The super connectivity of the pancake graphs and the super laceability of the star graphs

Cheng-Kuan Lin\textsuperscript{a}, Hua-Min Huang\textsuperscript{a}, Lih-Hsing Hsu\textsuperscript{b,\ast}

\textsuperscript{a}Department of Mathematics, National Central University, Chung-Li, Taiwan 32054, ROC
\textsuperscript{b}Information Engineering Department, Ta Hwa Institute of Technology, Hsinchu, Taiwan 307, ROC

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Abstract

A \textit{k}-container \(C(u, v)\) of a graph \(G\) is a set of \(k\)-disjoint paths joining \(u\) to \(v\). A \textit{k}-container \(C(u, v)\) of \(G\) is a \textit{k\textsuperscript{*}}-container if it contains all the vertices of \(G\). A graph \(G\) is \textit{k\textsuperscript{*}}-connected if there exists a \(k\textsuperscript{*}\)-container between any two distinct vertices. Let \(\kappa(G)\) be the connectivity of \(G\). A graph \(G\) is super connected if \(G\) is \(i\textsuperscript{*}\)-connected for all \(1 \leq i \leq \kappa(G)\). A bipartite graph \(G\) is \textit{k\textsuperscript{*}}-laceable if there exists a \(k\textsuperscript{*}\)-container between any two vertices from different parts of \(G\). A bipartite graph \(G\) is super laceable if \(G\) is \(i\textsuperscript{*}\)-laceable for all \(1 \leq i \leq \kappa(G)\). In this paper, we prove that the \(n\)-dimensional pancake graph \(P_n\) is super connected if and only if \(n \neq 3\) and the \(n\)-dimensional star graph \(S_n\) is super laceable if and only if \(n \neq 3\).

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1. Introduction

An interconnection network connects the processors of parallel computers. Its architecture can be represented as a graph in which the vertices correspond to processors and the edges correspond to connections. Hence, we use graphs and networks interchangeably. There are many mutually conflicting requirements in designing the topology for computer
networks. The $n$-cube is one of the most popular topologies [18]. The $n$-dimensional star network $S_n$ was proposed in [1] as “an attractive alternative to the $n$-cube” topology for interconnecting processors in parallel computers. Since its introduction, the network has received considerable attention. Akers et al. [1] showed that the star graphs are vertex transitive and edge transitive. The diameter and fault diameters were computed in [1,17,22,23].

In particular, Fragopoulou and Akl [7,8] studied the embedding of $(n - 1)$ directed edge-disjoint spanning trees on the star network $S_n$. These spanning trees are used in communication algorithms for star networks.

Akers et al. [1] also proposed another family of interesting graphs, the $n$-dimensional pancake graph $P_n$. They also showed that the pancake graphs are vertex transitive. Hung et al. [14] studied the hamiltonian connectivity on the faulty pancake graphs. The embedding of cycles and trees into the pancake graphs where discussed in [6,14,16]. Gates and Papadimitriou [10] studied the diameter of the pancake graphs. Until now, we do not know the exact value of the diameter of the pancake graphs [11].

For the graph definition and notation, we follow [3]. $G = (V, E)$ is a graph if $V$ is a finite set and $E$ is a subset of $\{(a, b) \mid (a, b)$ is an unordered pair of $V\}$. We say that $V$ is the vertex set and $E$ is the edge set. A path of length $k$ from $x$ to $y$ is a sequence of distinct vertices $\langle v_0, v_1, v_2, \ldots, v_k \rangle$, where $x = v_0$, $y = v_k$, and $(v_{i-1}, v_i) \in E$ for all $1 \leq i \leq k$. We also write the path $\langle v_0, v_1, \ldots, v_k \rangle$ as $\langle v_0, v_1, v_2, \ldots, v_k \rangle$, where $Q$ is a path from $v_i$ to $v_j$. Note that we allow $Q$ to be a path of length zero. We also write the path $\langle v_0, v_1, v_2, \ldots, v_k \rangle$ as $\langle v_0, v_1, v_2, \ldots, v_k \rangle$, where $Q_1$ is the path $\langle v_0, v_1, \ldots, v_i \rangle$ and $Q_2$ is the path $\langle v_j, v_{j+1}, \ldots, v_k \rangle$. We use $d(u, v)$ to denote the distance between $u$ and $v$, i.e., the length of the shortest path joining $u$ and $v$.

