Maximizing numerical radii of weighted shifts under weight permutations

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1. Introduction

Let $H$ be a complex Hilbert space and $A$ be a bounded linear operator on $H$. The numerical range of $A$ is defined by

$$W(A) \equiv \{\langle Ax, x \rangle : x \in H \text{ and } ||x|| = 1\},$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $H$ and $|| \cdot ||$ is the corresponding norm. $W(A)$ is always a nonempty, convex and bounded subset of $\mathbb{C}$. In addition, $W(A)$ is closed if $A$ is of finite rank. The numerical radius of $A$ is $w(A) \equiv \sup\{||z|| : z \in W(A)\}$.

The weighted cyclic matrix with the weight $w = (w_i)_{i=1}^n$, where $w_i \in \mathbb{C}$ for all $1 \leq i \leq n$, is the matrix

$$
\begin{pmatrix}
0 & w_1 & 0 & \cdots & 0 \\
0 & 0 & w_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
w_n & \cdots & 0 & w_{n-1} & 0
\end{pmatrix},
$$

and is regarded as a bounded linear operator on $\mathbb{C}^n$ (with the standard inner product) under the matrix multiplication. Let $S_n$ stand for the symmetric group on $1, 2, \ldots, n$ and $l \in S_n$. The weighted cyclic matrix with the weight $(w_{l(i)})_{i=1}^n$ is denoted...
by $A_{w,j}$ or simply $A_j$, that is,

$$A_{w,j} = A_j \equiv \begin{pmatrix} 0 & w(1) \\ \vdots & \ddots \\ w(n) & 0 \\ \end{pmatrix}.$$ 

In Section 2, we characterize those $\xi \in S_n$ such that $w(A_\xi) = \max_{i \in S_n} w(A_i)$. The characterizations for unilateral and bilateral weighted backward shifts are also obtained in Section 3.

Let $\text{Re} \ A \equiv (A + A^*)/2$ and $\rho(A)$ be the real part and the spectral radius of a matrix $A$, respectively. Suppose $A$ is the weighted cyclic matrix with the weight $w = (w_i)_{i=1}^n$, it is well-known that, for some real $\vartheta$, $e^{\vartheta t}A$ is unitarily equivalent to the weighted cyclic matrix with the weight $|w_i| = (|w_i|)_{i=1}^n$, which implies that their numerical radii coincide. When $w$ is nonnegative, that is, $w_i \geq 0$ for all $1 \leq i \leq n$, $\langle Ax, x \rangle \leq \langle A|x|, |x| \rangle$ for each $x = (x_1, \ldots, x_n)^T \in \mathbb{C}^n$, where $|x| \equiv (|x_1|, \ldots, |x_n|)^T$, and hence

$$w(A) = \max \langle Ax, x \rangle : x \text{ is a nonnegative unit vector in } \mathbb{C}^n = \rho(\text{Re} \ A),$$

the largest eigenvalue of $\text{Re} \ A$ (cf. [1, Proposition 3.3]). In addition, if at most one of these $w_i$’s is zero, then $\text{Re} \ A$ is nonnegative and unitarily irreducible, and there exists a unique positive unit vector $x$ such that $w(A) = \langle \text{Re} \ A \rangle y = \langle Ay, y \rangle$ by Perron–Frobenius theorem. On the other hand, if some of these $w_i$’s are zero, then $A$ and $e^{\vartheta t}A$ are unitarily equivalent for all real $\vartheta$ and hence $W(A)$ is a closed circular disc centered at the origin with radius $w(A)$ (cf. [2]). Moreover, if at least two of these $w_i$’s are zero, then $A$ is the direct sum of two weighted cyclic matrices and its numerical range equals the largest numerical range of these summands. Therefore, it is sufficient to consider the case that at most one of these $w_i$’s is zero when evaluating $w(A)$. For further information about numerical ranges and weighted cyclic matrices, we refer the readers to [1–8].

Let $l \in S_n$ and write $x_l \equiv (x_{l(1)} \cdots x_{l(n)})^T$, where $x = (x_1, \ldots, x_n)^T \in \mathbb{C}^n$. For any weight $w = (w_i)_{i=1}^n$, define $F_w \equiv \{ (ij) \in S_n : 1 \leq i < j \leq n \text{ and } |w_i| = |w_j| \}$, the subgroup of $S_n$ generated by the transpositions for which the weight $|w|$ is unchanged (we adopt that $F_w$ consists of only the identity permutation if these $|w_i|$’s are all distinct). Notice that $\psi \in F_w$ if and only if $|w_{\psi(j)}| = |w_i|$ for all $1 \leq i \leq n$. We observe that $w(A_{\psi,\varphi}) = w(A_{\varphi,\psi}) = w(A_{\varphi,\psi}) = w(A_{\varphi,\psi})$ for each $\psi \in F_w$. Moreover, set $\tau_n = \left( \begin{array}{cccc} 1 & 2 & \cdots & n \end{array} \right)$, $\sigma_n = \left( \begin{array}{cccc} 1 & 2 & \cdots & n \end{array} \right)$, and $\vartheta_n = \left( \begin{array}{cccc} 1 & 2 & \cdots & n \end{array} \right)$, the permutations which shift and reverse the order of $[1, \ldots, n]$, respectively, and define $H_\vartheta \equiv \langle \tau_n, \vartheta_n \rangle$, the subgroup of $S_n$ generated by $\tau_n$ and $\vartheta_n$. Because for each $x \in \mathbb{C}^n$, $(A_{\psi,\varphi}x) = (A_{\psi,\varphi}x) = (A_{\psi,\varphi}x) = (A_{\psi,\varphi}x)$, we obtain that $w(A_{\psi,\varphi}) = w(A_{\varphi,\psi})$ for all $\varphi \in H_\vartheta$. As a result, $w(A_{\psi,\varphi}) = w(A_{\varphi,\psi})$ for all $\psi \in F_w$ and $\varphi \in H_\vartheta$. In Section 2, we show that under the condition $|w_1| \geq |w_2| \geq \cdots \geq |w_n|$ with $w_{n-1} \neq 0$, $\xi \in S_n$ satisfies $w(A_\xi) = \max_{i \in S_n} w(A_i)$ if and only if $\xi = \sigma_n \vartheta_n \varphi$ for some $\psi \in F_w$ and $\varphi \in H_\vartheta$. In (Theorem 2.1). Here $\sigma_n$ is the permutation given by $\sigma_n(i) = 2i - 2 + 2i - 1$ if $1 \leq i \leq [n/2]$ and $\sigma_n(i) = 2i - 2 + 2i - 1$ if $[n/2] + 1 \leq i \leq n$, where $[\cdot]$ denotes the greatest integer function. Therefore,

$$\sigma_n = \begin{cases} \left( \begin{array}{cccc} 1 & 2 & \cdots & \lfloor n/2 \rfloor \end{array} \right) - 1 & \left( \begin{array}{cccc} \lfloor n/2 \rfloor & \lfloor n/2 \rfloor & \cdots & 1 \end{array} \right) + \left( \begin{array}{cccc} \lfloor n/2 \rfloor & \lfloor n/2 \rfloor & \cdots & 1 \end{array} \right) + \left( \begin{array}{cccc} \lfloor n/2 \rfloor & \lfloor n/2 \rfloor & \cdots & 1 \end{array} \right) + \left( \begin{array}{cccc} \lfloor n/2 \rfloor & \lfloor n/2 \rfloor & \cdots & 1 \end{array} \right) \right) \text{ if } n \text{ is odd}, \\
\left( \begin{array}{cccc} \lfloor n/2 \rfloor & \lfloor n/2 \rfloor & \cdots & 1 \end{array} \right) + \left( \begin{array}{cccc} \lfloor n/2 \rfloor & \lfloor n/2 \rfloor & \cdots & 1 \end{array} \right) + \left( \begin{array}{cccc} \lfloor n/2 \rfloor & \lfloor n/2 \rfloor & \cdots & 1 \end{array} \right) + \left( \begin{array}{cccc} \lfloor n/2 \rfloor & \lfloor n/2 \rfloor & \cdots & 1 \end{array} \right) \right) \text{ if } n \text{ is even}. 
\end{cases}$$

