** Set \(v_{h,n}(x) = -\log \mathbb{P}_\delta \exp\{-Z_{h,n}\}, \quad x \in B_n\). It follows from Eqs. (3) and (4) that \(v_{h,n}\) satisfies the equation

\[
v_{h,n}(x) + \int_{B_n} g_n(x, y)k(y)v_{h,n}(y)\,dy = h(x), \quad x \in B_n,
\]

where \(g_n(x, y)\) is the Green's function of \(L\) on \(B_n\). Set \(g_n(x, y) = 0\) if \(x \notin B_n\) or \(y \notin B_n\), and put \(v_{h,n}(x) = h(x)\) if \(x \notin B_n\). Then equation (9) becomes

\[
v_{h,n}(x) + \int_{\mathbb{R}^d} g_n(x, y)k(y)v_{h,n}(y)\,dy = h(x), \quad \forall x \in \mathbb{R}^d.
\]

Note that \(g_n(x, y) \uparrow g(x, y)\) as \(n \to \infty\), and, by Lemma 3.1, \(v_h(x) = \lim_{n \to \infty} v_{h,n}(x)\) for all \(x \in \mathbb{R}^d\). Therefore, for every \(x \in \mathbb{R}^d\),

\[
g_n(x, y)v_{h,n}(y) \to g(x, y)v_h(y) \quad \text{as} \quad n \to \infty.
\]

Since \(v_{h,n}(y) \leq \|h\|\), we have \(v_h(y) \leq \|h\|\) and so

\[
g_n(x, y)v_{h,n}(y)k(y) \leq cg(x, y)k(y).
\]

For every \(x \in \mathbb{R}^d\), condition (6) implies that \(g(x, y)k(y)\) is integrable. Letting \(n \to \infty\) in (10), the dominated convergence theorem implies that \(v_h\) satisfies Eq. (8).

For \(n > m\), both the functions \(v_{h,n}\) and \(-\log \mathbb{P}_\delta \exp\{-v_{h,n}(X_{t_m})\}\) are solutions of (5) in \(B_m\), and they have the same boundary values on \(\partial B_m\). The maximum principle (see, e.g. Dynkin (1991, Theorem 0.5)) implies that

\[
v_{h,n}(x) = -\log \mathbb{P}_\delta \exp\{-v_{h,n}(X_{t_m})\}, \quad x \in B_m.
\]
Letting \( n \to \infty \), we get \( \phi(x) = -\log P_{\delta_n} \exp\{-\langle v_h, X_{\tau_n} \rangle\} \) in \( B_n \). Theorem 2.1 implies that \( v_h \) is a solution of (5). To check \( v_h = h \) at \( \infty \), it suffices, by (8), to show that \( \int_{\mathbb{R}^d} g(x, y)k(y)dy \to 0 \) as \( \|x\| \to \infty \). Write

\[
\int_{\mathbb{R}^d} g(x, y)k(y)dy = A(x, r) + B(x, r),
\]

where

\[
A(x, r) = \int_{\|y\| \leq r} g(x, y)k(y)dy
\]

and

\[
B(x, r) = \int_{\|y\| > r} g(x, y)k(y)dy.
\]

Since \( k \) is bounded and \( g(x, y) \to 0 \) as \( \|x - y\| \to \infty \), for every \( r > 0 \), \( A(x, r) \) goes to 0 as \( \|x\| \to \infty \). Clearly condition (6) implies that \( \sup_{x \in \mathbb{R}^d} B(x, r) \to 0 \) as \( r \to \infty \). Letting \( \|x\| \to \infty \) and then \( r \to \infty \) in (11), we observe that \( \int g(x, y)k(y)dy \to 0 \) as \( \|x\| \to \infty \).

Let \( u \) be a solution of (1) with \( u = h \) at \( \infty \). Since \( Z < \infty \) and \( u = h \) at \( \infty \), \( \langle u, X_{\tau_n} \rangle \to Z_h \). By Theorem 2.2, \( u(x) = -\log P_{\delta_n} \exp\{-Z_h\} = v_h(x) \). □

**Remark.** (a) If \( L = \Delta \) and \( k \) is radial, then \( v_{\xi_n} \) is radial for every constant function \( h = c \) and so is \( v_c \).