A path of graph $G$ from $u$ to $v$ is a hamiltonian path if it contains all vertices of $G$. A graph $G$ is hamiltonian connected if there exists a hamiltonian path joining any two distinct vertices. A cycle is a path (except that the first vertex is the same as the last vertex) containing at least three vertices. A cycle of $G$ is a hamiltonian cycle if it contains all vertices. A graph is hamiltonian if it has a hamiltonian cycle.

The connectivity of $G$, $\kappa(G)$, is the minimum number of vertices whose removal leaves the remaining graph disconnected or trivial. It follows from Menger’s Theorem [20] that there are $k$ internal vertex-disjoint (abbreviated as disjoint) paths joining any two distinct vertices $u$ and $v$ for any $k \leq \kappa(G)$. A $k$-container $C(u, v)$ of $G$ is a set of $k$ disjoint paths joining $u$ to $v$. In this paper, we discuss another type of container. A $k$-container $C(u, v)$ is a $k^*$-container if it contains all vertices of $G$. A graph $G$ is $k^*$-connected if there exists a $k^*$-container between any two distinct vertices. In particular, a graph $G$ is $1^*$-connected if and only if it is hamiltonian connected, and a graph $G$ is $2^*$-connected if and only if it is hamiltonian. All $1^*$-connected graphs except that $K_1$ and $K_2$ are $2^*$-connected. The study of $k^*$-connected graphs is motivated by the globally connected graphs proposed by Albert et al. [2]. A graph $G$ is super connected if it is $i^*$-connected for all $1 \leq i \leq \kappa(G)$. In this paper, we will prove that the pancake graph $P_n$ is super connected if and only if $n \neq 3$.

A graph $G$ is bipartite if its vertex set can be partitioned into two subsets $V_1$ and $V_2$ such that every edge joins vertices between $V_1$ and $V_2$. Let $G$ be a $k$-connected bipartite graph with bipartition $V_1$ and $V_2$ such that $|V_1| \geq |V_2|$. Suppose that there exists a $k^*$-container $C(u, v) = \{P_1, P_2, \ldots, P_k\}$ in a bipartite graph joining $u$ to $v$ with $u, v \in V_1$. Obviously,
the number of vertices in $P_i$ is $2k_i + 1$ for some integer $k_i$. There are $k_i - 1$ vertices of $P_i$ in $V_1$ other than $u$ and $v$, and $k_i$ vertices of $P_i$ in $V_2$. As a consequence, $|V_1| = \sum_{i=1}^{k}(k_i - 1) + 2$ and $|V_2| = \sum_{i=1}^{k} k_i$. Therefore, any bipartite graph $G$ with $\kappa(G) \geq 3$ is not $k^*$-connected for any $3 \leq k \leq \kappa(G)$.

For this reason, a bipartite graph is $k^*$-laceable if there exists a $k^*$-container between any two vertices from different partite sets. Obviously, any bipartite $k^*$-laceable graph with $k \geq 2$ has the equal size of bipartition. A $1^*$-laceable graph is also known as hamiltonian laceable graph. Moreover, a graph $G$ is $2^*$-laceable if and only if it is hamiltonian. All $1^*$-laceable graphs except that $K_1$ and $K_2$ are $2^*$-laceable. A bipartite graph $G$ is super laceable if $G$ is $i^*$-laceable for all $1 \leq i \leq \kappa(G)$. In this paper, we will prove that the star graph $S_n$ is super laceable if and only if $n \neq 3$.

In the following section, we give the definition of the pancake graphs and discuss some of their properties. In Section 3, we prove that the pancake graph $P_n$ is super connected if and only if $n \neq 3$. The definition of the star graphs and some of their properties are presented in Section 4. In Section 5, we prove that the star graph $S_n$ is super laceable if and only if $n \neq 3$. In the final section, we discuss further research.

