Stability Analysis of Nonlinear Reactor Control Systems

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SUMMARY: A method for testing control system stability is presented, which is useful for finding the limit cycles of high order systems with multiple nonlinearities. The presented method is applied to the analysis of the control system of a material testing reactor, and a comparison with the method in a current literature is given.

I. INTRODUCTION

It is well known that a nuclear reactor control system usually consists of several nonlinearities such as saturation, backlash etc. The commonly used method for analysis is Nyquist diagram, which is not suitable for analyzing control systems with multiple nonlinearities, and also not suitable for finding the effects of the adjustable parameters. The main purpose of this paper is to present a stability-equation method and to apply this method to the analysis of a nonlinear reactor control system.

II. STABILITY-EQUATION METHOD FOR NONLINEAR SYSTEM ANALYSIS

The method proposed in this section is based upon a linearization technique; i.e., to replace the nonlinearities by their describing functions, and then a linearized characteristic equation can be obtained. After the characteristic equation is separated into two parts as

\[ F(s) = F_r + F_i = 0 \]  \hspace{1cm} (1)

where \( F_r \) and \( F_i \) are the real and imaginary parts of \( F(s) \) respectively (after the substitution \( s = jw \)), then a standard root-locus form can be written as

\[ \frac{F_r}{F_i} = -1 \]  \hspace{1cm} (2)

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It is readily shown that for all the characteristic roots in the left half of s-plane, all the poles \((p_i)\) and zeros \((z_i)\) of Eq. (2) must lie on the imaginary axis of s-plane and their absolute values are related as

\[ \cdots p_{-1}^< z_{-1}^< p_0^< z_1^< p_1^< z_2^< \cdots \]  

(3)

From Eqs. (2) and (3), it can be seen that the stability limit is reached when a pole is equal to a zero; thus to test system stability becomes simply to find the real roots of \(F_n\) and \(F_r\). (In a later part of this paper, these two equations are called stability equations.) The following example is used as an illustration.

Example 1. Consider the system in Fig. 1, where \(N_1\) and \(N_2\) are the describing functions of the nonlinearities, the characteristic equation is

\[ s^3 + s^2 + N_1 (g - jb) s + 0.05N_1 (g - jb) = 0 \]  

(4)

After the substitution \(s = jw\), the stability equations are

\[ F_n = -w^2 + N_1 bw + 0.05N_1 g = 0 \]  

(5)

\[ F_r = -w^3 + N_1 gw - 0.05N_1 b = 0 \]  

(6)

which gives

\[ N_1 = \frac{(0.05 + w^2)}{g(0.0025 + w^2)} \frac{w^2}{b (0.0025 + w^2)} \]  

(7)

Let \(R = \frac{\beta}{2|\delta|}\), where \(\beta\) is the amount of backlash and \(|\delta|\) is the magnitude of the input signal to the backlash, then for each value of \(\beta\), the corresponding values of \(w\), \(R\), \(g\), \(b\), and \(E\) for making the system have a limit cycle can be found, and their relations can be represented by various curves. For example, the relations among \(E\), \(w\) and \(\beta\) are given in Fig. 2, which indicates that, for the considered system, if the amount of backlash is increased, the frequency of the limit cycle will be reduced and its magnitude will be increased.

Fig. 2 also indicates that the presented method is useful for adjusting parameters. In the considered system, if the open loop gain \((k)\) is adjustable, then for any fixed value of \(\beta\) the relations among \(w\), \(E\) and \(k\) can be found using the same method.
LIMIT CYCLE STABILITY ANALYSIS

After a limit cycle is found, its stability characteristics should be defined. The main purpose of this section is to present a method for finding the stability characteristics of a limit cycle using the stability-equation method.