(b) Under the same assumptions on \( k(x) \) as in Example 1, Kawano (1984) and Cheng and Ni (1992) obtained similar results for \( L = \Delta \). By using Brownian path integration and potential theory, Zhao (1993) studied related problems for \( L = \Delta \).

**Theorem 3.3.** If \( u \) is a bounded solution of (1), then \( v = h \) at \( \infty \) for some \( h \in H \) and \( v(x) = v_h(x) \).

**Proof.** Note that if \( u \) satisfies Eq. (8) for some \( h \in H \), our conclusions follow from the same arguments as in the proof of Theorem 3.2. Since \( u(x) = -\log P_{\delta_n} \exp\{-\langle u, X_{\tau_n} \rangle\} \), (3) and (4) imply that \( u \) satisfies the equation

\[
u(x) + \int_{B_n} g_n(x, y)k(y)u^2(y)dy = h_n(x), \quad x \in B_n,
\]

where \( h_n(x) = \Pi_x[u(\xi_{\tau_n})] \). Clearly \( Lh_n = 0 \) and \( h_n(x) = \Pi_x[u(\xi_{\tau_n})] \leqslant \|w\| \) for all \( n \). By passing \( n \) to the limit in (12), \( h(x) = \lim_{n \to \infty} h_n(x) \) exists for all \( x \in \mathbb{R}^d \). Therefore, \( h \) is bounded and \( Lh = 0 \) in \( \mathbb{R}^d \). By passing to the limit in (12) again, \( u \) satisfies Eq. (8). □

**Remark.** The special case of Theorem 3.2, where \( L = \Delta \) and \( k \) satisfies the conditions in Example 1, was observed earlier by Cheng and Ni (1992).
4. Unbounded solutions of $Lu = ku^\alpha$

Lemma 4.1. (a) Let $B = \{x; \|x - x_0\| < R\}$ and

$$u(x) = \lambda (R^2 - r^2)^{-\frac{\alpha}{2-\alpha}}$$

where $\lambda$ is a positive constant and $r = \|x - x_0\|$. We have

$$\lim_{x \to a, x \in B} u(x) = \infty$$

for all $a \in \partial B$, and

$$Lu - ku^\alpha \leq 0 \text{ in } B$$

for some $\lambda$ depending only on $\alpha$, the dimension $d$ and the upper bounds for $\tilde{a}_{ij} = \frac{a_{ij}}{k}$ and $\tilde{b}_i = \frac{b_i}{k}$ in $B$.

(b) If $B \subset D$ for some open set $D$, and $v$ is a solution of (5) in $D$, then $v \leq u$ in $B$.

Proof. (a) is quoted from Dynkin (1991, Lemma 3.1) and (b) follows easily from the maximum principle. $\square$

Proof of Theorem II. (a) For $x \in B_n$ and $m > 0$, set $I_{n,m}(x) = -\log P_{\delta_x} \exp\{-mZ_{1,n}\}$. Theorem 2.1 implies that $I_{n,m}$ satisfies (5) in $B_n$ and $I_{n,m} = m$ on $\partial B_n$. By the maximum principle, $I_{n,m}$ is increasing in $m$. Therefore, for every $x \in B_n$,

$$I_n(x) = \lim_{m \to \infty} I_{n,m}(x)$$

exists. Clearly $I_n(x) = -\log P_{\delta_x} \{Z_{1,n} = 0\}$ and $I_n = \infty$ on $\partial B_n$. Let $B$ be an arbitrary bounded open ball and let $\tau$ be the first exit time of $\xi$ from $B$. If $\tilde{B} \subset B_n$ for some $n$, Theorem 2.1 and the maximum principle imply that

$$I_{n,m}(x) = -\log P_{\delta_x} \exp\{-\langle I_{n,m}, X_\tau \rangle\}, x \in B.$$  (13)

Note that, by Lemma 4.1, $|I_{n,m}| \leq c$ in $\tilde{B}$, for all $m$. Letting $m$ go to $\infty$ in (13), we have

$$I_n(x) = -\log P_{\delta_x} \exp\{-\langle I_n, X_\tau \rangle\}.$$  (14)

Clearly $I(x) = \lim_{n \to \infty} I_n(x)$ and, letting $n \to \infty$ in (14), we observe $I(x) = -\log P_{\delta_x} \exp\{-\langle I, X_\tau \rangle\}$. By Theorem 2.1, $I$ satisfies Eq. (1).