2. The pancake graphs

Let $n$ be a positive integer. We use $\langle n \rangle$ to denote the set $\{1, 2, \ldots, n\}$. The $n$-dimensional pancake graph, denoted by $P_n$, is a graph with the vertex set $V(P_n) = \{u_1 u_2 \ldots u_n \mid u_i \in \langle n \rangle$ and $u_i \neq u_j$ for $i \neq j\}$. The adjacency is defined as follows: $u_1 u_2 \ldots u_i \ldots u_n$ is adjacent to $v_1 v_2 \ldots v_i \ldots v_n$ through an edge of dimension $i$ with $2 \leq i \leq n$ if $v_j = u_{i-j+1}$ for all $1 \leq j \leq i$ and $v_j = u_j$ for all $i < j \leq n$. We will use bold face to denote a vertex of $P_n$. Hence, $u_1, u_2, \ldots, u_n$ denote a sequence of vertices in $P_n$. In particular, $e$ denotes the vertex $12 \ldots n$. By definition, $P_n$ is an $(n-1)$-regular graph with $n!$ vertices.

Let $u = u_1 u_2 \ldots u_n$ be any vertex of $P_n$. We use $(u)_i$ to denote the $i$th component $u_i$ of $u$, and use $P_{n}^{(i)}$ to denote the $i$th subgraph of $P_n$ induced by those vertices $u$ with $(u)_i = i$. Obviously, $P_n$ can be decomposed into $n$ vertex disjoint subgraphs $P_{n}^{(i)}$ for every $i \in \langle n \rangle$ such that each $P_{n}^{(i)}$ is isomorphic to $P_{n-1}$. Thus, the pancake graph can be constructed recursively. Let $H \subseteq \langle n \rangle$, we use $P_{n}^{H}$ to denote the subgraph of $P_n$ induced by $\cup_{i \in H} V(P_{n}^{(i)})$. By definition, there is exactly one neighbor $v$ of $u$ such that $u$ and $v$ are adjacent through an $i$-dimensional edge with $2 \leq i \leq n$. For this reason, we use $(u)^i$ to denote the unique $i$-neighbor of $u$. We have $((u)^i)^i = u$ and $(u)^{n} \in P_{n}^{(\{u\})}$. For $1 \leq i, j \leq n$ and $i \neq j$, we use $E^{i,j}$ to denote the set of edges between $P_{n}^{(i)}$ and $P_{n}^{(j)}$. The pancake graphs $P_2, P_3, P_4$ are shown in Fig. 1 for illustration.

The following theorem is proved by Hung et al. [14].

**Theorem 1** (Hung et al. [14]). $P_n$ is $1^*$-connected if $n \neq 3$, and $P_n$ is $2^*$-connected if $n \geq 3$.

**Lemma 1.** Assume that $n \geq 3$, $|E^{i,j}| = (n-2)!$ for any $1 \leq i, j \leq n$ with $i \neq j$.
Lemma 2. Let \( u \) and \( v \) be any two distinct vertices of \( P_n \) with \( d(u, v) \leq 2 \). Then \( (u)_1 \neq (v)_1 \). Moreover, \( \{(u)_i \mid 2 \leq i \leq n - 1\} = \langle n \rangle - \{(u)_1, (u)_n\} \) if \( n \geq 3 \).

Lemma 3. Let \( n \geq 5 \) and \( H = \{i_1, i_2, \ldots, i_m\} \) be any nonempty subset of \( \langle n \rangle \). There is a hamiltonian path of \( P_n^H \) joining any vertex \( u \in P_n^{[i_1]} \) to any other vertex \( v \in P_n^{[i_m]} \).

Proof. Note that \( P_n^{[i_j]} \) is isomorphic to \( P_{n-1} \) for every \( 1 \leq j \leq m \). We set \( x_1 = u \) and \( y_m = v \). By Theorem 1, this theorem holds for \( m = 1 \). Assume that \( m \geq 2 \). By Lemma 1, we choose \( (y_j, x_{j+1}) \in E^{i_j,i_{j+1}} \) with \( y_j \neq x_j \) and \( y_m \neq x_m \) for every \( 1 \leq j \leq m - 1 \). By Theorem 1, there is a hamiltonian path \( Q_j \) of \( P_n^{[i_j]} \) joining \( x_j \) to \( y_j \) for every \( 1 \leq j \leq m \). The path \( (x_1, Q_1, y_1, x_2, Q_2, y_2, \ldots, x_m, Q_m, y_m) \) forms a desired path. \( \square \)

3. The super connectivity of the pancake graphs

Lemma 4. Let \( n \geq 5 \). Let \( u \) and \( v \) be any two distinct vertices in \( P_n^{[t]} \) for some \( t \in \langle n \rangle \). If \( P_{n-1} \) is \( k^* \)-connected, then there is a \((k+1)^*\)-container of \( P_n \) between \( u \) and \( v \).