One could interpret $\sigma_n$ as a rearrangement which centralizes those $w_i$’s with largest absolute values. Furthermore, if $w$ is nonnegative, the unique positive unit vector $y$ satisfying $w(A_{\sigma_n}) = \langle A_{\sigma_n}y, y \rangle$ must be of the form $y = x_{\sigma_n}$, where $x = (x_1, \ldots, x_n)^T$ with $x_1 \geq x_2 \geq \cdots \geq x_n > 0$ and

$$\sigma_n = \begin{cases} \sigma_n \vartheta_n \equiv \left( \begin{array}{cccc} 1 & 2 & \cdots & \lfloor n/2 \rfloor \end{array} \right) - 1 & \left( \begin{array}{cccc} \lfloor n/2 \rfloor & \lfloor n/2 \rfloor & \cdots & 1 \end{array} \right) + \left( \begin{array}{cccc} \lfloor n/2 \rfloor & \lfloor n/2 \rfloor & \cdots & 1 \end{array} \right) + \left( \begin{array}{cccc} \lfloor n/2 \rfloor & \lfloor n/2 \rfloor & \cdots & 1 \end{array} \right) + \left( \begin{array}{cccc} \lfloor n/2 \rfloor & \lfloor n/2 \rfloor & \cdots & 1 \end{array} \right) \right) \text{ if } n \text{ is odd}, \\
\sigma_n \vartheta_n \equiv \left( \begin{array}{cccc} 1 & 2 & \cdots & \lfloor n/2 \rfloor \end{array} \right) - 1 & \left( \begin{array}{cccc} \lfloor n/2 \rfloor & \lfloor n/2 \rfloor & \cdots & 1 \end{array} \right) + \left( \begin{array}{cccc} \lfloor n/2 \rfloor & \lfloor n/2 \rfloor & \cdots & 1 \end{array} \right) + \left( \begin{array}{cccc} \lfloor n/2 \rfloor & \lfloor n/2 \rfloor & \cdots & 1 \end{array} \right) \right) \text{ if } n \text{ is even}. 
\end{cases}$$

In Section 3, the comparison of numerical radii of unilateral and bilateral weighted (backward) shifts is considered. Let $w = (w_i)_{i=0}^\infty$ (resp., $w = (w_i)_{i=-\infty}^\infty = (\cdots, w_{-1}, w_0, w_1, \cdots)$), where the $w_i$’s in $C$ for $i \in \mathbb{N}^0 \equiv \{0\} \cup \mathbb{N}$ (resp., $i \in \mathbb{Z}$), are bounded. Here we underline the 0th component of $w$ for the bilateral case. The unilateral (resp., bilateral) weighted backward shift with the weight $w$ is the bounded linear operator on $\ell^2 \equiv \{(x_0, x_1, \cdots) : |x_i|^2 < \infty\}$. (resp.,
\[ \ell^2(Z) \equiv \{ (\cdots, x_0, x_1, \cdots) : \sum_{i=-\infty}^{\infty} |x_i|^2 < \infty \} \] with the matrix representation

\[
A = \begin{pmatrix}
0 & w_0 & w_1 & 0 & \cdots \\
& 0 & w_1 & 0 & \ddots \\
& & \ddots & \ddots & \ddots \\
& & & \ddots & w_{-1} \\
& & & & 0 \\
& & & & & w_0 \\
\end{pmatrix}
\]

resp.,

\[
B = \begin{pmatrix}
\cdots & w_{-1} & 0 & \cdots \\
\cdots & 0 & w_0 & \cdots \\
\cdots & \ddots & \ddots & \ddots \\
\cdots & \ddots & \ddots & \ddots \\
\cdots & \ddots & \ddots & \ddots \\
\cdots & \ddots & \ddots & \ddots \\
\end{pmatrix}
\]

where the 0th component of a vector in \( \ell^2(Z) \) and the \((0, 0)\)-entry of an infinite matrix for the bilateral case is underlined. Its numerical range is also a circular disc centered at the origin with radius \( w(A) \) (resp., \( w(B) \)) (cf. [28]). Let \( S_{n0} \) (resp., \( S_{n2} \)) stand for the collection of all rearrangements of the elements of \( \mathbb{N}^0 \) (resp., \( \mathbb{Z} \)). If \( l \in S_{n0} \) (resp., \( \ell \in S_{n2} \)), the unilateral (resp., bilateral) weighted backward shift with the weight \( (w_{(l(i))})_{i=0}^{\infty} \) (resp., \( (w_{(l(i))})_{i=-\infty}^{\infty} \)) is denoted by \( A_{w_{(l)}} \) or simply \( A_l \) (resp., \( B_{w_{(l)}} \) or simply \( B_l \)). We will show that there exists a \( \zeta \in S_{n0} \) satisfying \( w(A_{\zeta}) = \sup_{l \in S_{n0}} w(A_l) \) if and only if \( \sup_{l \in S_{n0}} |w_{(l(k))}| = \lim_{p \to -\infty} |w_{(l(k))}| \) (Theorem 3.1). A corresponding result for the bilateral case is also obtained (Theorem 3.2). We also represent the values \( \sup_{l \in S_{n0}} w(A_l) \) and \( \sup_{l \in S_{n2}} w(B_l) \) as the numerical radius of some bilateral weighted backward shift \( U_w \).

2. Weighted cyclic matrices

Throughout this section, for each \( x = (x_1, \cdots, x_n)^T \in \mathbb{C}^n \), we define \( \hat{x} = (\hat{x}_1, \cdots, \hat{x}_n)^T \) and \( \tilde{x} = (\tilde{x}_1, \cdots, \tilde{x}_n)^T \) by

\[
\hat{x}_i = \begin{cases} 
    x_1 & \text{if } i = 1, \\
    x_{i-1} & \text{if } 2 \leq i \leq n,
\end{cases}
\]

and

\[
\tilde{x}_i = \begin{cases} 
    x_{i+1} & \text{if } 1 \leq i \leq n-1, \\
    x_n & \text{if } i = n.
\end{cases}
\]

We also define \( l(n + 1) \equiv l(1) \) for each \( l \in S_n \). The following theorem is the main result of this section.

**Theorem 2.1.** Let \( w = (w_i)_{i=1}^{\infty} \) with \( |w_1| \geq |w_2| \geq \cdots \geq |w_n| \) and \( w_{n-1} \neq 0 \). We have

\[
\max_{l \in S_n} w(A_l) = w(A_{\sigma_n}).
\]

Moreover, the following statements are true:

(a) \( w(A_{\zeta}) = \max_{l \in S_{n0}} w(A_l) \) for some \( \zeta \in S_{n0} \) if and only if \( \zeta = \psi \sigma_n \varphi \) for some \( \psi \in F_w \) and \( \varphi \in H_n \) and

(b) if \( w \) is nonnegative, then there exists a unique positive unit vector \( x = (x_1, \cdots, x_n)^T \) with \( x_1 \geq x_2 \geq \cdots \geq x_n > 0 \) such that

\[
w(A_{\sigma_n}) = \langle A_{\sigma_n} x_{\sigma_n}, x_{\sigma_n} \rangle = \sum_{i=1}^{n} w_{(\sigma_n(i))} \hat{x}_i x_i.
\]

In a word, Theorem 2.1 states that a permutation \( \zeta \in S_{n0} \) satisfies \( A_{\zeta} \) attains the maximal numerical radius among all \( l \in S_n \) if and only if there is a \( \varphi \in H_n \) such that the sequence \( d_l \equiv |w_{(\varphi)(i)}|, l = 1, 2, \ldots, n \), satisfies \( d_{1} \geq d_{r-1} \geq d_{r+1} \geq d_{r+2} \geq d_{r+3} \geq \cdots \) with \( r = \lfloor n/2 \rfloor + 1 \). We establish Theorem 2.1 via a series of lemmas. The first involves an interesting inequality which is quite well known; for example, see [9, Lemma 3.6]. We give a short proof for completeness.

**Lemma 2.2.** Let \( u = (u_j)_{j=1}^{n} \), \( v = (v_i)_{i=1}^{n} \) and \( w = (w_i)_{i=1}^{\infty} \) be \( n \)-tuples with nonnegative components and with \( w_i \)'s nonincreasing. Suppose that \( \sum_{i=1}^{k} u_i \leq \sum_{i=1}^{k} v_i \) for all \( 1 \leq k \leq n \). Then

\[
\sum_{i=1}^{k} u_i w_i \leq \sum_{i=1}^{k} v_i w_i
\]

for all \( 1 \leq k \leq n \). Furthermore, under the condition \( \sum_{i=1}^{n} u_i w_i = \sum_{i=1}^{n} v_i w_i \), we have \( \sum_{i=1}^{s} u_i = \sum_{i=1}^{s} v_i \) for all \( 1 \leq s < n \) with \( w_s > w_{s+1} \), and \( \sum_{i=1}^{n} u_i = \sum_{i=1}^{n} v_i \) if \( w_n \neq 0 \). In particular, \( u_i = v_i \) for all \( 1 \leq i \leq n \) if \( w_i \)'s are positive and strictly decreasing.

**Proof.** The fact \( u_i w_i \leq v_i w_i \) is trivial. Fix \( k = 2, \ldots, n \). Summing by parts, we have

\[
\sum_{i=1}^{k} (v_i - u_i) w_i = \sum_{i=1}^{k-1} (w_i - w_{i+1}) \sum_{j=1}^{i} (v_j - u_j) + w_k \sum_{i=1}^{k} (v_i - u_i) \geq 0.
\]

Hence the desired inequality follows. Under the condition \( \sum_{i=1}^{n} u_i w_i = \sum_{i=1}^{n} v_i w_i \), the inequality in (1) is an equality when \( k = n \). We get that \( \sum_{i=1}^{s} u_i = \sum_{i=1}^{s} v_i \) for all \( 1 \leq s < n \) with \( w_s > w_{s+1} \), and \( \sum_{i=1}^{n} u_i = \sum_{i=1}^{n} v_i \) if \( w_n \neq 0 \). Finally, the last statement holds since \( \sum_{i=1}^{s} u_i = \sum_{i=1}^{s} v_i \) for all \( 1 \leq s \leq n \) in such a case. \( \square \)
Combining the inequality in Lemma 2.2 and the effect of permutations, we obtain the next lemma, which plays an important role when deriving the main theorem.

