Dynamics and thermal instability of magnetic flux in type-II superconductors

B. Ya. Shapiro, 1,* I. Shapiro, 1 B. Rosenstein, 2 and F. Bass 1

1Department of Physics, Bar Ilan University, Ramat Gan 52100, Israel
2Department of Electrophysics, National Chiao Tung University, Hsinchu, Taiwan, Republic of China.

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In recent experiments, trapped magnetic flux is initially generated by abrupt laser heating of a strip of a type-II superconducting film subjected to a weak magnetic field. We study herein the nonequilibrium penetration of the flux into the Meissner state area. Effects of the heat dissipation and transport on the motion and stability of the interface between the magnetic flux and flux-free domains are considered. It is shown that the magnetic induction and the temperature have the form of a shock wave moving with constant velocity as large as that corresponding to the depairing current. In the vicinity of the front, superconductivity is suppressed by strong screening currents. The front velocity is determined by the Joule heat caused by the electric current in the normal domain at the flux front. The stability of the shock wave solution is investigated both analytically and numerically. For sufficiently small heat diffusion constant a finger shaped thermal instability is found.

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I. INTRODUCTION

The dynamics of magnetic flux penetration into a type-II superconductor and its instabilities have been studied by a variety of techniques over the years, (see Ref. 1, and references therein). Magneto-optics experiments 2 demonstrate that in a wide range of situations there exists a well-defined interface between the magnetic flux penetrating into the sample and the flux-free Meissner state. Improvements to these magneto-optical techniques have revealed a wide class of instabilities, including magnetic macroturbulence 3,4 and a dendritic instability. 5 The instability of the magnetic flux and flux avalanches are observed both in anisotropic high temperature superconductor 4 and in an isotropic material like Nb. 5

Traditionally there are three possible scenarios in which the instabilities could arise. The standard thermomagnetic instabilities appear when the critical vortex state is perturbed locally by the heat released by a moving vortex. This dissipation leads to the thermal softening of the vortex system which in turn is responsible for the instability. 1 In this case the instability develops around a well defined interface (front) between the magnetic flux penetrating into the sample and the flux-free Meissner state. Improvements to these magneto-optical techniques have revealed a wide class of instabilities, including magnetic macroturbulence 3,4 and a dendritic instability. 5 The instability of the magnetic flux and flux avalanches are observed both in anisotropic high temperature superconductor 4 and in an isotropic material like Nb. 5

In this case the excess heat released at the front is caused by vortex-antivortex annihilation. Yet another type of instability occurs in strongly anisotropic superconductors. 5,9 In this case the stationary vortex-antivortex interface is destroyed by the Thomas-Kelvin instability.

Recently, a type of flux instability was observed experimentally. In these experiments superconductivity was locally destroyed in a completely nonadiabatic fashion by a femtosecond laser pulse. 10 The pulse clearly forces the system out of thermal equilibrium. The superconductivity is destroyed inside a narrow strip of a YBCO film subjected to a magnetic field perpendicular to the film. The field does not exceed the first critical field \$H_{c1}\$, so that initially fluxons cannot penetrate the rest of the sample. Therefore the magnetic flux initially fills the normal domain. Recovery of superconductivity occurs in two stages. Once the short pulse is over, the strip cools and the flux nucleates into a dense system of Abrikosov vortices. The characteristic time of that stage is microscopic, of order of the Ginzburg-Landau (GL) relaxation time (appearing in the time dependent GL equations) \$t_{GL} \sim 10^{-10}$ s. This process has been studied by us some time ago 11 and we do not address this stage in the present paper since it was shown that no instability is originated at this stage.

On the larger (mesoscopic) time scale the rapidly created vortices are pushed into the superconducting part of the sample. The fluxons move very fast with velocities of order of \$10^5 \text{ cm/s}$ (in YBCO). 10 The flux flow currents, \(J\), in this case are much higher than the critical current \(J_c\) typical for the thermodynamic Bean model critical state, but smaller (although not much smaller) than the depairing current \(J_d\): \(J_d > J \approx J_c\). Just after the vortex nucleation stage the magnetic flux forms a rapidly moving front. This highly nonequilibrium relaxation dynamics is very different from the essentially adiabatic dynamics of the critical state discussed earlier. The front line shape is not always stable: sometimes it dynamically develops dendriticlike structures. 12

The existence of the sharp and typically straight front can be in principle understood in the framework of the theory of nonlinear magnetic flux diffusion. 13,14 Geshkenbein et al. considered the flux diffusion in the creep regime, while Shapiro et al. 14 considered the flux flow regime. In both cases the temperature gradient effects were neglected and no instability of the front was predicted, namely, it was shown that corrugation of the front line is unfavorable. The front velocity under these assumptions decreases with time. 14 However, corrugation of the front is typically caused by thermal effects, 1 hence, one expects that in the case of fast dynamics of the front, these effects are even more important.

