A HYBRID METHOD FOR SOLVING PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS

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Abstract—The temperature distribution across an one-dimensional thermally conductive, uniform slab was found at discrete space points using continuous-time, discrete-space (CTDS) technique on the University of Cincinnati AD-4 hybrid computer facilities. In addition, a pure analog computer solution was also investigated, so that a comparison can be made between these two methods. The agreement between the analog and hybrid methods is satisfactorily good.

INTRODUCTION

A hybrid computer is in principle a combination of a digital and an analog computer together with some interface devices in such a way that these two can operate as one unit. Analog computers are particularly well suited for the solution of large set of coupled simultaneous ordinary differential equations. These equations can be solved simultaneously on the analog computer at high speed without computing penalty. On the other hand, digital computers are capable of preforming logic decisions both quickly and accurately. By combining analog computer's ability to quickly solve simultaneous differential equations with a digital computer's prodigious memory and ability to rapidly make complex logic decisions, one can obtain a hybrid system that combines the best features of both analog and digital machines.

The interface devices of a hybrid computer include analog to digital converters (ADC), digital to analog converter (DAC) and some logic control signals. The analog to digital converters provide for the conversion of analog signals present at the analog console to sixteen digital words. A multiplexer provides for scanning sixteen such analog signals in succession and stores in the digital memory for later playback. The digital to analog converters are really digital registers that control electronic switches attached to resistor networks. The output terminal of the resistor networks can be connected to the summing junctions of computing amplifiers and thereby provide a very high speed technique for transferring a digital value held in the digital computer to a corresponding analog value which then appears at the output terminal of the computing operational amplifier. Thus in summary, digital or analog information can be accepted by the system and can be exchanged by the system at high speed under the control of digital logic signals or by digital computer program written in the convenient language of Fortran. Of course, the analog or
digital computer of the hybrid system can work as a separate computer independent of the other if desired. Fig. 1 shows a schematic diagram for a hybrid computer.

![Schematic Diagram](image)

**Fig. 1. Hybrid Computer Schematic Diagram**

For a number of years it has been suggested that one of the fruitful field of application for hybrid computation might lie in the study of distributed systems. In principle, the combination of analog speed with the memory and logic capabilities of digital computers should make it possible to solve partial differential equations both efficiently and rapidly. One of the most promising technique for hybrid solution of partial differential equations involves in a CTDS (continuous-time discrete-space) analog mechanization of a space nodal point (or a group of nodal points) with the digital computer used for storage and playback of guessed or previously computed solutions of the surrounding nodal points. The nodal point simulated on the analog computer is then integrated over the entire region to obtain the solution for the first major iteration. Iterations will continue until solutions converge to a satisfactory degree of accuracy.

In the digital computer solution of a partial differential equations, both time and space variables are discretized by applying finite difference method. The number of difference equations equal the number of space nodal points times the number of time increments. In the analog computer solution of partial differential equations, the equations can be approximated by a set of coupled ordinary differential equations and can be solved simultaneously. However, unless the problem is simple, and the number of nodal points is small, pure analog methods frequently requires large quantities of hardware and pure digital solutions are always extremely time-consuming, thereby justifying the idea of hybrid computation.

**ONE-DIMENSIONAL DIFFUSION EQUATIONS**

An infinite one-dimensional thermally conductive, uniform slab of length L is initially at a temperature distribution of 10°C. Its left boundary at \( x=0 \) is held at zero temperature and the right boundary at \( x=L \) is held at 100°C. The temperature temperature distribution along this slab is described by the equation

\[
\frac{\partial^2 T(x, t)}{\partial x^2} = \frac{\partial T(x, t)}{\partial t} \tag{1}
\]

with the boundary conditions
The temperature is to be evaluated at \( n \) equally-spaced discrete nodal points, i.e. the spatial variable \( x \) is considered to be discrete. Applying the central finite difference approximation with respect to \( x \) and preserving time as a continuous variable, the temperature at the \( i \)th nodal point becomes:

\[
\begin{align*}
\frac{dT_i(t)}{dt} &= \frac{1}{(\Delta x)^2} \left[ T_{i+1}^{(t)} - 2T_i^{(t)} + T_{i-1}^{(t)} \right] \\
&= \frac{1}{(\Delta x)^2} \left[ T_i^{(t+1)} - 2T_i^{(t)} + T_i^{(t-1)} \right] \\
&\text{for } i = 1, 2, \ldots, n
\end{align*}
\]  