As mentioned before, a limit cycle exists whenever there is a pole \( p_i \) equal to a zero \( z_i \); thus the stability characteristics of a limit cycle can be defined if the variations of the pole and zero in the neighbourhood of the limit cycle (due to the variation of \( E \)) can be defined. On the other hand, since the stability equations are functions of frequency, the effect of \( w \) should be considered also. A method of using partial derivatives to find the effects of the variations of \( E \) and \( w \) upon the real roots of stability equations is given as follows:

Using Taylor’s series, for incrementals of \( E \) (\( \Delta E \)) and \( w \) (\( \Delta w \)), the stability equations can be written approximately as

\[
F_n(w + \Delta w, |E| + \Delta E) = F_n(w, |E|) + \frac{\partial F_n}{\partial w} \Delta w + \frac{\partial F_n}{\partial E} \Delta E = 0
\]

\[
F_l(w + \Delta w, |E| + \Delta E) = F_l(w, |E|) + \frac{\partial F_l}{\partial w} \Delta w + \frac{\partial F_l}{\partial E} \Delta E = 0
\]

Since \( F_n(w, |E|) = 0 \) and \( F_l(w, |E|) = 0 \) at the limit cycle, thus

\[
\Delta w_n = -\frac{\partial F_n}{\partial E} \Delta E / \frac{\partial F_n}{\partial w}
\]

\[
\Delta w_l = -\frac{\partial F_l}{\partial E} \Delta E / \frac{\partial F_l}{\partial w}
\]

where \( \Delta w_n \) & \( \Delta w_l \) represent the variations of the roots of \( F_n \) and \( F_l \) in the neighbourhood of the limit cycle respectively.

For the example in the last section, if \( \beta = 0.2075 \), then as indicated by the dotted lines in Fig. 2, a limit cycle with \( w = 3.14 \) \( R = 0.2573 \) and \( |E| = 0.1033 \) exists. The partial derivatives are found as

\[
\frac{\partial F_n}{\partial w} = -2w + N_4b + N_4w \frac{\partial b}{\partial w} + 0.05N_4 \frac{\partial g}{\partial w}
\]
\[
\frac{\partial F_i}{\partial w} = -3w^3 + N_1g + N_1w \frac{\partial g}{\partial w} - 0.05N_1 \frac{\partial b}{\partial w} \tag{13}
\]

\[
\frac{\partial F_R}{\partial |E|} = bw \frac{\partial N_1}{\partial |E|} + N_1 \frac{\partial b}{\partial |E|} + 0.05g \frac{\partial N_1}{\partial |E|} + 0.05N_1 \frac{\partial g}{\partial |E|} \tag{14}
\]

\[
\frac{\partial F_i}{\partial |E|} = gw \frac{\partial N_1}{\partial |E|} + N_1 \frac{\partial g}{\partial |E|} - 0.05N_1 \frac{\partial b}{\partial |E|} - 0.05b \frac{\partial N_1}{\partial |E|} \tag{15}
\]

where

\[
\frac{\partial g}{\partial w} = \frac{\partial g}{\partial R} \frac{\partial R}{\partial w}, \quad \frac{\partial g}{\partial |E|} = \frac{\partial g}{\partial R} \frac{\partial R}{\partial |E|} \tag{16}
\]

\[
\frac{\partial b}{\partial w} = \frac{\partial b}{\partial R} \frac{\partial R}{\partial w}, \quad \frac{\partial b}{\partial |E|} = \frac{\partial b}{\partial R} \frac{\partial R}{\partial |E|} \tag{17}
\]

and where

\[
R = \frac{\beta}{2 |\beta|} = 8w \sqrt{1 - \frac{\epsilon_n^2}{4|E|^2}} \tag{18}
\]

\[
\frac{\partial R}{\partial w} \approx -\frac{\beta \pi}{8w^2} \tag{19}
\]

\[
\frac{\partial R}{\partial |E|} \approx -\frac{\pi \beta \epsilon_n^2}{32w |E|^3} \tag{20}
\]