In the present paper we study both numerically and analytically the dynamics of the nonadiabatically created magnetic flux in sufficiently thick (thickness larger than the magnetic penetration length) superconducting films. In particular, effects of dissipation and the heat transport on the motion

\(H_{c1}\)

\(J_d\)

\(J_c\)

\(J\)

\(t_{GL}\)

\$t_{GL} \sim 10^{-10}$ s

\$10^5 \text{ cm/s}$

\(J_d > J \approx J_c\)

\(J_d\)

\(J_c\)

\(J\)

\$t_{GL} \sim 10^{-10}$ s

\$10^5 \text{ cm/s}$

\(J_d > J \approx J_c\)

\(J_d\)

\(J_c\)

\(J\)

\$t_{GL} \sim 10^{-10}$ s

\$10^5 \text{ cm/s}$

\(J_d > J \approx J_c\)
and stability of the flux front are considered. It is shown that the Joule heat released at the flux front can produce front propagation at constant velocity inside the type-II superconductor. Heating of the front by the moving magnetic flux is essential. We found that for certain voltage-current characteristics of the superconductor in its resistive state, the magnetic induction penetrating a flux-free superconductor forms a sharp front. Strong superconducting currents in the vicinity of the front suppress superconductivity in this area and create a normal domain at the front. The interface moves with constant velocity which is completely determined by the Joule heat released in the normal domain at the leading edge of the front. The straight front line shows an instability with respect to local temperature fluctuations. In fact an excessive local temperature at the front leads to excessive Joule heat released there and in turn increases the local front velocity in the area of the fluctuation. The hydrodynamical tangential instability of the flux front destroys the flat front. Numerical simulation of the exact set of nonlinear equations allows us to study the evolution of the instability and demonstrates the emergence and development of the corrugated interface.

II. MODEL AND BASIC EQUATIONS

A. Hydrodynamics of the vortex matter

(for the slab geometry)

Let us consider a typical experimental situation (see Ref. 12), when a relatively thick (with thickness larger than magnetic penetration depth $\lambda$) type-II superconducting film is subjected to a weak external magnetic field ($B < B_\text{c1}$). The magnetic induction $B$ therefore has only a $z$ component $B_z = B$ and all dependencies on the $z$ coordinate can be neglected. The two dimensional vortex systems is described by the magnetic induction $B(r,t)$ and the temperature profile $T(r,t)$, where $r = (x,y)$ is a two dimensional vector. To derive the hydrodynamic equations one starts from the continuity equation for the fluxon density $n(r,t) = \sum_{\alpha} \delta(r - r_\alpha(t))$ and the flux current $I_i(r,t) = \sum_{\alpha} v_i^\alpha(t) \delta(r - r_\alpha(t))$. Here $i = x, y$ and $\alpha = 1, \ldots, N$ labels the fluxons. The continuity equation

$$\frac{\partial n}{\partial t} = -\nabla I_i$$

supplemented by the constitutive relation

$$I_i(r,t) = D_i(r,t)\nabla n(r,t)$$

leads to the flux diffusion equation

$$\frac{\partial n(r,t)}{\partial t} = -\nabla [D_i(r,t)\nabla n(r,t)].$$

Since $n(r,t) = B(r,t)/\phi_0$, where $\phi_0$ is the unit flux, the Maxwell equation

$$-\frac{1}{c} \frac{\partial B}{\partial t} = \epsilon_{ij} \nabla E_j$$

leads to the identification $E_i = (c/\phi_0) e_{ij} D_j(r,t) \nabla n(r,t)$, while $\epsilon_{ij}$ is the antisymmetric tensor. Since in the mixed state of the type-II superconductor $E = R J$, where $R(B,T)$ is the resistivity, one obtains $R = (4\pi/\phi_0^2)D_i$. The electric current density in turn is equal to $J_i = (c/4\pi) e_{ij} \nabla B$. The flux diffusion equation then takes the form

$$\frac{4\pi}{c^2} \frac{\partial B}{\partial t} = \frac{\partial}{\partial x} \left[ R \frac{\partial B}{\partial x} \right] + \frac{\partial}{\partial y} \left[ R \frac{\partial B}{\partial y} \right].$$

(5)

The function $R(B,T)$ will be phenomenologically defined in the next subsection. In the normal state the same equation applies with the normal state resistivity.

Now we turn to the heat transport equation, identical to the conventional normal state heat balance equation

$$C \frac{\partial T}{\partial t} = D \nabla^2 T + J \cdot E(B,T) - \gamma (T - T_0).$$

(6)

Here $C$ is the heat capacity and $D$ is the heat diffusion constant, $T_0$ is the temperature of the cooling liquid with $\gamma = 1/t_r$ being the heat relaxation constant, when $t_r$ is the heat relaxation time. The first term on the right hand side is the heat conduction, the second is the Joule heat, and the third describes the heat exchange between the slab and the cooling liquid. The Joule heat term consists of two different contributions. In the mixed state it is dominated by the motion of the magnetic flux, while in the normal metal when the superconductivity is suppressed by the currents, one has usual Ohmic resistance losses.