**Lemma 2.3.** Let $x = (x_1 \cdots x_n)^T$ be nonnegative and nonincreasing, and let $l, \eta \in S_n$. For each $k = 1, 2, \ldots, n$, there exist $\nu, l_1, l_2 \in S_n$ such that $x_{l(l(i)v(i))}x_{l(l(i)v(i)+1))} = \tilde{x}_{l_1(i)}\tilde{x}_{l_2(i)}$ for all $1 \leq i \leq n$, $l_1(1) > l_1(2) > \cdots > l_1(k)$ and

$$\sum_{i=1}^{k} x_{l(i)v(i)}x_{l(i)v(i)+1)} = \sum_{i=1}^{k} \tilde{x}_{l_1(i)}\tilde{x}_{l_2(i)} \leq \sum_{i=1}^{k} \tilde{x}_i\tilde{x}_i.$$

**Proof.** Let $l, \eta \in S_n$. Assume that $l(p_0) = 1$ and $l(p_1) = n$ for some $1 \leq p_0, p_1 \leq n$. We only consider the case $p_0 < p_1$; the case $p_0 > p_1$ can be obtained analogously. There exist $\phi, \psi \in S_n$ such that

$$\tilde{x}_{\phi(i)} = \begin{cases} x_{l(i)} & \text{if } 1 \leq i < p_0 \text{ or } p_1 \leq i \leq n, \\ x_{l(i+1)} & \text{if } p_0 \leq i < p_1, \end{cases}$$

and

$$\tilde{x}_{\psi(i)} = \begin{cases} x_{l(i+1)} & \text{if } 1 \leq i < p_0 \text{ or } p_1 \leq i \leq n, \\ x_{l(i)} & \text{if } p_0 \leq i < p_1. \end{cases}$$

We obtain that $x_{l(i)v(i)}x_{l(i)v(i)+1)} = \tilde{x}_{\phi(i)}\tilde{x}_{\phi(i)}$ for all $1 \leq i \leq n$. Fix $k = 1, 2, \ldots, n$. There is a $\nu \in S_n$ satisfying $1 \leq \nu(i) \leq k$ for all $1 \leq i \leq n$ and $\phi(\nu(v(1))) > \phi(\eta(\nu(2))) > \cdots > \phi(\eta(v(k)))$. Let $l_1 = \phi\nu$ and $l_2 = \psi\eta$. It is obvious that $x_{l(\nu(v(i)))x_{l(\nu(v(i)+1))} = x_{l(i)}\tilde{x}_{i}}$ for all $1 \leq i \leq n$. Moreover, $\tilde{x}_i \geq \tilde{x}_{l(i)}$ for all $1 \leq i \leq k$ since $\tilde{x}$ is nonincreasing. This shows that

$$\sum_{i=1}^{k} x_{l(i)v(i)}x_{l(i)v(i)+1)} = \sum_{i=1}^{k} \tilde{x}_{\phi(i)}\tilde{x}_{\phi(i)} = \sum_{i=1}^{k} \tilde{x}_{\phi(i)}\tilde{x}_{\phi(i)} = \sum_{i=1}^{k} \tilde{x}_{l_1(i)}\tilde{x}_{l_2(i)} \leq \sum_{i=1}^{k} \tilde{x}_i\tilde{x}_i. \quad (2)$$

Besides, the fact that $\tilde{x}$ is nonincreasing implies that $\sum_{i=1}^{j} \tilde{x}_i \leq \sum_{i=1}^{j} \tilde{x}_i$ for all $1 \leq j \leq k$. Therefore, $\sum_{i=1}^{j} \tilde{x}_i \tilde{x}_i \leq \sum_{i=1}^{j} \tilde{x}_i \tilde{x}_i$ for all $1 \leq j \leq k$ by Lemma 2.2. Combining this with (2), we obtain that $\sum_{i=1}^{k} x_{l(i)v(i)}x_{l(i)v(i)+1)} \leq \sum_{i=1}^{k} \tilde{x}_i\tilde{x}_i$. Thus the proof is completed. $\square$

In brief, the preceding lemma says that $(\tilde{x}_{\tilde{x}_i})^n_{i=1}$ weakly majorizes $(x_{l(i)v(i)}x_{l(i)v(i)+1)}^n_{i=1}$ (cf. [10, Definition 4.3.24]).

**Lemma 2.4.** Let $0 < \alpha < \beta < y$. There is a $\beta > 0$ such that $\beta < \alpha$ and

$$2y^2 = (y + \beta)^2 + (y - \alpha)^2. \quad (3)$$

Moreover, $\beta/\alpha \to 1$ as $\alpha \to 0^+$. \newline

**Proof.** Since $y^2 > (y - \alpha)^2$, (3) holds for some $\beta > 0$. Expanding (3) and dividing it by $\alpha$, we obtain $2y^2 = 2y(\beta/\alpha) + \beta(\beta/\alpha)$, which implies that $\beta/\alpha < 1$ and $\beta/\alpha \to 1$ as $\alpha \to 0^+$. \square

We apply Lemma 2.4 to derive Lemma 2.5, which describes the relation between the weight $w$ and the unit vector $x$ such that $w(A_n^\alpha) = \langle A_n^\alpha x_n^\alpha, x_n^\alpha \rangle$.

**Lemma 2.5.** Let $w_1 \geq w_2 \geq \cdots \geq w_n \geq 0$ with $w_{n-1} \neq 0$ and let $x = (x_1 \cdots x_n)^T$ with $x_1 \geq x_2 \geq \cdots \geq x_n > 0$ be a positive unit vector such that $\langle A_n^\alpha x_n^\alpha, x_n^\alpha \rangle = w(A_n^\alpha)$. Suppose that $x_{n} = x_{n+1}$ for some $1 \leq i_0 < n$.

(a) If $i_0$ is even, then $w_i = w_{i+1}$ for all odd indices $i$ and $x_i = x_{i+1}$ for all even indices.

(b) If $i_0$ is odd, then $w_i = w_{i+1}$ for all even indices $i$ and $x_i = x_{i+1}$ for all odd indices.

**Proof.** We only derive (a); (b) can be obtained analogously. (a) will be established if the following statement is true: if $i_0$ is even, then $w_{i_0-1} = w_{i_0}, w_{i_0+1} = w_{i_0+2}$ (for $i_0 \leq n - 2), x_{i_0-2} = x_{i_0-1}$ (for $i_0 \geq 4$) and $x_{i_0+2} = x_{i_0+3}$ (for $i_0 \leq n - 3$). We prove the cases $4 \leq i_0 \leq n - 3$ and $i_0 = n - 1$. The cases $i_0 = 2$ and $i_0 = n - 2$ can be proven in a similar way. If $4 \leq i_0 \leq n - 3$ and the assertion fails, then $(w_{i_0-1}x_{i_0-2} + w_{i_0+1}x_{i_0+2}) - (w_{i_0}x_{i_0-1} + w_{i_0+2}x_{i_0+3}) > 0$. Set $y \equiv x_{n} = x_{n+1}$ and let $\alpha, \beta$ be defined as in Lemma 2.4. We have $0 < \beta < \alpha < \beta$ and $2y^2 = (y + \beta)^2 + (y - \alpha)^2$. Because $\beta/\alpha \to 1$ as $\alpha \to 0^+$, we may assume that $\alpha$ is sufficiently small such that $(w_{i_0-1}x_{i_0-2} + w_{i_0+1}x_{i_0+2})/\alpha - (w_{i_0}x_{i_0-1} + w_{i_0+2}x_{i_0+3}) > 0$. Define $z = (z_1 \cdots z_n)^T$ by

$$z_i = \begin{cases} x_i & \text{if } i \neq i_0, i_0 + 1, \\ y + \beta & \text{if } i = i_0, \\ y - \alpha & \text{if } i = i_0 + 1. \end{cases}$$
Then $z$ is a unit vector. We get that
\[
\begin{align*}
w(A_{\sigma_i}) &\geq \langle A_{\sigma_i}z_{\sigma_i}, z_{\sigma_i} \rangle = \sum_{i=1}^{n} w_i z_i^2 i \\
&= \sum_{i=1}^{n} w_i z_i + (w_{i0} - 1)x_{i0} - 2 + w_{i0} + 1x_{i0} + 2) \beta - (w_{i0}x_{i0} - 1 + w_{i0} + 2x_{i0} + 3) \alpha \\
&> \sum_{i=1}^{n} w_i z_i = \langle A_{\sigma_i}x_{\sigma_i}, x_{\sigma_i} \rangle = w(A_{\sigma_i}),
\end{align*}
\]
a contradiction. Hence our assertion holds. Suppose $i_0 = n - 1$. If $w_{n-2} \neq w_{n-1}$ or $x_{n-3} \neq x_{n-2}$, then $w_{n-2}x_{n-3} > w_{n-1}x_{n-2}$.