Proof. Since \( P_n^{[t]} \) is isomorphic to \( P_{n-1} \), there is a \( k^* \)-container \( \{Q_1, Q_2, \ldots, Q_k\} \) of \( P_n^{[t]} \) joining \( u \) to \( v \). We need to find a \((k+1)^*\)-container of \( P_n \) joining \( u \) to \( v \). We set \( p = (u)_1 \) and \( q = (v)_1 \).

Case 1: \( p = q \). Thus, \( (u)^n \) and \( (v)^n \) are in \( P_n^{[p]} \). By Lemma 3, there is a hamiltonian path \( Q \) of \( P_n^{[p]} \) joining \( (u)^n \) to \( (v)^n \). We write \( Q \) as \( ((u)^n, Q', y, z, (v)^n) \). By Lemma 2, \( (y)_1 \neq (z)_1, (y)_1 \neq t \), and \( (z)_1 \neq t \). By Lemma 3, there is a hamiltonian path \( R \) of \( P_n^{[n-(t,p)]} \) joining \( (y)^n \) to \( (z)^n \). We set \( Q_{k+1} \) as \( (u, (u)^n, Q', y, (y)^n, R, (z)^n, z, (v)^n, v) \).
Then \( \{ Q_1, Q_2, \ldots, Q_{k+1} \} \) forms a \((k + 1)^{st}\)-container of \( P_n \) joining \( u \) to \( v \). See Fig. 2a for illustration.

**Case 2:** \( p \neq q \). Thus, \((u)^n\) and \((v)^n\) are in different subgraphs \( P_n^{[p]} \) and \( P_n^{[q]} \). By Lemma 3, there is a hamiltonian path \( Q \) of \( P_n^{[n]-[p]} \) joining \((u)^n\) to \((v)^n\). We set \( Q_{k+1} \) as \((u, (u)^n, Q, (v)^n, v)\). Then \( \{ Q_1, Q_2, \ldots, Q_{k+1} \} \) forms a \((k + 1)^{st}\)-container of \( P_n \) joining \( u \) to \( v \). See Fig. 2b for illustration.

Thus, the theorem is proved. □

**Lemma 5.** Let \( n \geq 5 \) and \( k \) be any positive integer with \( 3 \leq k \leq n - 1 \). Let \( u \) be any vertex in \( P_n^{[s]} \) and \( v \) be any vertex in \( P_n^{[r]} \) such that \( s \neq t \). Suppose that \( P_{n-1} \) is \( k^{st}\)-connected. Then there is a \( k^{st}\)-container of \( P_n \) between \( u \) and \( v \) not using the edge \((u, v)\) if \((u, v) \in E(P_n)\).