In the geometry we consider (see Fig. 3), the dependence of both the temperature and the magnetic induction on $z$ can be neglected. The magnetic induction is independent of $z$, since thickness of the thick film (slab) in the $z$ direction is assumed to be larger than the magnetic penetration length $\lambda$, while the temperature is uniform in the $z$ direction, despite the presence of the last term, since the thermal diffusion length is typically much larger than the film’s thickness. The detailed argumentation is presented in Ref. 15.

B. Resistivity at high currents

As a rule, the nonlinear resistivity $R(J,B,T)$ is a complicated function of magnetic field, current and temperature, see Fig. 1. In this work we will be interested mainly in resistivity at currents much larger than the critical current $J_c$, when the pinned vortices are released. The vortex resistivity grows quickly above $J_c$, either exponentially or as a power $R \sim J^\mu$ with large $\mu$. In this relatively low current regime the dependence of the resistivity on magnetic induction $B$ is very smooth (roughly linear). However, when the current approaches the depairing current $J_d$ the power $\mu$ becomes smaller and resistivity strongly depends on $B$.

Recently detailed measurements of the $I$–$V$ characteristics of Nb films at high current density of order $10^6$ A/cm$^2$ were performed. Near the depairing current it has the form

$$R(B,T) = R_n(T) \left( \frac{J}{J_d(T,B)} \right)^\mu.$$  

(7)

Here $R_n(T)$ is the normal state resistivity. The dependence of the depairing current $J_d$ on magnetic field and temperature can be fitted well by the following form:
FIG. 1. Schematic plot of the nonlinear resistivity of a type-II superconductor in the mixed state as a function of current. The resistivity is zero below the critical current \( J_c \), exponentially small in the flux creep regime just above \( J_c \), and evolves into a power function in the flux flow regime. At the depairing current it merges with the Ohmic normal state resistivity.

\[
J_d(T,B) = J_{d0} \Delta \left[ \frac{B_c^2(T)}{B} \right]^{\nu/\mu}.
\]

The upper critical field depends on temperature as 
\( B_c(T) = B_c(T) \), where we assumed that dimensionless temperature \( \theta = T/T_c \) is not far from 1, namely \( \Delta = 1 - \theta \) is small.

When the current exceeds \( J_d(B,T) \), the electric field is continuous, the resistivity saturates at its’ normal value \( R_n(B,T) \). The derivative of \( R \) appearing in the nonlinear flux diffusion Eq. (5) is discontinuous. We fitted the \( I-V \) curves of Nb and obtained \( \mu = 1.5 \) with temperature independent \( R_n \). For Nb at fields of the order of \( B_c \), we obtain the best fit \( \nu = 1.3 \). The values of other material parameters are: 
\( B_c(0) = 4.43 \) T, \( R_n = 9.9 \) \( \mu \Omega \) cm and \( T_c = 8.6 \) K. These were measured directly. The obtain the best fit for the constant 
\( J_{d0} = 9.2 \cdot 10^6 \) A/cm\(^2\). See Fig. 2 for a sample of data taken at 
\( T = 7.8 \) K, \( \Delta = 0.9 \).

Of course the exponents depend on material and weakly depend on field for larger magnetic fields. The power law however generally holds. Examples include YBCO well

FIG. 2. A fit of the resistivity dependence on the current density of Ref. 16 to the model resistivity Eqs. (7),(8) with exponents \( \nu = 1.3 \), \( \mu = 1.5 \). Magnetic field is 20 mT (circles), 30 mT (stars) and 40 mT (squares).

above \( B_c \) (see Ref. 17) in which the power law is clearly seen, but \( \mu = 2 \), \( \nu = 2 \). The corresponding data on high \( T_c \) superconductors are not yet available for fields below \( B_c \), to our knowledge, and therefore we treat the powers as phenomenological parameters (see also Refs. 14 and 18). An additional difference between the conventional and the high \( T_c \) materials is that the normal state conductivity in high \( T_c \) cuprates is linear ("strange metal").

C. Boundary and initial conditions

In a typical experiment\(^1\)\(^2\) the heat of the laser beam suppresses superconductivity in a narrow strip of width \( l \) dividing the sample into two equal superconducting parts of length \( L_x \) on both sides of the irradiated strip. Magnetic flux promptly fills the normal area and forms a nonequilibrium vortex strip state. Subsequently the laser is switched off and sample is cooled (see Fig. 3). The set of Eqs. (5) and (6) must be supplemented by the initial and boundary conditions in the center of the sample and on the sample’s edges. The initial temperature is assumed to be homogeneous

\[
T(x, y, t = 0) = T_0,
\]

where \( T_0 \) is the temperature of the cooling liquid. Magnetic field fills the irradiated area of width \( l \) and magnetic flux of magnitude

\[
\Phi = 2lB_0 \int B(x, y, t) dx dy
\]

is assumed to be trapped in superconductor and conserved. Here \( L_x \) is width of the sample. The boundary conditions for temperature are

\[
T(x = \pm L_x, y = 0, t) = T_0.
\]

An alternative boundary condition for the magnetic induction which we consider independently is fixed magnetic field at the center \( B(x=0) = B_0 \), while \( B(x=\pm L_x) = 0 \).
III. STRAIGHT FLUX FRONT FOR $\mu=0$

A. Asymptotics in the superconducting phase

When the boundary conditions are independent of $y$ (see notations in Fig. 3), the front is straight and the problem becomes one dimensional. We start with a case when the resistivity depends only on magnetic induction. Hence, now we consider $\mu=0$, returning to the general case in Sec. IV A. In addition we initially solve a simplified set dropping the relaxation term $\Gamma=0$ and diffusion $\kappa=0$. This assumption will be supported a posteriori by calculating the terms’ effects and computing with the numerical solution.