Set $y \equiv x_{n-1} = x_n$. By Lemma 2.4, there exist $\alpha$ and $\beta$ such that $0 < \beta < \alpha < y, 2\gamma^2 = (y + \beta)^2 + (y - \alpha)^2$ and $w_{n-2}x_{n-3}(\beta/\alpha) - w_{n-1}x_{n-2} - w_n(((\beta/\alpha) - 1)y - \beta) > 0$. Define $z' = (z'_1 \cdots z'_n)^T$ by
\[
\begin{align*}
z'_i &= \begin{cases}
  x_i & \text{if } 1 \leq i \leq n - 2, \\
  y + \beta & \text{if } i = n - 1, \\
  y - \alpha & \text{if } i = n.
\end{cases}
\end{align*}
\]
Then $z'$ is a unit vector and
\[
\begin{align*}
w(A_{\sigma_i}) &\geq \langle A_{\sigma_i}z'_{\sigma_i}, z'_{\sigma_i} \rangle = \sum_{i=1}^{n} w_i z'_i + w_{n-2}x_{n-3}\beta - w_{n-1}x_{n-2}\alpha + w_n(((\beta/\alpha) - 1)y - \beta) \\
&> \sum_{i=1}^{n} w_i z_i = \langle A_{\sigma_i}x_{\sigma_i}, x_{\sigma_i} \rangle = w(A_{\sigma_i}).
\end{align*}
\]
This is impossible and thus $w_{n-2} = w_{n-1}$ and $x_{n-3} = x_{n-2}$. \hfill \Box

Now we are ready to prove Theorem 2.1.

**Proof of Theorem 2.1.** Without loss of generality, we may assume that $w$ is nonnegative. Let $y = (y_1 \cdots y_n)^T$ be a nonnegative unit vector with $y_1 \geq y_2 \geq \cdots \geq y_n$. Suppose that $l$ and $l_1$ are in $S_n$. Notice that $\langle A_{l}y_l, y_l \rangle = \sum_{i=1}^{n} w_i(y_i(l)l_i(l+1) = \sum_{i=1}^{n} w_iy_i(l+1)l_i(l+1)$. Since $\sum_{i=1}^{k} y_i(l+1)l_i(l+1) \leq \sum_{i=1}^{k} \tilde{y}_i$ for all $1 \leq k \leq n$ by Lemma 2.3 and $w_1 \geq w_2 \geq \cdots \geq w_n \geq 0$, we have
\[
\begin{align*}
\langle A_{l}y_l, y_l \rangle &\leq \sum_{i=1}^{n} w_i \tilde{y}_i = \langle A_{\sigma_i}y_{\sigma_i}, y_{\sigma_i} \rangle 
\end{align*}
\]
by Lemma 2.2. Consequently, $\max_{\sigma_i \in S_n} w(A_{\sigma_i}) = w(A_{\sigma_1})$.

(b) The irreducibility of $\text{Re} A_{\sigma_1}$ and (4) guarantee the existence of a unique positive unit vector $x = (x_1 \cdots x_n)^T$ with $x_1 \geq x_2 \geq \cdots \geq x_n > 0$ such that $w(A_{\sigma_1}) = \langle A_{\sigma_1}x_{\sigma_1}, x_{\sigma_1} \rangle = \sum_{i=1}^{n} w_i x_i^2$.

(a) We only need to prove the sufficiency. Suppose that $\xi \in S_n$ satisfies $w(A_{\xi}) = \max_{\xi \in S_n} w(A_{\xi})$. Because of the irreducibility of $\text{Re} A_{\xi}$, there exist a positive unit vector $z = (z_1 \cdots z_n)^T$ with $z_1 \geq z_2 \geq \cdots \geq z_n > 0$ and an $l_2 \in S_n$ such that $w(A_{\xi}) = \langle A_{\xi}z_{l_2}, z_{l_2} \rangle$. From (4),
\[
\begin{align*}
w(A_{\xi}) &= \langle A_{\xi}z_{l_2}, z_{l_2} \rangle = \sum_{i=1}^{n} w_i z_i l_i(z_{l_2}(i+1) = \sum_{i=1}^{n} w_i z_i(l+1)z_{l_2}(i+1) \\
&\leq \sum_{i=1}^{n} z_i z_i = \langle A_{\sigma_1}z_{\sigma_1}, z_{\sigma_1} \rangle \leq w(A_{\sigma_1}) = \max w(A_{\xi}) = w(A_{\xi}).
\end{align*}
\]
This shows that the inequalities involved are actually equalities and hence $\sum_{i=1}^{n} z_i z_i(l+1)z_{l_2}(i+1) = \sum_{i=1}^{n} w_i z_i z_i$. We show that $\sum_{i=1}^{n} z_i z_i(l+1)z_{l_2}(i+1) = \sum_{i=1}^{n} z_i z_i$. If $w_\xi \neq 0$, then this equality is confirmed by Lemma 2.2. Suppose next that $w_\xi = 0$. Since $w_{n-1} \neq 0$, $\sum_{i=1}^{n} z_i z_i(l+1)z_{l_2}(i+1) = \sum_{i=1}^{n} z_i z_i \xi$ by Lemma 2.2 again and we obtain $z_i z_i(l+1)z_{l_2}(i+1) \leq \xi z_i z_i = z_i z_i$. The situation that $z_i z_i(l+1)z_{l_2}(i+1) = z_i z_i$ only occurs when $z_i z_i(l+1)z_{l_2}(i+1) = z_i z_i$. Because of $l_2(l+1) \neq l_2(l+1) + i$, we get that $z_i = z_i$, a contradiction. This ensures that $\sum_{i=1}^{n} z_i z_i(l+1)z_{l_2}(i+1) = \sum_{i=1}^{n} z_i z_i$. From Lemma 2.3, there exist $\nu, \rho \in S_n$ such that $z_i z_i(l+1)z_{l_2}(i+1) = \xi z_i z_i$ for all $1 \leq i \leq n$ and
\[
\sum_{i=1}^{n} z_i z_i = \sum_{i=1}^{n} \xi z_i z_i.
\]
We claim that $l_2 = \sigma' \varphi$ for some $\varphi \in H_n$. First consider the case that the $z_i$’s are distinct. Since $\tilde{z}_1 > \tilde{z}_2 > \cdots > \tilde{z}_{n-1} = \tilde{z}_n > 0$, we get that $\sum_{s=1}^{n} \tilde{z}_s \tilde{z}_i = \sum_{s=1}^{n} \tilde{z}_s \tilde{z}_i$ for all $1 \leq s \leq n$ with $s \neq n - 1$ by (5) and Lemma 2.2. This is equivalent to $\tilde{z}_s \tilde{z}_i = \tilde{z}_i$ for $1 \leq i \leq n - 2$ and $\tilde{z}_s \tilde{z}_i \in \{\tilde{z}_1, \tilde{z}_n\} = \{\tilde{z}_1, \tilde{z}_n\}$. We derive that $(z_{l_2})_{i=1}^{n}$ and $(\tilde{z}_i \tilde{z}_s \tilde{z}_i)_{i=1}^{n}$ consist of the same components (up to permutation), which are

$$Z_{z_1 z_2}, Z_{z_1 z_4}, \ldots, Z_{z_1 z_{n-2}}, Z_{z_{n-2} z_2}, Z_{z_{n-2} z_4}, \ldots, Z_{z_{n-2} z_{n-1}}, Z_{z_{n-1} z_2}, Z_{z_{n-1} z_4}, \ldots, Z_{z_{n-1} z_{n-1}}$$

with $\rho(n-1), \rho(n-1) \in \{z_{n-2}, z_{n-1}\}$. This only occurs when $z_{l_2} = (z_{l_2(1)} \cdots z_{l_2(n)})^T$ is such that $z_2$ and $z_3$ are in the positions near $z_1$, and $z_4$ is next to $z_3$ and on the opposite side of $z_1$, etc. Hence, we get that $l_2 = \sigma' \varphi$ for some $\varphi \in H_n$. Now suppose that $z_0 = z_{l_0+1}$ for some even index $l_0$. By Lemma 2.5, $z_i = z_{l_i+1}$ for all even indices $i$. We may assume that $z_1 > z_{l_1+1}$ for all odd indices $i$ since otherwise, by Lemma 2.5 again, $w$ and $z$ are both constant vectors and this theorem obviously holds. From (5) and Lemma 2.2, we have $\sum_{i=1}^{n} \tilde{z}_s \tilde{z}_i = \sum_{i=1}^{n} \tilde{z}_s \tilde{z}_i$ for $s = n$ and for all even indices $s$ with $2 \leq s \leq n - 1$. Hence, $\tilde{z}_s \tilde{z}_i = \tilde{z}_i \tilde{z}_i$ for all odd indices $i$ with $1 \leq i \leq n$, and, in addition, $\tilde{z}_s \tilde{z}_i = z_{l_1+1}$ when $n$ is even, and $\tilde{z}_s \tilde{z}_i = z_{l_1+1}$ when $n$ is odd. Under the condition that $n$ is even, we obtain $\tilde{z}_s \tilde{z}_i = \tilde{z}_i$ for all $1 \leq i \leq n$, and the same argument above leads us to $l_2 = \sigma' \varphi$ for some $\varphi \in H_n$ (here we do not distinguish $z_i$ and $z_{l_i+1}$ for even indices $i$ since $z_i = z_{l_i+1}$ in this case). When $n$ is odd, we have $\tilde{z}_s \tilde{z}_i = \tilde{z}_i$ for all $1 \leq i \leq n$, and the same argument above leads us to $l_2 = \sigma' \varphi$ for some $\varphi \in H_n$ (here we do not distinguish $z_i$ and $z_{l_i+1}$ for even indices $i$). For the case that $z_{l_0} = z_{l_0+1}$ for some index $l_0$, the same result can be obtained analogously. Therefore,

$$\langle A \varphi, z_{l_2} \rangle = \langle A \varphi, z_{l_2} \rangle = \langle A \varphi, z_{l_2} \rangle$$

and

$$\langle A \varphi, z_{l_2} \rangle = \langle A \varphi, z_{l_2} \rangle = \langle A \varphi, z_{l_2} \rangle$$