For the considered limit cycle, the results are

\[
\frac{\partial g}{\partial R} \approx -1.15, \quad \frac{\partial b}{\partial R} \approx +0.6, \quad \frac{\partial R}{\partial w} \approx -0.00826,
\]

\[
\frac{\partial R}{\partial |E|} \approx -0.00236, \quad \frac{\partial N_1}{\partial |E|} \approx -120
\]

\[
\frac{\partial g}{\partial w} = 9.5 \times 10^{-3}, \quad \frac{\partial g}{\partial |E|} = 2.72 \times 10^{-3}, \quad \frac{\partial b}{\partial w} = -4.95 \times 10^{-3}
\]

\[
\frac{\partial b}{\partial |E|} = -1.41 \times 10^{-3}
\]

thus

\[
\frac{\partial F_R}{\partial w} = -3.49, \quad \frac{\partial F_R}{\partial |E|} = -95
\]

\[
\frac{\partial F_i}{\partial w} \approx 19.2, \quad \frac{\partial F_i}{\partial |E|} = -303
\]

which give

\[
\Delta w_R = -27.2 \Delta E \tag{21}
\]

\[
\Delta w_I = -15.8 \Delta E \tag{22}
\]

Hence for a positive incremental \( \Delta E \) in \( E \), the pole and zero distribution
of the stability equations becomes

\[ p_1 = -3.165, \quad z_{-1} = -0.16, \quad p_0 = -0.025, \quad z_1 \leq 3.14, \quad p_1 > 3.14 \]

which represents a stable system; i.e., \( E \) will be brought back to its original value (0.1033). Therefore, the limit cycle is stable.

The considered example has been simulated with a digital computer, and the results are checked with those obtained from theoretical analysis.

**STABILITY ANALYSIS OF A NONLINEAR REACTOR CONTROL SYSTEM**

The design of a nonlinear reactor control system has been stated in some detail in reference 3, where a system with two nonlinearities separated by a linear section has been analyzed using Nyquist diagram and describing function methods; and for avoiding the complexity in analysis, the authors always approximate the ON-OFF element with a pure gain. Even so, the analyses show great complexity, especially when the effects of the adjustable parameters are considered.

In this section, the stability-equation method is applied for the analysis of the aforementioned system.

The considered system is given in Fig. 3, where the transfer functions are

(a) Reactor:

\[
1 \frac{R_n}{n_0} G_n = \frac{1}{n_0} \frac{\delta n}{\delta k} = \frac{E}{\frac{104(s+3.01)}{s} (s+1.14)(s+0.301)(s+0.111)(s+0.0305)(s+0.0124)} \]

\[
\frac{1}{s} (s+64.4)(s+2.9)(s+1.02)(s+0.195)(s+0.0681)(s+0.0143)
\]

\[
(23)
\]

(b) Motor:

\[
K_m G_m = \frac{\delta k}{\mu/\mu_0} = \frac{M_0}{s(s+\tau s)}
\]

\[
(24)
\]

where \( M_0 \) is the reactivity ramp for the motors and \( \tau = 0.05 \) second is the motor time constant.

(c) The ON-OFF error unit

\[
K_0 G_e = \frac{\mu/\mu_0}{E} = \frac{4}{g_0 \pi} R' \sqrt{1-R'^2}
\]

\[
(25)
\]
where \( \epsilon_0 \) is the dead zone amplitude, and \( R' = \frac{\epsilon_0}{|E|} \) is a parameter.

(d) The backlash

\[
K_bG_n = g - jb
\]  

(26)

where \( g \) and \( b \) are the real and imaginary parts of the describing function of the backlash respectively.