Looking for a solution of Eqs. (15) and (16) in the form

$$ \begin{align*}
 b & = b_s(X), \\
 \Delta & = \Delta_s(X),
\end{align*} $$

where $X=x-Vt$ is the distance from the interface and $V$ is the interface velocity, one obtains

$$ \begin{align*}
 -V^2 \frac{db_s}{dX} & = \frac{d}{dX} \left( \frac{b_s}{\Delta_s} \right)^* \frac{db_s}{dX}, \\
 V \frac{d\Delta_s}{dX} & = P_J.
\end{align*} $$

Here the Joule power density is $P_J=\rho J^2$. Let us first investigate the asymptotics of $b_s(X)$ in the vicinity of the front $X \rightarrow 0$. In the cold superconductor, the magnetic field vanishes. Therefore formally (ignoring formation of the very narrow normal region near the front which will be discussed in the next subsection) we look at the magnetic field $b_s(X)$ as a power with coefficient dependent on velocity only for $X < 0$:

$$ b_s(X) = A(V)|X|^{\alpha}. $$

The temperature is assumed to be of the form

$$ \Delta_s(X) = \Delta_{s0} - \Delta_{s1}|V||X|^{\beta}. $$

Substituting the Ansatz Eqs. (25) and (26) into Eqs. (23) and (24), one obtains on the superconducting side of the front $X < 0$:

$$ \begin{align*}
 V & = \Delta_{s0}(\nu V)^{1/\nu}, \\
 A & = \Delta_{s0}(\nu V)^{2/\nu}, \\
 \Delta_{s1} & = \frac{1}{2} \Delta_{s0}^2 (\nu V)^{2/\nu},
\end{align*} $$

which is satisfied for

$$ \begin{align*}
 \alpha & = 1/\nu, \\
 \beta & = 2/\nu.
\end{align*} $$

The electric current $j$ is $\partial b_s/\partial x$ formally diverges as $|X|^{1/\nu-1}$ at the front for $\nu > 1$. Of course the divergence is intercepted by the phase transition into the normal state creating the “hot” region of presumably small width $w_n$ determined by the condition that the depairing current is reached

$$ j(X = -w_n) = j_f = \Delta_{s0}(\nu V w_n)^{1/\nu}. $$

There is also dissipation in the superconducting part of a larger width $w_f$. The expression for the Joule heat term.
caused by the magnetic flux motion everywhere, not necessarily close to the front interface, diverges at the front as [see Eq. (25)] \( P_j \propto |X|^{2\nu - 1} \) for \( \nu > 2 \) only. Its integral, however, always converges.

To determine \( V, w_n \), and other characteristics of the front motion we need the solution in the normal domain. This and its matching with the asymptotics in the superconductor is considered next.

**B. Solution in normal domain for the temperature independent resistivity**

In the normal domain we assume first we assume for simplicity that \( \rho_n(T) \) = const in addition to the previously used simplification \( \kappa = \Gamma = 0 \). The nonlinear wave Ansatz

\[
 b = b_n(X), \quad \Delta = \Delta_n(X)
\]

will be initially used to find the current density \( j_n = (db_n/dX) \). Substitution of Eq. (32) into the normal state Eqs. (19) and (20) leads to the following set in terms of the front variable \( X = x - Vt \):

\[
 -V j_n = \rho_n \frac{dj_n}{dX},
\]

\[
 V \frac{d\Delta_n}{dX} = \rho_n j_n^2.
\]

The first equation has a solution

\[
 j_n(X) = j_n0 \exp \left[ -\frac{XV}{\rho_n} \right] = j_n0 \left( 1 - \frac{XV}{\rho_n} \right).
\]

The approximate form is generally valid since \( (|X|/V) \rho_n < (w_n/V) \ll 1 \) as will be justified \textit{a posteriori}. Then the heat transfer equation and the boundary condition \( \Delta_n(X=0) = \Delta_0 \) gives

\[
 \Delta_n(X) = \Delta_0 - \frac{\rho_n j_n0^2}{2V^2} \left\{ \exp \left[ -\frac{2XV}{\rho_n} \right] - 1 \right\} \approx \Delta_0 + \frac{j_n0^2}{V} X.
\]

In this region most of the heat is released

\[
 \Xi_n = \int_{-w_n}^{0} \rho_n(\theta) \left( \frac{\partial b_n}{\partial X} \right)^2 dX \approx \rho_n j_n0^2 w_n.
\]

We will use this result later.