It can be verified that $z_{l_2} = z_{l_2}$ for all $1 \leq i \leq n$. This yields $\sum_{i=1}^{n} w_{\varphi}(i) \tilde{z}_i = \sum_{i=1}^{n} w_{\varphi}(i) \tilde{z}_i$, where $\psi = \varphi \sigma^{-1}$. With the help of Lemmas 2.2 and 2.5, we get $w_{\varphi}(i) = w_i$ for all $1 \leq i \leq n$. Hence $\psi \in F_w$ and $\zeta = \psi \sigma \varphi$ as asserted. □

### 3. Unilateral and bilateral weighted backward shifts

In the preceding section, we deal with finite weighted cyclic matrices. Related theory for infinite matrices, that is, unilateral and bilateral weighted backward shifts, is developed in this section. Before exhibiting the main theorems, we introduce a bilateral weighted backward shift which is frequently used latter. Suppose that $w = (w_i)_{i=-\infty}^{\infty}$ (resp., $w = (w_i)_{i=-\infty}^{\infty}$) is bounded. We define a nonnegative sequence $(u_i)_{i=-\infty}^{\infty}$ as follows. Let $s = \lim \sup_{k \to \infty} |w_k|$ (resp., $s = \max \{\lim \sup_{k \to \infty} |w_k|, \lim \sup_{k \to \infty} |w_k|\}$). If $\lim_{k \to \infty} |w_k| = s$, then select $u_i = s$ for all $i \geq 0$. If the set $\Omega = \{i \in N^0 : |w_i| > s\}$ (resp., $\Omega = \{i \in \mathbb{Z} : |w_i| > s\}$) is finite, we can arrange $\{(u_i)_{i \in \Omega}\}$ in nonincreasing order, say $|w_0| \geq |w_1| \geq \cdots \geq |w_n|$, and select

$$u_i = \begin{cases} |w_i| & \text{if } 0 \leq i \leq n, \\ s & \text{if } i > n. \end{cases}$$

If $\Omega$ is infinite, then there exists a one-to-one correspondence $\lambda : N^0 \to \Omega$ such that $|w_{\lambda(0)}| \geq |w_{\lambda(1)}| \geq \cdots$ and we select

$$u_i = |w_{\lambda(i)}| \text{ for all } i \geq 0.$$ Obviously, $(u_i)_{i=-\infty}^{\infty}$ is nonincreasing. Let $\sigma : \mathbb{Z} \to N^0$ be the one-to-one correspondence satisfying $\sigma(i) = 2i$ if $i \geq 0$ and $\sigma(i) = -2i - 1$ if $i < 0$, that is, $\sigma(0) = 0, \sigma(-1) = 1, \sigma(1) = 2, \sigma(-2) = 3, \sigma(2) = 4, \ldots$. We define $u_w$ as the bilateral weighted backward shift with the weight $(u_{\lambda(i)})_{i=-\infty}^{\infty}$ (resp., $(u_{\lambda(i)})_{i=-\infty}^{\infty}$). Our main results are Theorems 3.1 and 3.2. They deal with the unilateral and bilateral weighted backward shifts, respectively.
Theorem 3.1. Let \( w = (w_i)_{i=0}^{\infty} \) be bounded. We have
\[
\sup_{i \in S_{\varnothing}} w(A_i) = w(U_w).
\]
In addition, \( w(A_i) = \sup_{i \in S_{\varnothing}} w(A_i) \) for some \( \xi \in S_{\varnothing} \) if and only if \( \sup_{i \in S_{\varnothing}} |w_i| = \limsup_{k \to \infty} |w_k| = s \), and \( \sup_{i \in S_{\varnothing}} w(A_i) = s \) in this case.

Theorem 3.2. Let \( w = (w_i)_{i=-\infty}^{\infty} \) be bounded. We have
\[
\sup_{i \in S_{\xi}} w(B_i) = w(U_w).
\]
In addition, \( w(B_i) = \sup_{i \in S_{\xi}} w(B_i) \) for some \( \xi \in S_{\xi} \) if and only if one of the following conditions holds:
1. \( \sup_{i \in S_{\xi}} |w_i| = \max \{ \limsup_{|k| \to \infty} |w_k|, \limsup_{|k| \to -\infty} |w_k| \} \equiv s \),
2. \( |w_i| \to \limsup_{|k| \to \infty} |w_k| = \limsup_{|k| \to -\infty} |w_k| \) for all \( i \in S_{\xi} \), and
3. \( |w_i| \geq \limsup_{|k| \to \infty} |w_k| = \limsup_{|k| \to -\infty} |w_k| = s \) for all \( i \in S_{\xi} \) and there is an \( i_0 \in \mathbb{N} \) such that \( |w_i| = s \) for all \( |i| \geq i_0 \).

In case (1), we have \( \sup_{i \in S_{\xi}} w(B_i) = s \).

To establish Theorems 3.1 and 3.2, we introduce several lemmas, some of which are interesting on their own. Because the numerical ranges of unilateral (resp., bilateral) weighted backward shifts with the weight \( (w_i)_{i=0}^{\infty} \) (resp., \( (w_i)_{i=-\infty}^{\infty} \)) and with the weight \( (|w_i|)_{i=0}^{\infty} \) (resp., \( (|w_i|)_{i=-\infty}^{\infty} \)) are the same, we only need to consider the unilateral and bilateral weighted backward shifts with nonnegative weights. We may further assume that all the \( w_i \)'s are nonzero. Otherwise, they are the direct sum of weighted cyclic matrices or unilateral weighted backward shifts, and their numerical ranges are the largest numerical ranges of these summands. For any \( B = (B_{ij})_{i,j=0}^{\infty} \) (resp., \( B = (B_{ij})_{i,j=-\infty}^{\infty} \)) and \( 0 \leq m \leq n \leq \infty \) (resp., \( -\infty \leq m \leq n \leq \infty \)), we let \( B[m, n] \) denote the matrix \( (B_{ij})_{i,j=m}^{n} \).

Lemma 3.3. Let \( w = (w_i)_{i=0}^{\infty} \) (resp., \( w = (w_i)_{i=-\infty}^{\infty} \)) be positive and bounded. We have
\[
\sup_{i \in S_{\varnothing}} w(A_i) = w(U_w) \quad \text{(resp., } \sup_{i \in S_{\xi}} w(B_i) = w(U_w) \text{)}.
\]

Proof. Suppose \( i \in S_{\varnothing} \) (resp., \( i \in S_{\xi} \)). For a fixed \( n \in \mathbb{N} \), \( A_i[i, 2n \cdot ] \) (resp., \( B_i[i, 2n \cdot ] \)) is of the form
\[
A_i = \begin{pmatrix}
0 & w_{i(0)} & & \\
0 & 0 & \ddots & \\
& \ddots & \ddots & w_{i(2n-1)} \\
& & 0 & w_{i(n)}
\end{pmatrix},
\]
resp.,
\[
B_i = \begin{pmatrix}
0 & \cdots & \cdots & w_{i(n-1)} & w_{i(n)} \\
0 & \cdots & \cdots & 0 & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & 0 & \cdots \\
0 & \cdots & \cdots & 0 & \cdots \\
\end{pmatrix}.
\]