**Proof.** Since \( |E^{[s]}| = (n - 2)! \leq 6 \), we can choose a vertex \( y \) in \( P_n^{[s]} \) − \{u\} and a vertex \( z \) in \( P_n^{[r]} \) − \{v\} with \((y, z) \in E^{[s,r]}\). Note that \( P_n^{[s]} \) and \( P_n^{[r]} \) are both isomorphic to \( P_{n-1} \). Let \( \{R_1, R_2, \ldots, R_k\} \) be a \( k^{st}\)-container of \( P_n^{[s]} \) joining \( u \) to \( y \), and \( \{H_1, H_2, \ldots, H_k\} \) be a \( k^{st}\)-container of \( P_n^{[r]} \) joining \( z \) to \( v \). We write \( R_i = (u, R'_i, y_i, y) \) and \( H_i = (z, z_i, H'_i, v) \). (Note that \( y_1 = u \) if the length of \( R'_i \) is zero and \( z_i = v \) if the length of \( H'_i \) is zero.) Let \( I = \{ y_i \mid 1 \leq i \leq k \} \) and \( J = \{ z_i \mid 1 \leq i \leq k \} \). Note that \((y_1) = (y)_j \) for some \( j \in \{2, 3, \ldots, n - 1\} \), and \((y)_i \neq (y)_m \) if \( i \neq m \). By Lemma 2, \((y_i) \mid 1 \leq i \leq k \) \( \cap \{ s, t \} \) = \emptys. Similarly, \((z_i) \mid 1 \leq i \leq k \) \( \cap \{ s, t \} \) = \emptys. Let \( A = \{ y_i \mid y_i \in I \) and there exists an element \( z_j \in J \) such that \((y_i) = (z_j) \}. \) Then we relabel the indices of \( I \) and \( J \) such that \((y_i) = (z_i) \) if \( 1 \leq i \leq |A| \). We set \( X \) as \((y_i) \mid 1 \leq i \leq k - 2 \) \( \cup \{ z_i \mid 1 \leq i \leq k - 2 \} \cup \{ s, t \} \). By Lemma 3, there is a hamiltonian path \( T_i \) of \( P_n^{[y_j), (z_i)]} \) joining \((y_i)^n \) to \((z_i)^n \) for every \( 1 \leq i \leq k - 2 \), and there is a hamiltonian path \( T_{k-1} \) of \( P_n^{[y_j) \cup \{s, t\}] \) joining \((y_{k-1})^n \) to \((z_{k-1})^n \). (Note that \((y_{k-1}), (z_{k-1}) = (y_{k-1}) \) if \((y_{k-1}) = (z_{k-1}) \).) We set

\[
Q_i = (u, R'_i, y_i, (y_i)^n, T_i, (z_i)^n, z_i, H'_i, v) \text{ for } 1 \leq i \leq k - 2, \\
Q_{k-1} = (u, R'_k, y_{k-1}, (y_{k-1})^n, T_{k-1}, (z_k)^n, z_k, H'_k, v), \text{ and} \\
Q_k = (u, R'_k, y_k, z, z_{k-1}, H'_k, v).
\]

It is easy to check that \( \{ Q_1, Q_2, \ldots, Q_k \} \) forms a \( k^{st}\)-container of \( P_n \) joining \( u \) to \( v \) not using the edge \((u, v)\) if \((u, v) \in E(P_n)\). See Fig. 3 for illustration. □
Theorem 2. $P_n$ is $(n-1)^*\text{-connected}$ if $n \geq 2$.

Proof. It is easy to see that $P_2$ is $1^*\text{-connected}$ and $P_3$ is $2^*\text{-connected}$. Since the $P_4$ is vertex transitive, we claim that $P_4$ is $3^*\text{-connected}$ by listing all $3^*\text{-containers}$ from 1234 to any vertex as follows:

\[
\begin{align*}
\{1234, 2134, 3412\} & \\
\{1234, 3214, 4123, 2143, 3412, 4312\} & \\
\{1234, 3214, 3421\} & \\
\{1234, 4123, 2143, 4231\} & \\
\end{align*}
\]
Assume that $P_k$ is $(k-1)^*$-connected for every $4 \leq k \leq n-1$. Let $u$ and $v$ be any two distinct vertices of $P_n$ with $u \in P_n^{[s]}$ and $v \in P_n^{[t]}$. We need to find an $(n-1)^*$-container between $u$ and $v$ of $P_n$. Suppose that $s = t$. By Lemma 4, there is an $(n-1)^*$-container of $P_n$ joining $u$ to $v$. Thus, we assume that $s \neq t$. We set $p = (u)_1$ and $q = (v)_1$.

**Case 1:** $p = t$ and $q = s$. Thus, $(u)^p \in P_n^{[t]}$ and $(v)^q \in P_n^{[s]}$.

*Subcase 1.1:* $u = (v)^q$. Thus, $(u, v) \in E(P_n)$. By Lemma 5, there is an $(n-2)^*$-container \{ $Q_1, Q_2, \ldots, Q_{n-2}$ \} of $P_n$ joining $u$ to $v$ not using the edge $(u, v)$. We set $Q_{n-1}$ as $(u, v)$. Then \{ $Q_1, Q_2, \ldots, Q_{n-1}$ \} forms an $(n-1)^*$-container of $P_n$ joining $u$ to $v$.