**C. Matching solutions on the superconductor-normal interface and the flux front velocity**

The current, temperature, and the temperature gradient are all continuous on the superconductor-normal interface located at \( X = -w_n \). Consequently the current on the normal side approaches the same depairing current as that on the superconducting side, see Eq. (31). The temperature matching conditions are

\[
 \Delta(-w_n) = \Delta_0 - \frac{j_n^2 w_n}{V} = \Delta_n0.
\]

**D. The macroscopic description of the normal domain**

Since the normal domain is very narrow, it is more convenient to avoid explicit matching in simulations treating...
instead the heat release phenomenologically. In this approach the width of the normal domain is considered to be smaller than any other relevant scale and the normal part of the Joule heat term in the heat diffusion Eq. (16) is replaced by a delta function. This is equivalent to boundary condition on the front in which the normal domain contribution $\Xi_n$ calculated in Eq. (43) is added. The fine structure of the front is ignored in such an approach but it still provides a simple relation between the temperature difference between the Meissner domain and the mixed state domain $[\theta]$.

$$ V \simeq \frac{\Xi_n}{[\theta]} \quad (45) $$

This is obtained by integration of the heat transfer Eq. (16) in the vicinity of the front.

The temperature jump at the front $[\theta]$ however cannot be calculated in the framework of such a simple phenomenological theory and has to be obtained from the microscopic theory [see Eq. (44)]. This allows us to relate the temperature jump across the front to the microscopic parameters of the problem

$$ [\theta] = \frac{\rho_n \Delta^2}{\nu}, \quad (46) $$

where resistivity of the normal domain $\rho_n$ is a parameter the microscopic model. This relation significantly simplifies the numerical simulation in which appearance of a singular shock wave naturally increases complexity. The simulation will go beyond the limit $\kappa=\Gamma=0$ treated analytically earlier.

E. Numerical solution for magnetic flux conserving boundary conditions

The set of the scaled one dimensional Eqs. (15) and (16) for resistivity in the form of Eq. (17) in the superconducting domain was solved numerically using the Euler method. The normal domain was not directly simulated and matched. Instead we used the phenomenological relations described in the previous subsection to set the boundary condition on the front. Parameters describing the, numerical “experiment” were chosen to be: $\mu=0$, $v=5$, $\Gamma=0$, and $\kappa$ in the range 0.01–0.1. Size of the system is $L_s/x^*=200$. The boundary conditions are: the total flux $\Phi/(B^*x^*)$ in the range 0.5–2.5, temperature of the cold superconductor $\theta_0=0.7$:

$$ \theta(x=-200) = \theta(x=200) = \theta_0 \quad (47) $$

The normal phase was not simulated since it can be integrated analytically. The transition to the normal state at depairing current was taken into account by holding constant the normal domain Joule heat dissipation $\Xi_n$ for values in the range $5 \cdot 10^{-2}$–2.

The results of the numerical solution are presented in Figs. 5–7. The evolution of the magnetic induction is presented in Fig. 5 for the following values of the flux and heat diffusion constant: (a) $\Phi=0.5$, $\kappa=0.1$, (b) $\Phi=0.5$, $\kappa=0.01$, and (c) $\Phi=2.4$, $\kappa=0.05$. The value of $\Xi_n$ was kept fixed at $\Xi_n=0.5$. Different curves represent successive times with intervals of $\Delta t=2.5 t^*$ between them. Velocity of the sharp front is constant and is plotted as a function of $\Xi_n$ in Fig. 6 for $\Phi=2.4$ and $\kappa=0.05$. The temperature front moves together with the flux front velocity. The data are presented for the same times as for the magnetic induction. It demonstrates that the front interface velocity $V$ is linearly dependent on $\Xi_n$. The dependence of $\Phi$ is negligible. The results closely follow Eq. (44) obtained analytically for $\kappa=0$ and confirms the general physical picture proposed in the previous section that the velocity of the shock wave is universal in a sense.
that it depends only the heat released in the normal domain. The simulation reveals that the evolution is qualitatively the same for other values of the parameters.

The dynamics of the temperature distribution $\theta(x,t)$ is presented in Fig. 7 and has a form of a thermal shock wave. Two sets of parameters were simulated: (a) $\Phi=0.5$, $\kappa=0.1$, and (b) $\Phi=0.5$, $\kappa=0.01$. The maximum of temperature $\theta$ in this wave is reached at the interface between the superconducting and normal domains in the vicinity of the magnetic flux front. As we discussed in the previous section, the current is maximal in the normal domain which is narrow. We found in all the cases studied that the Joule heat released in the mixed state domain (see Fig. 4) does not exceed $1\%$ of that in the normal domain. Note a curious feature of Fig. 7 that all the curves intersect at a certain point.

IV. GENERALIZATIONS: THE $\mu \neq 0$ RESISTIVITY AND THE CONSTANT MAGNETIC FIELD BOUNDARY CONDITION

A. More general $I-V$ $\mu \neq 0$

Although in real samples resistance in the resistive mixed state might be a more complicated function than it was assumed earlier, the model representation in the form of Eqs. (7) and (8) with arbitrary critical exponents $\nu$ and $\mu$ is a robust and experimentally justified way to treat the problem. In such a case the main conclusions obtained for resistance with $\mu=0$, remain valid for some special relations between the critical exponents only.