Let \( (c_{i})_{i=1}^{2n+1} \) be the rearrangement of \( \{w_{i(0)} \}_{i=0}^{2n-1} \cup \{0\} \) (resp., \( \{w_{i(0)} \}_{i=-n-1}^{n} \cup \{0\} \)) in nonincreasing order and set
\[
M_{2n+1} \equiv \begin{pmatrix}
0 & c_{\sigma_{2n+1}(1)} & & \\
0 & \ddots & \ddots & \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & c_{\sigma_{2n+1}(2n)}
\end{pmatrix},
\]
where \( \sigma_{2n+1} \in S_{2n+1} \) is defined as in Section 1. Notice that \( c_{\sigma_{2n+1}(2n+1)} = c_{2n+1} = 0 \). From Theorem 2.1, \( w(A_i[0, 2n]) \leq w(M_{2n+1}) \) (resp., \( w(B_i[-n, n]) \leq w(M_{2n+1})) \). Since each entry of \( M_{2n+1} \) is less than or equal to the corresponding one of \( U_w[-n, n] \), [1, Corollary 3.6] implies that \( w(M_{2n+1}) \leq w(U_w[-n, n]) \). Hence,
\[
w(A_i) = w \left( \bigcup_{n=1}^{\infty} A_i[0, 2n] \right) \leq w \left( \bigcup_{n=1}^{\infty} U_w[-n, n] \right) = w(U_w),
\]
(resp., \( w(B_i) = w \left( \bigcup_{n=1}^{\infty} B_i[-n, n] \right) \leq w \left( \bigcup_{n=1}^{\infty} U_w[-n, n] \right) = w(U_w) \)).
by Wang and Wu [8, Lemma 2.6(a)]. We conclude that \( \sup_{t \in S_0} w(A_t) \leq w(U_w) \) (resp., \( \sup_{t \in S_z} w(B_t) \leq w(U_w) \)). It remains to prove the converse. Let \( \epsilon > 0 \). We may pick an \( n_0 \in \mathbb{N} \) such that \( w(U_w[-n_0, n_0]) > w(U_w) - \epsilon \) because \( w(U_w) = w(\bigcup_{n=1}^{\infty} U_w[-n, n]) \). We observe that the way to construct \( U_w \) guarantees the existence of some \( l' \in S_0 \) (resp., \( l' \in S_z \)) which satisfies \( w(A_{l'}[0, 2n_0]) > w(U_w[-n_0, n_0]) - \epsilon \) (resp., \( w(B_{l'}[-n_0, n_0]) > w(U_w[-n_0, n_0]) - \epsilon \)) since the eigenvalues of the real part of a finite matrix depends continuously on its entries (cf. [11]). Consequently,

\[
\begin{align*}
\sup_{t \in S_0} w(A_t) &\geq w(A_{l'}[0, 2n_0]) > w(U_w[-n_0, n_0]) - \epsilon > w(U_w) - 2\epsilon \\
\sup_{t \in S_z} w(B_t) &\geq w(B_{l'}[-n_0, n_0]) > w(U_w[-n_0, n_0]) - \epsilon > w(U_w) - 2\epsilon
\end{align*}
\]

and we obtain that \( \sup_{t \in S_0} w(A_t) \geq w(U_w) \) (resp., \( \sup_{t \in S_z} w(B_t) \geq w(U_w) \)). This completes the proof. \( \square \)

The next lemma provides a sufficient condition under which \( w(A_t) = \sup_{t \in S_0} w(A_t) \) (resp., \( w(B_t) = \sup_{t \in S_z} w(B_t) \)) for some \( \xi \in S_0 \) (resp., \( \zeta \in S_z \)). In order to simplify the notations, for each \( s \geq 0 \), let \( A(s) \) and \( B(s) \) be the unilateral and bilateral weighted backward shifts with the constant weight \( w_i \equiv s \), respectively, that is,

\[
A(s) = \begin{pmatrix} 0 & s & 0 & \cdots \\ s & 0 & s & \cdots \\ 0 & \cdots & \cdots & \cdots \\
\end{pmatrix} \quad \text{and} \quad B(s) = \begin{pmatrix} 0 & s & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ s & 0 & s & \cdots \\
\end{pmatrix}.
\]

We have \( w(A(s)) = w(B(s)) = s \) by [8, Corollary 4.7 and Theorem 4.9].

**Lemma 3.4.** Let \( w = (w_i)_{i=0}^{\infty} \) (resp., \( w = (w_i)_{i=-\infty}^{\infty} \)) be positive and bounded. If \( \sup_{k \in \mathbb{N}} w_k = \limsup_{k \to \infty} w_k \equiv s \) (resp., \( \sup_{k \in \mathbb{Z}} w_k = \max(\limsup_{k \to \infty} w_k, \limsup_{k \to -\infty} w_k) \equiv s \)), then there exists a \( \xi \in S_0 \) (resp., \( \zeta \in S_z \)) such that \( w(A_{\xi}) = \sup_{t \in S_0} w(A_t) = w(U_w) = s \) (resp., \( w(B_{\zeta}) = \sup_{t \in S_z} w(B_t) = w(U_w) = s \)).

**Proof.** We only prove the unilateral case; the bilateral case can be done analogously. Let \( w = (w_i)_{i=0}^{\infty} \) be positive, bounded and satisfy \( \sup_{k \in \mathbb{N}} w_k = \limsup_{k \to \infty} w_k \equiv s \). We get that \( U_w = A(s) \) and therefore, \( \sup_{t \in S_0} w(A_t) = w(U_w) = s \) by Lemma 3.3. Now pick a subsequence \( (n_i)_{i=0}^{\infty} \) of \( N_0 \) such that \( w_{n_i} \to s \) as \( i \to \infty \). Let \( \xi : N_0 \to N_0 \) be any one-to-one correspondence with \( \xi(2^k + j) = n_{2^k+j} \) for all \( k = 0, 1, \ldots \) and all \( j \) satisfying \( 0 \leq j < 2^k \). For each \( p = 1, 2, \ldots \), define \( x^{(p)}_i = \xi^{(p)}_i \) by

\[
x^{(p)}_i = \begin{cases} 2^{-p/2} & \text{if } 2^p \leq i < 2^{p+1} + 2^p - 1, \\
0 & \text{otherwise}.
\end{cases}
\]

We have \( \|x^{(p)}_i\| = 1 \) and

\[
(A_{\xi} x^{(p)}_i, x^{(p)}_i) = \frac{1}{2^p} (w_{n_{2^p}} + w_{n_{2^p+1}} + \cdots + w_{n_{2^{p+1}}}) \to s \quad \text{as} \quad p \to \infty.
\]

Hence \( w(A_{\xi}) = s \) as required. \( \square \)

For any complex sequence \( x = (x_i)_{i=0}^{\infty} \), we define \( \tilde{x} = (\tilde{x}_i)_{i=0}^{\infty} \) and \( \tilde{x} = (\tilde{x}_i)_{i=0}^{\infty} \) by

\[
\tilde{x}_i = \begin{cases} x_0 & \text{if } i = 0, \\
x_{i-1} & \text{if } i \geq 1,
\end{cases} \quad \text{and} \quad \tilde{x}_i = x_{i-1} \quad \text{for all } i \geq 0.
\]

A result which is parallel to Lemma 2.3 is also obtained.

**Lemma 3.5.** Let \( x = (x_i)_{i=0}^{\infty} \) be nonnegative and nonincreasing, and let \( 1, \eta \in S_0 \). For each \( k = 0, 1, \ldots \), we have

\[
\sum_{i=0}^{k} x_{\eta(i)} x_{\eta(i)+1} \leq \sum_{i=0}^{k} \tilde{x}_i \tilde{x}_i.
\]

In addition, if \( x \) is positive and \( x_i \to 0 \) as \( i \to \infty \), then there exist \( k_0 \in \mathbb{N} \) and \( \mu > 0 \) such that \( \sum_{i=0}^{k} x_{\eta(i)} x_{\eta(i)+1} + \mu < \sum_{i=0}^{k} \tilde{x}_i \tilde{x}_i \) for all \( k \geq k_0 \).
Proof. The first part of this proof is similar to the one for Lemma 2.3. Let \( l, \eta \in S_{\phi} \) and suppose \( l(p_0) = 0 \). Define \( y = (y_i)_{i=0}^{\infty} \) by \( y \equiv x \) if \( p_0 = 0 \), and \( y \equiv (x_0, x_1, x_2, \ldots, x_{l(0)+1}, \ldots) \) if \( p_0 > 0 \). That is, \( y \) is obtained by deleting the \( l(0) + 1 \)st component of \( x \). There exist \( \phi, \psi \in S_{\phi} \) satisfying

\[
\tilde{x}_\phi(i) = \begin{cases} 
  x_0 & \text{if } 0 \leq i < p_0, \\
  x_{l(i+1)} & \text{if } i \geq p_0,
\end{cases}
\quad \text{and} \quad
\tilde{y}_\psi(i) = \begin{cases} 
  x_{0(i+1)} & \text{if } 0 \leq i < p_0, \\
  x_{0(i)} & \text{if } i \geq p_0.
\end{cases}
\]