*Subcase 1.2:* $u \neq (v)^q$. We set $y = (v)^q$ and $z = (u)^p$. Let \{ $R_1, R_2, \ldots, R_{n-2}$ \} be an $(n-2)^*$-container of $P_n^{[s]}$ joining $u$ to $y$, and let \{ $H_1, H_2, \ldots, H_{n-2}$ \} be an $(n-2)^*$-container of $P_n^{[t]}$ joining $z$ to $v$. We write $R_i = (u, r_i, y_i, y)$ and $H_i = (z, z_i, h_i, v)$. We set $I = \{ (y)_i \mid 1 \leq i \leq n-2 \}$ and $J = \{ (z)_j \mid 1 \leq j \leq n-2 \}$. Note that $(y)_i = (y)^i_j$ for some $j \in \{ 2, 3, \ldots, n-1 \}$, and $(y)^n_k \neq (y)^{n-1}_k$ if $k \neq l$. By Lemma 2, $I = \{ (y)_i \mid 2 \leq i \leq n-1 \}$ =

| {1234, 2134, (4312), (1324), (2431), (1234), (3124), (3241), (2341)} | {1234, 2134, (3124), (2413), (2431), (1234), (3124), (3241), (2341)} |
| {1234, 2134, (3124), (2413), (2431), (1234), (3124), (3241), (2341)} | {1234, 2134, (3124), (2413), (2431), (1234), (3124), (3241), (2341)} |
| {1234, 2134, (3124), (2413), (2431), (1234), (3124), (3241), (2341)} | {1234, 2134, (3124), (2413), (2431), (1234), (3124), (3241), (2341)} |
| {1234, 2134, (3124), (2413), (2431), (1234), (3124), (3241), (2341)} | {1234, 2134, (3124), (2413), (2431), (1234), (3124), (3241), (2341)} |
| {1234, 2134, (3124), (2413), (2431), (1234), (3124), (3241), (2341)} | {1234, 2134, (3124), (2413), (2431), (1234), (3124), (3241), (2341)} |
| {1234, 2134, (3124), (2413), (2431), (1234), (3124), (3241), (2341)} | {1234, 2134, (3124), (2413), (2431), (1234), (3124), (3241), (2341)} |
| {1234, 2134, (3124), (2413), (2431), (1234), (3124), (3241), (2341)} | {1234, 2134, (3124), (2413), (2431), (1234), (3124), (3241), (2341)} |
| {1234, 2134, (3124), (2413), (2431), (1234), (3124), (3241), (2341)} | {1234, 2134, (3124), (2413), (2431), (1234), (3124), (3241), (2341)} |
| {1234, 2134, (3124), (2413), (2431), (1234), (3124), (3241), (2341)} | {1234, 2134, (3124), (2413), (2431), (1234), (3124), (3241), (2341)} |
| {1234, 2134, (3124), (2413), (2431), (1234), (3124), (3241), (2341)} | {1234, 2134, (3124), (2413), (2431), (1234), (3124), (3241), (2341)} |
| {1234, 2134, (3124), (2413), (2431), (1234), (3124), (3241), (2341)} | {1234, 2134, (3124), (2413), (2431), (1234), (3124), (3241), (2341)} |
Fig. 4. Illustration for Theorem 2.

\( \langle n \rangle - \{s, t\} \). Similarly, \( J = \langle n \rangle - \{s, t\} \). We have \( I = J \). Without loss of generality, we assume that \( (y_i)_1 = (z_i)_1 \) for every \( 1 \leq i \leq n - 2 \). By Lemma 3, there is a Hamiltonian path \( T_i \) of \( P_n(\{y\}_i) \) joining \( (y)_i \) to \( (z)_i \) for every \( 1 \leq i \leq n - 4 \), and there is a Hamiltonian path \( T_{n-3} \) of \( P_{n}(\{y_{n-3},(y_{n-2})_i\}) \) joining \( (y_{n-3})_i \) to \( (z_{n-2})_i \). We set 

\[
Q_i = \langle u, R'_i, y_i, T_i, (y)_i, z_i, H'_i, v \rangle \quad \text{for} \quad 1 \leq i \leq n - 4,
Q_{n-3} = \langle u, R'_{n-3}, y_{n-3}, (y_{n-3})_i, T_{n-3}, (z_{n-2})_i, z_{n-2}, H'_{n-3}, v \rangle,
Q_{n-2} = \langle u, z, z_{n-3}, H'_{n-3}, v \rangle, \quad \text{and}
Q_{n-1} = \langle u, R'_{n-2}, y_{n-2}, y, v \rangle.
\]

Then \( \{Q_1, Q_2, \ldots, Q_{n-1}\} \) forms an \( (n - 1)^* \)-container of \( P_n \) joining \( u \) to \( v \). See Fig. 4a for illustration.