The characteristic equation is

\[
1 + \frac{1}{n_0}K_bG_nK_cG_pK_nG_n = 0
\]  

(27)

Let \( K = M_0K_cG_0 \times 10^4 \), then Eq. (27) gives

\[
s^8 + 88.6s^8 + 1647s^7 + (20Kg + 5750 - j20Kb)s^6 + (92.2Kg + 5310 - j92.2Kb)s^5 + (107.4Kg + 1146 - 107.4Kb)s^4 + (35.5Kg + 65.9 - j35.5Kb)s^3 + (3.67Kg + 0.72 - j3.67Kb)s^2 + (0.1098Kg - j0.1098Kb)s + (0.000866Kg - j0.000866Kb) = 0
\]  

(28)

Thus the stability equations are

\[
F_1 = w^8 - 1467w^7 + 20Kbw^6 + (92.2Kg + 5310)w^5 - 107.4Kbw^4 - (3.5Kg + 66)w^3 + 3.67Kbw^2 + 0.1098Kgw - 0.000866Kb = 0
\]  

(29)

\[
K_R = 88.6w^6 - (20Kg + 5750)w^5 + 92.2Kbw^4 + (107.4Kg + 1146)w^4 - 35.5Kbw^3 - (3.67Kg + 0.72)w^2 + 0.1098Kbw + 0.000866Kg = 0
\]  

(30)

In order to determine a limit cycle, Eqs. (29) and (30) should be solved simultaneously. Rewrite these two equations as

\[
-K = \frac{w^8 - 1467w^7 + 5310w^6 - 66w^3}{(92.2w^8 - 35.5w^8 + 0.1098w + 20w^8 - 107.4w^4 + 3.67w^2 - 0.000866)b}
\]  

\[
= \frac{A}{C_0 + Db}
\]  

(31)

and

\[
-K = \frac{88.6w^6 - 5750w^6 + 1146w^4 - 0.72w^2}{- (20w^8 - 107.4w^4 + 3.67w^2 - 0.000866) + (92.2w^8 - 35.5w^2 + 0.1098w)b}
\]  

\[
= \frac{B}{C_b - D_g}
\]  

(32)
Since \( g \) and \( b \) are functions of \( |\nu'| \); i.e.,

\[
g = g( |\nu'| ) \tag{33}
\]

\[
b = b( |\nu'| ) \tag{34}
\]

where

\[
|\nu'| = \frac{10^{-4}K}{w\sqrt{1 + \pi^2 w^2}} |E| \tag{35}
\]

and

\[
K = 10^4 M_0 K c G_c = 10^4 M_0 \frac{4}{|E| \pi} \sqrt{1 - \left( \frac{\epsilon_0}{|E|} \right)^2} \tag{36}
\]

thus for different values of \( M_0, E_0 \) and \( \beta \), the values of \( w, |E| \), \( g, b \) and \( K \) which satisfy Eqs. (31) to (36) simultaneously can be found using a digital computer. A flow chart for doing this is given in Fig. 4, and some of the computed results are given in Fig. 5.

For example, if \( M_0 = 1.5 \times 10^{-4}, \epsilon_0 = 0.005, \beta = 0.23^\circ \), which define point \( M \) in Fig. 5, then the corresponding values of \( w, |E|, g, b \) and \( K \) are

\[w = 0.066, |E| = 0.43, K = 4.44, g = 0.258, \text{ and } b = 0.269\]

The pole-zero distribution of the stability equations is given in Fig. 6, which indicates that the considered system has a limit cycle at \( w = 0.066 \). In order to test stability of this limit cycle, the method presented in the last section is applied. The calculations of the partial derivatives are given in the appendix, and the results are

\[1.211 \Delta w_n = -0.1391 \Delta E = 0 \tag{37}\]

\[-4.417 \Delta w_t = -0.0288 \Delta E = 0 \tag{38}\]

It can be seen that, for an incremental of \( |E| \), the zero of \( F_n = 0 \) (originally at \( w = 0.066 \)) tends to increase, and the zero of \( F_t = 0 \) (originally at \( w = 0.066 \)) tends to decrease; thus the pole and zero distribution becomes alternative in sequence, and the system is stable. Therefore, the considered limit cycle is stable.