It is trivial that \( x_0(i+1)\tilde{x}_{\phi(i)} = \tilde{x}_{\phi(i)}y_{\psi(i)} \) for all \( i \in \mathbb{N}^0 \). Fix \( k = 0, 1, \ldots \). We may choose \( a \in \mathbb{N}^0 \) satisfying \( 0 \leq v(i) \leq k \) for all \( 0 \leq i \leq k \) and \( \phi(\eta(v(0))) > \phi(\eta(v(1))) > \cdots > \phi(\eta(v(k))) \). Because \( \tilde{x}_i \geq \tilde{x}_{\phi(i)}y_{\psi(i)} \) for all \( 0 \leq i \leq k \), we have

\[
\sum_{i=0}^{k} x_{\eta(i)}x_{\eta(i)+1} = \sum_{i=0}^{k} \tilde{x}_{\phi(i)}y_{\psi(i)} = \sum_{i=0}^{k} \tilde{x}_{\phi(i)}y_{\psi(i)} \leq \sum_{i=0}^{k} \tilde{x}_iy_{\psi(i)}.
\]

where \( l_1 = \phi \eta \nu \) and \( l_2 = \psi \eta \nu \). Obviously, \( y_i \leq \tilde{x}_i \) for all \( i \in \mathbb{N}^0 \) and this implies \( \sum_{i=0}^{j} \tilde{x}_iy_{\psi(i)} \leq \sum_{i=0}^{j} \tilde{x}_i \) for all \( 0 \leq j \leq k \). By Lemma 2.2, we obtain

\[
\sum_{i=0}^{k} x_{\eta(i)}x_{\eta(i)+1} \leq \sum_{i=0}^{k} \tilde{x}_iy_{\psi(i)} \leq \sum_{i=0}^{k} \tilde{x}_i \tilde{x}_i.
\]

Now assume that \( x_i > 0 \) for all \( i \in \mathbb{N}^0 \) and \( x_i \to 0 \) as \( i \to \infty \). There are \( t_0, t_1 \in \mathbb{N} \) such that \( l(0) < t_0 \leq t_1 - 2, x_{t_0-1} > x_{t_0} \) and \( x_{t_1-1} > x_{t_1} \). If \( k \geq t_1 \), then from (1) and (6),

\[
\sum_{i=0}^{k} \tilde{x}_iy_{\psi(i)} - \sum_{i=0}^{k} x_{\eta(i)}x_{\eta(i)+1} \geq \sum_{i=0}^{k} \tilde{x}_iy_{\psi(i)} - \sum_{i=0}^{k} \tilde{x}_iy_{\psi(i)} \leq \sum_{i=0}^{k} \tilde{x}_iy_{\psi(i)} \geq (x_{t_1-1} - x_{t_1})x_{t_0-1} - x_{t_0} > 0.
\]

This completes the proof. \( \square \)

Lemma 3.5 tells us that for a nonnegative and nonincreasing sequence \( x = (x_i)_{i=0}^{\infty}, (\tilde{x}_i)_{i=0}^{\infty} \) weakly majorizes \((x_{\eta(i)}x_{\eta(i)+1})_{i=0}^{\infty} \). In addition, if \( x \) is positive and \( x_i \to 0 \) as \( i \to \infty \), \( \sum_{i=0}^{k} (\tilde{x}_i - x_{\eta(i)}x_{\eta(i)+1}) > 0 \) is away from 0 for sufficiently large \( k \). In Lemma 3.6, we describe a necessary condition for the existence of some \( \zeta \in S_{\phi} \) such that \( w(A_\zeta) = \sup_{l \in S_{\phi}} w(A_l) \).

**Lemma 3.6.** Let \( w = (w_i)_{i=0}^{\infty} \) be positive and bounded. If \( w(A_\zeta) = \sup_{l \in S_{\phi}} w(A_l) \) for some \( \zeta \in S_{\phi} \), then \( W(A_\zeta) \) is open.

**Proof.** We show that if \( W(A_\zeta) \) is closed, then there exists a \( \zeta' \in S_{\phi} \) such that \( w(A_\zeta') < w(A_\zeta) \). From [8, Proposition 2.1(b)], there exist a positive unit vector \( x = (x_i)_{i=0}^{\infty} \) with \( x_0 \geq x_1 \geq \cdots \) and \( l_i \in S_{\phi} \) satisfying \( w(A_i) = \langle A_i x_1, x_1 \rangle = \sum_{i=0}^{\infty} w_{\zeta(i)}x_1x_{\eta(0)i}x_{\eta(0)i+1} \). Since \( x_i \to 0 \) as \( i \to \infty \), we have \( x_{\eta(0)i}x_{\eta(0)i+1} \to 0 \) as \( i \to \infty \). This guarantees the existence of some \( t \in S_{\phi} \) such that \( (x_{\eta(0)i}x_{\eta(0)i+1})_{i=0}^{\infty} \) is nonincreasing. By Lemma 3.5, \( \sum_{i=0}^{\infty} x_{\eta(0)i}x_{\eta(0)i+1} \leq \sum_{i=0}^{\infty} \tilde{x}_i \tilde{x}_i \) for all \( k = 0, 1, \ldots \), and there exist \( k_0 \in \mathbb{N} \) and \( \mu > 0 \) such that \( \sum_{i=0}^{k_0} x_{\eta(0)i}x_{\eta(0)i+1} + \mu < \sum_{i=0}^{\infty} \tilde{x}_i \tilde{x}_i \) for all \( \mu \geq 0 \). Now we may pick an \( N \in \mathbb{N} \) satisfying \( N > k_0 \) and \( \sum_{i=0}^{N} w_{\zeta(i)}x_{\eta(0)i}x_{\eta(0)i+1} \geq (\min_{l \leq k_0} w_{\zeta(i)}) \mu \). Let \( \nu \in S_{\phi} \) be such that \( 0 \leq \nu(i) \leq N \) for all \( 0 \leq i \leq N \) and \( w_{\zeta(0)} \geq w_{\zeta(0)} \geq \cdots \geq w_{\zeta(n)} \). We have \( w_{\zeta(0)} \geq \min_{l \leq k_0} w_{\zeta(i)} \) and \( \sum_{i=0}^{N} x_{\eta(0)i}x_{\eta(0)i+1} \leq \sum_{i=0}^{\infty} x_{\eta(0)i}x_{\eta(0)i+1} \) for all \( k = 0, 1, \ldots \). From (1),

\[
\sum_{i=0}^{N} w_{\zeta(0)} \tilde{x}_i \tilde{x}_i - \sum_{i=0}^{N} w_{\zeta(i)}x_{\eta(0)i}x_{\eta(0)i+1} = \sum_{i=0}^{N} w_{\zeta(0)} \tilde{x}_i \tilde{x}_i - \sum_{i=0}^{N} w_{\zeta(i)}x_{\eta(0)i}x_{\eta(0)i+1} = \sum_{i=0}^{N} (w_{\zeta(0)} - w_{\zeta(i)}) \tilde{x}_i \tilde{x}_i - \sum_{i=0}^{N} w_{\zeta(i)}x_{\eta(0)i}x_{\eta(0)i+1} = \sum_{i=0}^{N} \mu \tilde{x}_i \tilde{x}_i \geq \sum_{i=0}^{N} \mu \tilde{x}_i \tilde{x}_i.
\]


Therefore, if $\zeta' \in S_{\alpha_0}$ is the rearrangement such that $A_{w',\sigma_{N+1}}$ is a principal submatrix of $A_{\zeta'}$, then $w(A_{\zeta}) \geq \sum_{i=0}^{N} w_{(i)} \hat{x}_i > w(A_{\zeta'})$ as asserted. □

**Proof of Theorem 3.1.** We may suppose that $w$ is positive. With the help of Lemmas 3.3 and 3.4, we only need to show that if $w(A_{\zeta}) = \sup_{i \in S_{\alpha_0}} w(A_i)$ for some $\zeta \in S_{\alpha_0}$, then $\sup_{k \geq 0} w_k = \limsup_{k \to \infty} w_k$. If otherwise, then there exists an $i_0 \in N^0$ such that $w_0 = \sup_{k \in N} w_k > \limsup_{k \to \infty} w_k \equiv s$. By multiplying a suitable scalar, we may assume that $w(A_{\zeta}) = 1$. This implies $1 = w(A_{\zeta}) = w(U_w) \geq w(B(s)) = s$. Suppose that $s = 1$. Let $w' = (w')_0^\infty$ be defined by $w'_0 = w_0$ and $w'_i = 1$ for all $i \geq 1$. We get $1 = w(U_w) \geq w(U_{w'}) = (w_0^2 + 1)/2w_0 > 1$ by Wang and Wu [8, Theorem 4.9(b)], which is impossible. Suppose $s < 1$ and let $S_0 \in (s, 1)$. From Lemma 3.6, $W(A_{\zeta})$ is open and hence $W(A_{\zeta}) = W(A_{\zeta}[m, \infty))$ for all $m \in N$ by Wang and Wu [8, Proposition 2.4]. We get $w(A_{\zeta}) = w(A_{\zeta}[m_0, \infty)) \leq w(A(S_0)) = s_0 < 1$ for sufficiently large $m_0$, a contradiction. This finishes the proof. □

To prove Theorem 3.2, we need the necessary and sufficient conditions for the existence of some $\zeta \in S_{\alpha_0}$ which satisfies $U_w = B_{\zeta}$.