(2)

where

\[ T_i(t) = \text{Temperature at the } i \text{th nodal point.} \]

\[ \Delta x = \text{The distance between two nodal points.} \]

**HYBRID COMPUTER SET-UP AND SOLUTION**

The analog circuit for solving the resulting \( n \) simultaneous differential equations of eqn. (2) all in parallel is shown in Fig. 2. Although the all-parallel analog approach is quite feasible for one-dimensional diffusion equations with a small

![Fig. 2. Analog Circuit For Solving One-dimensional Heat Diffusion Equation with 5 Space Nodal Points.](image-url)
number of nodal points, it soon becomes impractical due to the analog hardware limitations when a large number of nodal points, say, 50 points, is required.

An alternative is to set up only one nodal point on the analog patchboard of the hybrid computer with the digital computer used for storage and playback of guessed or previously computed solutions of surrounding nodal points. In this CTDS hybrid method, the set of \( n \) equations are solved iteratively, one at a time. Thus in Fig. 3 the single integrator circuit is used to solve eqn. (2) at the \( i \)th nodal point with the voltages \( T_{i-1} \) and \( T_{i+1} \) reproduced from a previous solution. This circuit is first used to compute \( T_1 \) with \( T_0 \) given by prescribed boundary condition and \( T_0 \) given by the initial condition. The resulting solution \( T_1 \) is then sampled and converted to digital words and stored in digital computer. Next the circuit is used to compute \( T_2 \) at the second nodal point with \( T_1 \) obtained by playing back the just-computed solution through a DAC (digital to analog converter) and \( T_0 \) given by the initial condition. The resulting solution \( T_2 \) is then converted to digital form again and stored for playback. The process is repeated, using the circuit shown in Fig. 3 to compute \( T_3, T_4, \ldots, T_n \) respectively. This completes a major iteration.

Fig. 3. Hybrid Computer Circuit for Solving One-dimensional Heat Equations.

The second major iteration is begun with the circuit again representing \( T_1 \), but this time \( T_2 \) was the solution obtained previously in the first major iteration. This single integrator circuit is then iterated sequentially across the one-dimensional slab over and over again until all the \( T_i \)'s converge to within a small prescribed value between successive iterations. In this arrangement, the temperature at the \( i \)th nodal point for the \( k \)th major iteration is represented as:

\[
\frac{dT_i^k}{dt} = \frac{1}{(\Delta x)^2} \left( T_{i-1}^k - 2T_i^k + T_{i+1}^k \right) \tag{3}
\]

where

\( i \) = Number of nodal points.
$k=$Number of major iterations.

It is always wanted to estimate the time needed to achieve the solution. This is a direct function of the number of iterations necessary for convergence. The speed of convergence depends upon the number of nodal points, and the initial guess used to start the iteration procedure.

RESULTS OF HYBRID CALCULATION

A flow chart for hybrid program used for solving the foregoing heat flow problem is shown in fig. 4. The hybrid program written in the convenient language of FORTRAN is listed in appendix. The solutions were calculated on the University of Cincinnati hybrid facilities which is a combination of an AD-4 analog computer and an IBM 1130 digital computer. All the hybrid communication routines and the addresses of the hybrid devices used in the program were written by the author according to the "HYBRID COMMUNICATION ROUTINES OPERATORS'S MANUAL.