Since the effects of the adjustable parameters on system stability can be found using a digital computer and the stability characteristics of any limit cycle can be defined, the presented method is useful for the analysis and design of control systems with a digital computer.
CONCLUSIONS

The method of testing stability in high order systems with multiple nonlinearities, presented in this paper, has the advantage of reducing the required work for finding the limit cycles and its-stability characteristics. The presented method is suitable for use with a digital computer to find the effects of various adjustable parameters on system stability. In comparison with the commonly used graphical method,\(^8\) the superior characteristics of the presented method are quite evident.

Since the basic approach used in this paper is linearization, same limitations are encountered as the use of describing function method; i.e., the nonlinearities must be separated by linear transfer functions with enough low pass characteristics.

APPENDIX

From Eqs. (8) and (9)

\[
F_n(w_0 + \Delta w, |E|_0 + \Delta E) = F_n(w_0, |E|_0) + \left( \frac{\partial F_n}{\partial w} \right) \Delta w + \left( \frac{\partial F_n}{\partial |E|} \right) \Delta E = 0 \tag{A}
\]

\[
F_i(w_0 + \Delta w, |E|_0 + \Delta E) = F_i(w_0, |E|_0) + \left( \frac{\partial F_i}{\partial w} \right) \Delta w + \left( \frac{\partial F_i}{\partial |E|} \right) \Delta E = 0 \tag{B}
\]

The partial derivatives are

\[
\frac{\partial F_n}{\partial w} = 8 \times 88.6w^7 - 6(20Kg + 5750)w^5 + 5 \times 92.2Kbw^4 + 4(107.4Kg + 1146)w^5 - 3 \times 35.5Kbw^2 - 2(3.67Kg + 0.72)w + 0.1098Kb
\]

\[
-20w^6 \frac{\partial Kg}{\partial w} + 92.2w^5 \frac{\partial Kb}{\partial w} + 107.4w^4 \frac{\partial Kg}{\partial w} - 35.5w^3 \frac{\partial Kb}{\partial w}
\]

\[
-3.67w^2 \frac{\partial Kg}{\partial w} + 0.1098w \frac{\partial Kb}{\partial w} + 0.000866 \frac{\partial Kg}{\partial w}
\]

\[
\frac{\partial F_n}{\partial |E|} = -20w^6 \frac{\partial Kg}{\partial |E|} + 92.2w^5 \frac{\partial Kb}{\partial |E|} + 107.4w^4 \frac{\partial Kg}{\partial |E|} - 35.5w^3 \frac{\partial Kb}{\partial |E|}
\]

\[
-3.67w^2 \frac{\partial Kg}{\partial |E|} + 0.1098w \frac{\partial Kb}{\partial |E|} + 0.000866 \frac{\partial Kb}{\partial |E|}
\]
\[
\frac{\partial F_i}{\partial w} = 9w^4 - 7 \times 1.467w^4 + 6 \times 20Kbw^4 + 5(92.2Kg + 5310)w^4 - 4 \\
\times 107.4Kbw^4 - 3(35.5Kg + 66)w^2 + 3.67 \times 2Kbw + 0.1098Kg \\
+ 20w^6 \frac{\partial K_b}{\partial w} + 92.2w^6 \frac{\partial K_g}{\partial w} - 107.4w^4 \frac{\partial K_b}{\partial w} - 35.5w^4 \frac{\partial K_g}{\partial w} \\
+ 3.67w^2 \frac{\partial K_b}{\partial w} + 0.1098w \frac{\partial K_g}{\partial w} - 0.000866 \frac{\partial K_b}{\partial w} (E)
\]

\[
\frac{\partial F_i}{\partial |E|} = 20w^6 \frac{\partial K_b}{\partial |E|} + 92.2w^6 \frac{\partial K_g}{\partial |E|} - 107.4w^4 \frac{\partial K_b}{\partial |E|} - 35.5w^4 \frac{\partial K_g}{\partial |E|} \\
+ 3.67w^2 \frac{\partial K_b}{\partial |E|} + 0.1098w \frac{\partial K_g}{\partial |E|} - 0.000866 \frac{\partial K_b}{\partial |E|} (F)
\]