Assume $u$ is a solution of (1). Since $I_n = \infty$ on $\partial B_n$, the maximum principle implies that $u \leq I_n$ in $B_n$ and so $u \leq I$.

(b) Write $v_c$ for $v_h$ in Theorem 3.2 if $h$ is a constant function $c$. Clearly $v_c(x) \uparrow J(x)$ as $c \uparrow \infty$ and $J(x) \to \infty$ as $\|x\| \to \infty$. Since $v_c(x) = -\log P_{\delta_x} \exp\{-\langle v_c, X_\tau \rangle\}$ in $B_n$, we have $J(x) = -\log P_{\delta_x} \exp\{-\langle J, X_\tau \rangle\}$ in $B_n$. By Lemma 4.1, $J$ is bounded on $\partial B_n$, and so Theorem 2.1 implies that $J$ is a solution of (1).
Assume \( u \in E \). For every \( c > 0 \), \( \langle u, X_{t_n} \rangle \geq cZ_{1,n} \) for sufficiently large \( n \). For any \( c > 0 \),

\[
  u(x) = \lim_{n \to \infty} - \log P_{\delta_n} \exp\{-\langle u, X_{t_n} \rangle\} \\
  \geq \lim_{n \to \infty} - \log P_{\delta_n} \exp\{-cZ_{1,n}\} = v_c(x).
\]

Letting \( c \uparrow \infty, u(x) \geq J(x) \) for all \( x \in \mathbb{R}^d \). Therefore \( J \) is the minimal element in \( E \). \( \square \)

**Remark.** (a) Assume \( L = \Delta \) and \( k \) is radial. Then both \( I_{n,m} \) and \( I_n \) are radial. Therefore \( I \) is radial. Clearly \( J \) is radial.

(b) Let \( k_1, k_2 \) be two bounded strictly positive continuous functions on \( \mathbb{R}^d \) which both satisfy condition (6). Assume further that \( k_1(x) \leq k_2(x) \) for all \( x \). For \( s = 1, 2 \), let \( I_{s,n,m}, I_s \), and \( J_s \) denote \( I_{n,m}, I_n \), and \( J \) respectively, with \( k \) replaced by \( k_s \). For \( x \in B_n \), we have

\[
  L I_{2,n,m} - k_2 I_{2,n,m}^2 = 0 = L I_{1,n,m} - k_1 I_{1,n,m}^2 \geq L I_{1,n,m} - k_2 I_{1,n,m}^2.
\]

The Maximum principle implies that \( I_{2,n,m} \leq I_{1,n,m} \) on \( B_n \). Therefore \( I_2 \leq I_1 \). Similar arguments imply that \( J_2 \leq J_1 \).

Denote by \(|E|\) the cardinality of \( E \). By Theorem II, \(|E| \geq 1 \).

**Theorem 4.2.** The following three statements are equivalent.

(a) \(|E| = 1 \).

(b) For every measure \( \mu \in M \) with compact support, we have

\[
  P_{\mu}[Z_{1,n} \to 0] = P_{\mu}[Z_{1,n} = 0 \text{ for sufficiently large } n].
\]

(c) There exists a constant \( c \) such that

\[
  I(x) \leq cJ(x) \quad \text{for } x \text{ sufficiently large},
\]

where \( I \) and \( J \) are functions in Theorem II.

**Proof.** Note that for every \( \mu \in M \) with compact support, we have

\[
  P_{\mu}[Z = 0] = \exp\{-\langle I, \mu \rangle\}
\]

and

\[
  P_{\mu}[Z_{1,n} = 0 \text{ for sufficiently large } n] = \exp\{-\langle I, \mu \rangle\}.
\]

Therefore (a) and (b) are equivalent. Clearly (b) implies (c). Assume that (c) holds.