Assuming $b_1$ and $\Delta_s$, in the vicinity of the front ($X \rightarrow 0$) in the form of the Eqs. (25) and (26) one obtains asymptotically for $A(V)$ and $\alpha$:

$$\alpha = \frac{\mu + 1}{\nu + \mu} ; \quad A(V) = \Delta_0 V^{1[(\nu+\mu)/(\mu+\nu)]}.$$

(48)

The electric current now behaves as

$$j \propto |X|^{(1-\nu)/(\mu+\nu)},$$

(49)

and still diverges for $\nu > 1$. This condition is independent of $\mu$, although the power in Eq. (49) depends on $\mu$. In the case $\nu < 1$ there is no normal domain and one can neglect the Joule heat. Hence, the temperature gradients are small and it suffices to consider the flux dynamics described by Eq. (15) with temperature fixed at $\theta_s$. Looking for an exact solution in a form

$$b = b_1 t^{-\alpha} f(\zeta),$$

(50)

where $\zeta = b_2 x / l^2$, we obtain for $b_1$ and $f(\zeta)$ (see Refs. 7 and 19), under the flux conservation law boundary condition $\Phi = \int dxdB(x,t)$:

$$\alpha = \beta = 1/(2\mu + 2 + \nu) ; \quad b_1 = \Phi^{(\mu+2)/(2\mu+2+\nu)} ;$$

$$b_2 = \Phi^{-(\nu+\mu)/(2\mu+2+\nu)}.$$

heat released in the normal domain. Other features are also
indicating that the front velocity is governed solely by the Joule
flux in Fig. 6. This is consistent with our analytic result pre-
h\[f(\zeta) = \left\{ \frac{\mu + \nu}{\mu + 2} \left( \frac{1}{2\mu + 2 + \nu} \right)^{1/(\mu+1)} \right\}^{(\mu+2)/((\mu+1))} \\
\times \left\{ 1 - \left( \frac{\zeta}{\zeta_f} \right)^{(\mu+2)/\mu} \right\}^{(1+\mu/\mu+\nu)}, \]
\[\zeta_f^{(2\mu+2)/\mu} = \frac{\mu + 1}{\mu + 2} \left( \frac{\mu + \nu}{\mu + 2} \right) \left( \frac{1}{2 + 2\mu + \nu} \right)^{1/(\nu+\mu)} \]
\times B \left[ \frac{2 + 2\mu + \nu \mu + 1}{\mu + \nu}, \frac{\nu + 1}{\mu + 2} \right], \]
where B is the beta function. The flux front moves with
velocity $V_f(t) = dx/dt \propto \nu^{-1}$ decaying with time. In the ab-
ence of the excessive heat released at the flux front the flux
front in this case is completely stable.

B. Constant magnetic field

In certain cases similar phenomena will occur when flux
is not conserved. Examples include narrow stripes, fields
larger than $H_{c1}$, etc. This does not mean that the effect dis-
appears since magnetic flux generally forms a thermomag-
netic shock wave. The main prerequisite is a phase transition
from superconductor to normal metal resulting in a sharp
flux front. This case was studied numerically for constant
magnetic field (in units of $B'$) $b = 0.05$ and parameters $\nu = 5,$
$\kappa = 0.05$, $\Gamma = 0$, and $\Xi = 0.5$. The profile of the magnetic field
and the temperature shock waves are presented in Figs.
8(a),8(b), where different curves correspond (from left to
right) to various times: $t = 0$, $5$, $10$, $15$... (in the $t'$ units).
It is important to note that, when the simulation was done for
different $\Xi$, the dependence was linear like for the constant
flux in Fig. 6. This is consistent with our analytic result pre-
dicting that the front velocity is governed solely by the Joule
heat released in the normal domain. Other features are also
independent of boundary conditions.

V. INSTABILITY OF THE STRAIGHT FRONT

A. Linear stability analysis for $\kappa = \Gamma = 0$

The dependence of the front velocity on the Joule heat
released near the interface can lead to an instability of the
straight front. Perturbations like a slight spatial distribution
of the sample parameters (resistance, for example) can trig-
gr the front instability. Keeping the normal resistivity in the
form $\rho_n = \rho_0 + \rho_1 \theta(x,t)$ we look for a solution of the corru-
gated front in the normal domain as
\[b = b_n(x - Vt) + \eta(x,y,t),\]
\[\theta = \theta_n(x - Vt) + \zeta(x,y,t). \]
The leading order solution $\beta_n$ and $\theta_n$ for the set of basic Eqs.
(15) and (16) for $\rho_1 = 0$ were obtained in Sec. III, while cor-
rections to the first order in $\rho_1$ will not be required in the
stability analysis. The first order terms in perturbations $\eta$ and
$\zeta$ are

\[\frac{\partial \eta}{\partial t} = \rho_n(\theta_n) \nabla^2 \eta + \rho_1 \frac{\partial \theta_n}{\partial x} \frac{\partial \eta}{\partial x} + \rho_1 \frac{\partial^2 b_n}{\partial x^2} \zeta + \rho_1 \frac{\partial b_n}{\partial x} \frac{\partial \zeta}{\partial x}, \]
\[\frac{\partial \zeta}{\partial t} = 2\rho_n(\theta_n) \frac{\partial b_n}{\partial x} \frac{\partial \eta}{\partial x} + \rho_1 \left( \frac{\partial b_n}{\partial x} \right)^2 \zeta. \]