![Flow Chart]

Fig. 4. Hybrid Flow Chart For Solving One-dimensional parabolic Partial Differential Equations.
BY APPLIED DYNAMICS, INC. A detail explanation of the program is also provided in appendix.

Table 1 shows the number of iterations and total computer time needed for solving the heat flow problems for five space nodal points and for three space nodal points. Actually the total computer time includes the time for execution and the print-out time. It is seen both computer time and number of iterations increases with the increase in nodal points.

<table>
<thead>
<tr>
<th></th>
<th>Number of Iterations</th>
<th>Total Computer Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Five Stations</td>
<td>12</td>
<td>5 min. 20 sec.</td>
</tr>
<tr>
<td>Three Stations</td>
<td>6</td>
<td>3 min. 10 sec.</td>
</tr>
</tbody>
</table>

Fig. 5 shows the solutions of $T_x^i$ for a five nodal stations problem after various iterations. It can be seen from Fig. 5 that the solutions converge after 12 iterations. The numerical values for each nodal station after 12 iterations are shown in table 2.

Fig. 5. Iterative solutions for $T_x^i$ by hybrid computation
Table 2. Hybrid results for 5 nodal stations of an one-dimensional heat flow problem

<table>
<thead>
<tr>
<th>Time (Sec.)</th>
<th>T₁</th>
<th>T₂</th>
<th>T₃</th>
<th>T₄</th>
<th>T₅</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1000</td>
<td>2000</td>
<td>3000</td>
<td>4000</td>
<td>5000</td>
</tr>
<tr>
<td>0.8</td>
<td>1004</td>
<td>2024</td>
<td>3130</td>
<td>4549</td>
<td>6751</td>
</tr>
<tr>
<td>1.6</td>
<td>1038</td>
<td>2124</td>
<td>3433</td>
<td>5014</td>
<td>7323</td>
</tr>
<tr>
<td>2.4</td>
<td>1106</td>
<td>2311</td>
<td>3730</td>
<td>5480</td>
<td>7605</td>
</tr>
<tr>
<td>3.2</td>
<td>1190</td>
<td>2484</td>
<td>3978</td>
<td>5744</td>
<td>7782</td>
</tr>
<tr>
<td>4.0</td>
<td>1271</td>
<td>2638</td>
<td>4117</td>
<td>5937</td>
<td>7903</td>
</tr>
<tr>
<td>4.8</td>
<td>1343</td>
<td>2769</td>
<td>4338</td>
<td>6085</td>
<td>7992</td>
</tr>
<tr>
<td>5.6</td>
<td>1403</td>
<td>2876</td>
<td>4467</td>
<td>6199</td>
<td>8060</td>
</tr>
<tr>
<td>6.4</td>
<td>1453</td>
<td>2961</td>
<td>4569</td>
<td>6290</td>
<td>8113</td>
</tr>
<tr>
<td>7.2</td>
<td>1490</td>
<td>3030</td>
<td>4650</td>
<td>6362</td>
<td>8156</td>
</tr>
<tr>
<td>8.0</td>
<td>1502</td>
<td>3096</td>
<td>4703</td>
<td>6414</td>
<td>8188</td>
</tr>
</tbody>
</table>

Note: The exact value = The printout value × 10⁻². For example, 1000 is equivalent to 100.

Fig. 6. Analog Results for 5 Nodal Stations Heat Flow Problem
The results obtained from pure analog method for 5 stations problem also are shown in Fig. 6. Observation of the results in table 2 and Fig. 6 suggests that the agreement between hybrid method and pure analog method is satisfactorily good.

CONCLUSIONS

Although the problem solved is one-dimensional, extension to two-dimensional problems is quite simple by using the method of lines, i.e., solving the two-dimensional partial differential equations in parallel along one row of nodes and use previously-obtained solutions for the row above and row below. In this arrangement the iterative equation obtained by applying the central finite-difference approximation to the spatial variable \( x \) and \( y \) in the two-dimensional slab heat flow equations are:

\[
\frac{dT_{i,j}}{dt} = \frac{1}{h^2} \left[ T_{i+1,j} + T_{i-1,j} + T_{i,j-1} + T_{i,j+1} + T_{i+1,j+1} - 4T_{i,j} \right]
\]

\[ h = \Delta x = \Delta y \]

\[ k = \text{Number of iterations} \]

where \( i \) corresponds to \( x \) and \( j \) corresponds to \( y \).