**Lemma 3.7.** Let $w = (w_i)_{i=-\infty}^{\infty}$ be positive and bounded. The following statements are equivalent:

(a) $U_w = B_{\zeta}$ for some $\zeta \in S_{\alpha_0}$,

(b) there exists a one-to-one correspondence $\phi : N^0 \to Z$ such that $w_{\phi(0)} \geq w_{\phi(1)} \geq \cdots$, and

(c) the condition (2) or (3) in Theorem 3.2 holds.

**Proof.** (a) $\iff$ (b) If $U_w = B_{\zeta}$ for some $\zeta \in S_{\alpha_0}$, we have $w_{\phi(0)} \geq w_{\phi(1)} \geq \cdots$, where $\phi = \zeta \sigma^{-1}$ and $\sigma$ is defined in the first paragraph in this section, and $\phi$ is the desired one-to-one correspondence from $N^0$ to $Z$. Conversely, if (b) is true, then we select $\zeta = \sigma\phi \in S_{\alpha_0}$ and get $B_{\zeta} = U_w$.

(b) $\iff$ (c) Let $s \equiv \max\{\limsup_{k \to \infty} w_k, \limsup_{k \to -\infty} w_k\}$. Because the set $\{i \in Z : w_i > s' \}$ is finite for any $s' > s$, the implication (c) $\Rightarrow$ (b) is trivial. Now suppose that (b) holds. We first claim that $w_i \geq s$ for all $i \in Z$. If otherwise, then we may choose a $p \in N^0$ such that $w_p < s$ and $w = \phi(t)$ for some $t \in N^0$. Since there are infinitely many $w_i$’s greater than $w_p$, we can find some $t' \in N^0$ satisfying $t' > t$ and $w_{\phi(t')} = w_p = \phi(t)$, a contradiction. Hence, $w_i \geq s$ for all $i \in Z$. This implies that $\limsup_{k \to \infty} w_k = \limsup_{k \to -\infty} w_k = s$. Moreover, if $w_i = s$ for some $i_0 \in Z$ and the set $\{i \in Z : w_i > s\}$ is infinite, the same argument also leads us to a contradiction. Hence, (c) is true. □

**Lemma 3.8.** Let $B$ be the bilateral weighted backward shift with the positive and bounded weight $w = (w_i)_{i=-\infty}^{\infty}$. Suppose that $x^{(p)} = (x^{(p)}_i)_{i=-\infty}^{\infty}$, $p = 1, 2, \ldots$, are nonnegative unit vectors such that $\langle Bx^{(p)}, x^{(p)} \rangle \to w(B)$ as $p \to \infty$, and suppose that, for each $i \in N^0$, $x^{(p)}_i \to \alpha_i$ as $p \to \infty$ for some $\alpha_i \geq 0$. We have

(a) $\alpha_i \to 0$ as $i \to \pm \infty$, and

(b) if $\alpha_{i_0} = 0$ for some $i_0$, then $W(B)$ is open.

**Proof.** (a) By Fatou’s lemma, $\sum_{i=-\infty}^{\infty} \alpha_i^2 = \sum_{i=-\infty}^{\infty} \liminf_{p \to \infty} (x^{(p)}_i)^2 \leq \liminf_{p \to \infty} \sum_{i=-\infty}^{\infty} (x^{(p)}_i)^2 = 1$ (cf. [12, Theorems 5.17 and 10.29]). Therefore, $\sum_{i=-\infty}^{\infty} \alpha_i^2$ converges and $\alpha_i \to 0$ as $i \to \pm \infty$.

(b) Suppose $\alpha_{i_0} = 0$ for some $i_0$. From $\langle Bx^{(p)}, x^{(p)} \rangle \to w(B)$ and $x^{(p)}_i \to 0$ as $p \to \infty$, we get that $\langle \hat{B}x^{(p)}, x^{(p)} \rangle = \sum_{i=-\infty}^{-2} w_i x^{(p)}_{i+1} + \sum_{i=0}^{\infty} w_i x^{(p)}_{i+1} \to w(B)$ as $p \to \infty$, where $B$ is the bilateral weighted backward shift with the weight $(\cdots w_0 \cdot \cdots w_{i_0-2} \in \delta w_{i_0+1} \cdot \cdots)$ with $0 < \epsilon < w_{i_0-1}$ and $0 < \delta < w_{i_0}$. This implies that $w(B) \geq w(B)$. By [8, Proposition 2.5(a)], we have $w(B) \leq w(B)$ and hence $w(B) = w(B)$. Therefore, $W(B)$ is open by Wang and Wu [8, Proposition 2.5(b)]. □

We remark that the preceding lemma also holds for the unilateral case and the proof can be obtained analogously. It is now time to derive Theorem 3.2.
Proof of Theorem 3.2. We may assume that \( w \) is positive. From Lemmas 3.3, 3.4 and 3.7, we only need to show that if none of (1), (2) and (3) holds, then \( w(B_\zeta) < \sup_{\zeta \in S_2} w(B_\eta) \) for all \( \zeta \in S_2 \). Suppose that this is the case, and \( w(B_\zeta) = \sup_{\zeta \in S_2} w(B_\eta) \) for some \( \zeta \in S_2 \). Set \( s = \max\{\limsup_{k \to \infty} w_k, \limsup_{k \to \infty} w_k\} \). Then \( s > 0 \), and there exist indices \( i_1, i_2 \in \mathbb{N}^0 \) and a subsequence \( (\eta_j)_{j=0}^\infty \) of \( \mathbb{N}^0 \) such that \( w_{\eta(i_1)} > s \geq w_{\zeta(i_2)} \) and \( w_{\eta(i_0)} > w_{\zeta(i_1)} \) for all \( j = 0, 1, \ldots \). Let \( x^{(p)} = (x^{(p)}_i)_{i=-\infty}^\infty, p = 1, 2, \ldots \), be nonnegative unit vectors satisfying \( \langle B_\zeta x^{(p)}, x^{(p)} \rangle \to w(B_\zeta) \) as \( p \to \infty \). Applying the diagonal process, we may assume that, for each \( i \in \mathbb{N}^0 \), \( x^{(p)}_i \to \alpha_i \) as \( p \to \infty \) for some \( \alpha_i \geq 0 \). Lemma 3.8(a) ensures that \( \alpha_i \to 0 \) as \( i \to \pm \infty \). If \( \alpha_{i_0} = 0 \) for some \( i_0 \in \mathbb{N}^0 \), then Lemma 3.8(b) says that \( W(B_\zeta) \) is open, and \( w(B_\zeta) = w(B_{\zeta_0}) \) for all \( m \geq 1 \) by [8, Proposition 2.4], where \( B_{\zeta_0} \) is the bilateral weighted backward shift with the weight \((\cdots, w_{-m-1}, w_{-m}, w_m, w_{m+1}, \ldots)\). Let \( s_0 \) be such that \( s < s_0 < (w_{\eta(i_1)} + (s^2/w_{\zeta(i_1)})/2 \). We have \( w(B_\zeta) = w(B_{\eta_0}) \leq w(B_{\zeta_0}) = s_0 \) for sufficiently large \( m_0 \). On the other hand, define \( w' = (w'_n)_{n=-\infty}^\infty \) by \( w'_0 = w_{\eta(i_1)} \) and \( w'_i = s \) for all \( i \neq 0 \). By Lemma 3.3 and [8, Theorem 4.9(b)], we have
\[
 w(B_\zeta) = w(U_w) \geq w(U_{w'}) = s \cdot \left( \left( \frac{w_{\eta(i_1)}}{s} \right)^2 + 1 \right) / \left( 2 \cdot \left( \frac{w_{\eta(i_1)}}{s} \right) \right) = \frac{1}{2} \left( \frac{w_{\eta(i_1)} + \frac{s^2}{w_{\eta(i_1)}}}{w_{\eta(i_1)}} \right) > s_0,
\]
which is impossible. We conclude that \( \alpha_i > 0 \) for all \( i \in \mathbb{Z} \). Let \( \zeta_0 \) be the index such that \( \alpha_{\eta_0} \alpha_{\eta_0+1} < \alpha_i \alpha_{i+1} \). Define \( \zeta' \in S_2 \) by \( \zeta'(i_1) = \zeta(i_1) \), \( \zeta'(i_2) = \zeta(i_2) \) and \( \zeta'(i) = \zeta(i) \) for all \( i \neq i_1, i_0 \). From elementary calculation, we obtain
\[
\langle B_\zeta x^{(p)}, x^{(p)} \rangle = \langle B_\zeta x^{(p)}, x^{(p)} \rangle + (w_{\eta(i_1)} - w_{\eta(i_2)})x^{(p)}_{\eta_0}x^{(p)}_{\eta_0+1} + (w_{\eta(i_0)} - w_{\eta(i_1)})x^{(p)}_{\eta_0}x^{(p)}_{\eta_0+1} \to w(B_\zeta) + (w_{\eta(i_0)} - w_{\eta(i_1)})\alpha_{i_0} \alpha_{i_0+1} \to \infty \text{ as } p \to \infty.
\]
Therefore, \( w(B_{\zeta'}) > w(B_\zeta) \), a contradiction. This completes the proof. \( \square \)

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