Due to translation invariance of these eigenvalue equations in
time and the direction along the front $y$ one represents $\eta$, $\zeta$
in a form
\[\eta = \eta(x) \exp(\Omega t + k_y); \quad \zeta = \zeta(x) \exp(\Omega t + k_y). \]
Then the eigenvalue equations become one dimensional
\[\hat{L} \begin{bmatrix} \eta \\ \zeta \end{bmatrix} = \Omega \begin{bmatrix} \eta \\ \zeta \end{bmatrix}, \]
where

FIG. 8. Magnetic field at $x=0$ is constant. The curves cor-
respond to six different times from left to right with intervals of $\Delta t = 5 t'$ between them. Joule heat released at the front $\Xi = 0.5$. (a) The magnetic induction evolution and (b) the temperature shock wave.
\[
\hat{L} = \rho_j(\theta_n)\left(\frac{\partial^2}{\partial X^2} + \rho_j(\theta_n)\frac{\partial^2}{\partial X^2} - k_f^2\right) + \rho_j(\theta_n)\left(\frac{\partial^2}{\partial X^2} + \frac{\partial}{\partial X} + \frac{\partial^2}{\partial X^2}\right) + 2\rho_j(\theta_n)\frac{\partial^2}{\partial X^2} + \rho_j(\theta_n)\frac{\partial^2}{\partial X^2} + \rho_j(\theta_n)\frac{\partial^2}{\partial X^2}.
\]

(58)

Let us first consider the simpler case of conventional superconductors for which \(\rho_j = 0\). Substituting Eqs. (35), (36) into Eqs. (54) and (55) one obtains [replacing \(\partial / \partial X \rightarrow i k_f\)]

\[
\hat{L}_0 = \begin{bmatrix}
-\rho_0(k_f^2 + k_s^2) & 0 \\
-2i\rho_0 j_d k_f & 0
\end{bmatrix}
\]

(59)

The matrix \(\hat{L}_0\) has one stable \(\Omega_1 = -\rho_0(k_f^2 + k_s^2)\) and one marginal \(\Omega_2 = 0\) eigenvalues. This eigenvalue is highly degenerate: any temperature deviation \(\xi\) for \(\eta=0\) belongs to this subspace: \(\hat{L}_0 \xi = 0\). Strictly speaking the marginal eigenvalue \(\Omega_2\) calls for investigation beyond the linear stability analysis. However, we believe it is stable and, in any case, addition of the \(\rho_i\) term to resistivity removes the marginality and the degeneracy. To find the corrected eigenvalue \(\Omega_2\), one has to diagonalize on the corresponding subspace the operator

\[
\hat{L}_{ij} = \rho_j(\frac{\partial^2}{\partial X^2})^2.
\]

(60)

The derivative is nearly constant in the normal domain, see Eq. (35):

\[
\hat{L}_{ij} = \rho_j \frac{f^2}{\rho_0} \exp\left(-\frac{2XY}{\rho_0}\right) = \rho_j f^2.
\]

(61)

Consequently

\[
\Omega_2 = \rho_j f^2.
\]

(62)

which demonstrates the instability for any wave vector.

The physical reason for the instability is the positive feedback between temperature fluctuation at the front increasing in its turn both the Joule power at the front and its velocity. In fact, it is the well known hydrodynamics tangential instability of the flux front which is responsible for the front instability. Indeed, in this case warmer segments of the front move faster and can destroy the flat front line.

B. Stability in the general case: Numerical simulation

If the normal resistivity of the sample is temperature dependent and \(\kappa = 0\), then the normal domain in the front shows instability with respect to small temperature fluctuations with arbitrary wave vector. In this case the normal domain in the front shows instability with respect to small temperature fluctuations with arbitrary wave vectors. The dispersion appears for the nonzero heat diffusion coefficient. In fact, however, these small fluctuations cannot destroy the straight line front. It becomes unstable due to large amplitude fluctuations. Let us consider the evolution of the instability.

First of all the instability can develop when the characteristic time \(t_0 = 1/(\rho_j f^2)\) is smaller than the characteristic time of the heat absorption in the sample \(t_s = \Gamma^{-1}\). In addition the heat diffusion along the \(y\) axis can also affect the unstable fluctuations. In the latter case the requirement is: \(ut_0 > \sqrt{\kappa t_0}\). These two requirements allow us to determine the critical velocity of the fluctuation for the onset of the instability

\[
u > u_c = \min(\Gamma w, J_f, \kappa p_1).\]

(63)

In metals and alloys the normal state resistivity practically does not depend on temperature in the relevant temperature range. This means that \(\rho_j = 0\) and consequently no instability is expected.