In this case, the advantages of hybrid computation stand out very clearly. It allows more equations to be solved on the analog computer simultaneously without the increase of computer time. However, the computing time increases linearly in pure digital computation when the number of equations is increased.

By the investigation of this paper, it is concluded that the CTDS hybrid method is adequate to solve the one-dimensional parabolic partial differential equations because of its simplicity in analog patching and because of its quickness in convergence.

APPENDIX

THE HYBRID PROGRAM FOR SOLVING PARABOLIC PARTIAL
DIFFERENTIAL EQUATIONS WITH ONE SPATIAL DIMENSION

//JOB
//FOR
*LIST ALL
*ONE WORD INTEGRERS
*IOCS (1132 PRINTER, DISK, CARD, TYPEWRITER)
C HYBRID SOLUTION FOR PARTIAL DIFFERENTIAL EQUATIONS
DIMENSION IX (1000,7), IN(5)
DATA IX/1000*0, 1000*1000, 1000*2000, 1000*3000, 1000*4000,
1 1000*5000, 1000*1000/
DATA IN/1000, 2000, 3000, 4000, 5000/
DATA N, IA, IB, ID, IMP/0, 2222, 2223, 2224, 0240/
M=100
CALL INITA (E, N)
CALL HYTST (3, 0)
CALL RUN (E)
   DO 60 K=1, 15
   DO 20 I=1, 5
   CALL IC (E)
   CALL WAIT (300)
   DO 20 J=1, 1000
   CALL STIND (E, ID, IN(I))
   CALL SIND (E, IA, IX (J, I))
   CALL STIND (E, IB, IX (J, I+2))
   CALL OP (E)
   CALL WAIT (M)
   CALL HOLD (E)
   CALL READ (E, IMP, IX (J, I+1))
20 CONTINUE
   WRITE (3, 30) K
30 FORMAT (2X, 'MAJOR ITERATION', I3, //, 6X 'TI', 9X, 'T2',
       1 9X, 'T3', 9X, 'T4', 9X, 'T5')
   DO 40 L=1, 1000, 100
40 WRITE (3, 50) IX (L, 2), IX (L, 3) IX (L, 4), IX (L, 5), IX (L, 6)
50 FORMAT (3X, 5 (16, 5X))
60 CONTINUE
CALL STP (E)
CALL EXIT
END

The process of incrementing time is done by going from IC MODE into OP MODE for a length of time determined by M in the CALL WAIT (M) statement and then proceeding to HOLD MODE, wherein the value of Ti is picked off and stored in memory by ADC lines using the subroutine CALL READ. The initial conditions and the values of surrounding nodal stations are fed in by means of DAC'S using the subroutine CALL STIND. The subroutines CALL INITIA, CALL HYTST, are special subroutines used to initialize all the interface functions to zero state and provide to indicate the error signals for the users.

REFERENCES

INTRODUCTION

This paper is concerned with realization methods for the solution of systems described by differential and difference equations. The method described is based upon the realization theory of Hankel and Toeplitz matrices, and the solution of the problem is reduced to the construction of a system whose transfer function is a given complex rational function. The realization theory is based upon the theory of Hankel and Toeplitz matrices, which are used to represent the behavior of a system over a given time interval. The method described is particularly useful for the solution of systems described by difference equations, and is a generalization of the method described in a previous paper.

The realization method is based upon the use of Hankel and Toeplitz matrices, which are used to represent the behavior of a system over a given time interval. The method described is particularly useful for the solution of systems described by difference equations, and is a generalization of the method described in a previous paper.