To prove (a), it suffices to show that \( I = J \). Fix \( x \in \mathbb{R}^d \). By Theorem 2.2, both \( Z_I = \lim_{n \to \infty} \langle I, X_{t_n} \rangle \) and \( Z_J = \lim_{n \to \infty} \langle J, X_{t_n} \rangle \) exist \( P_{\delta_n} \)-a.s. Since \( J(x) \to \infty \) and \( J(x) \to \infty \), \( Z_I = J_I = \infty \) on \( \{ Z > 0 \} \). Combining with Theorem 2.2, we have

\[
  -\log P_{\delta_n}[Z = 0] = J(x) = -\log P_{\delta_n}[\exp\{-Z_J\}, Z = 0] \]

\[
  = -\log P_{\delta_n}[\exp\{-Z_J\}, Z = 0]
\]
Therefore $Z_t = 0$ on $\{Z = 0\}P_x$-a.s. By assumption, we have $\langle I, X_{tn} \rangle \leq c \langle J, X_{tn} \rangle$ for $n$ sufficiently large, which implies that $Z_t = 0$ on $\{Z = 0\}, P_x$-a.s. Therefore

$$I(x) = -\log P_{\delta_t} \exp \{-Z_t\}$$
$$= -\log P_{\delta_t}[\exp \{-Z_t\}, Z = 0]$$
$$= -\log P_{\delta_x}[Z = 0] = J(x).$$

If $L = \Delta$ and if $k$ is radial with $k(x) = c|x|^{-1}$, $l > 2$, for large $x$, then the first part of Theorem 4.3 of Cheng and Ni (1992), implies that for every radial solution $u$ of (1), we have $u(x) \sim |x|^\frac{l-2}{l-1}$ as $\|x\| \to \infty$. By using this observation and Theorem 4.2, we prove Theorem III.

**Proof of Theorem III.** By assumption, there exist two constants $c_1, c_2$ and two radial functions $k_1$ and $k_2$ with $k_1(x) = |x|^{-l} = k_2(x)$ for $x$ sufficiently large and

$$c_1 k_1(x) \leq k(x) \leq c_2 k_2(x) \quad \text{for all } x \in \mathbb{R}^d.$$

For $s = 1, 2$, let $I_s, J_s$ denote $I$ and $J$ respectively with $k$ replaced by $c_s k_s$. By Remark (b) following the proof of Theorem II, we have, for all $x$,

$$I_2(x) \leq I(x) \leq I_1(x)$$

and

$$J_2(x) \leq J(x) \leq J_1(x).$$

Since $I_s(x) \sim |x|^\frac{l-2}{l-1}$ and $J_s(x) \sim |x|^\frac{l-2}{l-1}$, $s = 1, 2$, as $\|x\| \to \infty$, we get

$$\frac{I(x)}{J(x)} \leq \frac{I_1(x)}{J_2(x)} \leq c$$

for $x$ sufficiently large. Our result follows from Theorem 4.2. □

**Remark.** In general we do not know if condition (6) is a sufficient condition for $|E| = 1$.

5. Application

The range $\mathcal{R}$ of the superdiffusion $X$ is the smallest closed subset of $\mathbb{R}^d$ such that $\mathcal{R}$ contains supports of $X_t$ for all $t \geq 0$. For constant $k$, Iscoe (1986) proved that $\mathcal{R}$ is compact a.s. for $L = \Delta$ and Dynkin (1991) observed this for general $L$.

**Theorem 5.1.** If $L = \Delta$ and $k(x) \sim |x|^{-l}$, $l > 2$, as $\|x\| \to \infty$, then for every $\mu \in M$ with compact support,

$$P_\mu[\mathcal{R} \text{is compact}] = \exp\{-\langle I, \mu \rangle\},$$

where $I$ is the unique unbounded solution of (1).
Proof. If \( \mu \in M \) has compact support, we have, \( P_\mu \)-a.s.,

\[
\{ \mathcal{A} \text{ is compact} \} = \bigcup_{n=1}^{\infty} \{ Z_{1,n} = 0 \}.
\]

(See Dynkin, 1991). Our statement follows from Theorem II and Theorem III.

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