The threshold in the fluctuation velocity \(u_c\) (which is proportional to the Joule heat released in the front) means that only a large temperature fluctuation can provide Joule heating necessary to destroy the planar front. Physically large amplitude fluctuations of the temperature at the front are nonuniform because they are caused by the spatial distribution of the impurities in the sample locally increasing resistivity and hence the Joule heat and velocity of the fluctuations in the front. Numerical simulations support this scenario.

In order to study the development of the instability for arbitrary \(\kappa\), the set of the Eqs. (15), (16) have been solved numerically. The Joule heat power \(\Xi_n\) released in the normal domain at the front has the following model form:

\[
\Xi_n(\theta) = \rho_j(\theta) = 1 + \alpha(\theta(x,y,t) - \theta_0).
\]

(64)

where initial temperature is perturbed in the region \(0 < x < 5, 4 < y < 5\), (temperature fluctuation \(\theta(x,y,t=0) = 0.88\)), while outside this region \(\theta(x,y,t=0) = \theta_0 = 0.7\). We chose \(\alpha = 14.5, \kappa = 0.05\) and 2.5. Physically this kind of fluctuation represents a local variation of the normal state resistivity (proportional to \(\Xi_n\)) when the front of the shock wave passes an inhomogeneity.

The evolution of a small fluctuation in two opposite limits is presented in Fig. 9. For small \(\kappa\), Fig. 9(a), an unstable pattern of the magnetic induction develops. It should be noted that the flux line lost its stability essentially immediately after the temperature fluctuation affected the system. For finite \(\kappa\) we observe that most of the \(\kappa = 0\) unstable modes are diffused away and do not develop into an instability of the system. For large \(\kappa\), Fig. 9(b), a similar perturbation relaxes into a straight line front and disappears in accord with the stability analysis.

VI. DISCUSSION

To summarize, we considered the formation and stability of the shock waves in the vortex matter under extreme conditions of the fast flux expansion into the Meissner state. In this case very strong screening currents significantly exceeding the critical current \(J_s\) flow in the mixed state. For such strong currents the vortex matter resistivity \(R\) has a form \(R \approx B^2 J_f\). We predict that when \(\nu > 1\) both the moving flux and the temperature profile form a sharp singular shock waves.
The condition for formation of the normal domain is completely determined by the Joule heat released in the normal domain at the front and hence on the normal resistivity of the sample. The flux front velocity has the form for \( \mu = 0 \) (in dimensional units) is

\[
V = [c R_n n_d/(1 - T_0/T_c) B^*] \times [B'/B_d(0)]^{\nu}.
\]

Taking for example material parameters of the optimally doped YBCO, \( J_d = 10^8 \text{ A/cm}^2 \), \( R_n = 2 \cdot 10^{-6} \text{ \Omega cm} \), \( C = 1 \text{ J/cm}^2 \text{ K} \), one obtains for the flux front velocity \( V \approx 10^5 \text{ cm/s} \), which is in a good agreement with experimental data.\(^{12}\) Note, however, that the value strongly depends on the exponents \( \mu \) and \( \nu \). The width of the normal stripe is 0.5 \( \mu \text{m} \).

The type of the voltage–current characteristic therefore is the decisive factor determining the flux front stability in type-II superconductors. The instability is developed when the voltage–current characteristics of the uniform superconductor in its resistive state provides sufficient screening currents at the moving flux front interface. The physical reason for the instability is very similar to a well known hydrodynamic instability,\(^{20}\) when different layers of the liquid move with different and parallel velocities. In fact it is the positive feedback between excessive local temperature at the front and Joule heat released there that leads to instability. The hydrodynamic tangential instability of the flux front destroys the flat front. The instability develops for the fluctuation velocities exceeding the critical value \( U > U_c = \text{min}[[c B'(1 - T_0/T_c)]/4 \pi n D(J_d/C)]^{-1} d [dR_n/dT]_{T_c} \). where \( D \) is the heat diffusion constant and \( t_f \) is the heat absorption time. Taking \( D = 30 \text{ J/(cm s K)} \) and \( t_f = 10^{-11} \text{ s} \), one estimates the two velocities as \( 5 \cdot 10^6 \text{ cm/s} \) and \( 2.6 \cdot 10^5 \text{ cm/s} \).

The avalanche-type instability appears when the moving flux front enters the area in which locally the normal resistivity is large. The experimental observation of the fast flux dynamics in YBCO has been carried out by Leiderer et al.\(^{12}\) The velocity of the front indeed has the universal character on the advanced stage of the instability and does not depend on initial magnetic gradients. The dendrite velocity in the later stages of disintegration of the front are expected to be of order of \( U_c \). This instability is not expected to arise in materials like Nb since \( dR_n/dT |_{T_c} \) is negligibly small and \( U_c \) vanishes.

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Author to whom correspondence should be addressed.