For the considered limit cycle at M (i.e., \(w = 0.066, |E| = 0.43, \beta = 0.23^o, M_0 = 1.5 \times 10^{-4}, K = 4.44, g = 0.258, b = 0.269),

\[
R = \frac{\beta}{2(\partial_r)} \cdot \frac{10^4w^2}{2 |E| K}
\]

then

\[
\frac{\partial R}{\partial w} = \frac{10^4w^2}{2 |E| K} = 10.52
\]

\[
\frac{\partial g}{\partial w} = \frac{\partial g}{\partial R} \frac{\partial R}{\partial w} = -1.3 \times 10.52 = -13.7
\]

\[
\frac{\partial b}{\partial w} = \frac{\partial b}{\partial R} \frac{\partial R}{\partial w} = -0.5 \times 10.52 = -5.26
\]

Since

\[
K = 10^4M_0 \frac{4}{|E| \pi \sqrt{1 - R^2}},
\]

\[
\pm 10^4M_0 \frac{4}{|E|}
\]

thus

\[
R = \frac{\beta w^2}{8M_0}
\]

which gives

\[
\frac{\partial R}{\partial |E|} = 0, \quad \frac{\partial g}{\partial |E|} = 0, \quad \frac{\partial b}{\partial |E|} = 0
\]

and

\[
\frac{\partial K}{\partial |E|} = - \frac{4 \times 10^4 \times 1.5 \times 10^{-4}}{\pi |E|^2} = -10.37, \quad \frac{\partial K}{\partial w} = 0
\]
similarly \( \frac{\partial b}{\partial |E|} = 0 \)

Thus \( \frac{\partial K_g}{\partial w} = K \frac{\partial g}{\partial w} = -13.7K \)

\( \frac{\partial K_b}{\partial w} = K \frac{\partial b}{\partial w} = -5.26K \)

\( \frac{\partial K_g}{\partial |E|} = g \frac{\partial K}{\partial |E|} = -10.37g \)

\( \frac{\partial K_b}{\partial |E|} = b \frac{\partial K}{\partial |E|} = -10.37b \)

Substituting into Eqs. (C) and (F), then

\( \frac{\partial F_g}{\partial w} = 1.211, \frac{\partial F_b}{\partial w} = -0.139, \frac{\partial F_i}{\partial w} = -4.417, \frac{\partial F_g}{\partial |E|} = -0.0288 \)

Hence Eqs. (37) and (38) are obtained.

**SYMBOLS**

s Laplace operator
w frequency
c output quantity
r input quantity
K gain of transfer function
p zero of \( E_t \)
z zero of \( F_b \)
\( F_n \) real part of characteristic equation
\( F_i \) imaginary part of characteristic equation
\( M_0 \) the velocity ramp of motors
\( \epsilon_0 \) dead-zone amplitude of the ON-OFF error unit

\[ R = \beta/2 |\gamma| \] parameter

\[ R' = \frac{\epsilon_0}{|E|} \]

\( g - jb \) describing function of a backlash-type nonlinearity
N  describing function of ON-OFF nonlinearity
s  input signal to backlash
E  input signal to the ON-OFF element, E = |E| sinωt
n0  steady-state power of reactor
β  magnitude of backlash

REFERENCES


Figure 2: Relations among $|E|, W$ and $\beta$.

Figure 3: Block diagram of a nonlinear reactor control system.
FIG. 4 FLOW CHART FOR FINDING THE LIMIT CYCLES

FIG. 5 RELATIONS AMONG $\xi$, $\beta$, $M_0$, AND $W$

FIG. 6 AN ILLUSTRATION OF POLE-ZERO DISTRIBUTION