**Case 2:** \( p = t \) and \( q \in \langle n \rangle - \{s, t\} \). Since \( |E^{s,q}| = (n - 2)! \geq 6 \), we can choose a vertex \( y \) in \( P_{n}^{(s)} \) - \( \{u\} \) with \( y)_i \in P_{n}^{(q)} \). We set \( z = (u)_i \in P_{n}^{(s)} \). Let \( \{R_1, R_2, \ldots, R_{n-2}\} \) be an \( (n - 2)^* \)-container of \( P_{n}^{(s)} \) joining \( u \) to \( y \), and \( \{H_1, H_2, \ldots, H_{n-2}\} \) be an \( (n - 2)^* \)-container of \( P_{n}^{(s)} \) joining \( z \) to \( v \). We write \( R_i = \langle u, R'_i, y_i, y \rangle \) and \( H_i = \langle z, z_i, H'_i, v \rangle \). We have \( \{(y_i)_1 | 1 \leq i \leq n - 2\} = \{(y)_i | 2 \leq i \leq n - 1\} \). By Lemma 2, \( \{(y)_i | 1 \leq i \leq n - 2\} = \langle n \rangle - \{s, q\} \). Similarly, \( \{(z)_i | 1 \leq i \leq n - 2\} = \langle n \rangle - \{s, t\} \). Without loss of generality, we
assume that \((y_1)_1 = (z_1)_1\) for every \(1 \leq i \leq n - 3\), \((y_{n-2})_1 = t\), and \((z_{n-2})_1 = q\). By Lemma 3, there is a hamiltonian path \(T_i\) of \(P_{n}^{(y_1)_n}\) joining \((y_i)_n\) to \((z_i)_n\) for every \(1 \leq i \leq n - 3\), and there is a hamiltonian path \(T_{n-2}\) of \(P_{n}^{(q)}\) joining \((y)_n\) to \((v)_n\). We set

\[
Q_i = (u, R'_i, y_1, (y_i)_n, T_i, (z_i)_n, H'_i, v) \quad \text{for } 1 \leq i \leq n - 3,
\]
\[
Q_{n-2} = (u, R'_{n-2}, y_{n-2}, y, (y)_n, T_{n-2}, (v)_n, v), \quad \text{and}
\]
\[
Q_{n-1} = (u, z, z_{n-2}, H'_{n-2}, v).
\]

Then \(\{Q_1, Q_2, \ldots, Q_{n-1}\}\) forms an \((n-1)^*\)-container of \(P_n\) joining \(u\) to \(v\). See Fig. 4b for illustration.

**Case 3:** \(p, q \in \langle n \rangle - \{s, t\}\). Since \(|E_{x,y}| = (n - 2)! \geq 6\), there exists an edge \((y, z)\) in \(E_{x,y}\) with \(y \in P_{n}^{(s)} - \{u\}\) and \(z \in P_{n}^{(t)} - \{v\}\). Let \(\{R_1, R_2, \ldots, R_{n-2}\}\) be an \((n-2)^*\)-container of \(P_{n}^{(s)}\) joining \(u\) to \(y\), and let \(\{H_1, H_2, \ldots, H_{n-2}\}\) be an \((n-2)^*\)-container of \(P_{n}^{(t)}\) joining \(z\) to \(v\). We write \(R_i = (u, R'_i, y_i, y)\) and \(H_i = (z, z_i, H'_i, v)\). We set \(I = \{(y_i)_1 | 1 \leq i \leq n-2\}\) and \(J = \{(z_i)_1 | 1 \leq i \leq n-2\}\). We have \(I = \{(y_i)_{1 \leq i \leq n-1}\}\). By Lemma 2, \(I = \langle n \rangle - \{s, t\}\). Similarly, \(J = \langle n \rangle - \{s, t\}\). We have \(I = J\). Without loss of generality, we assume that \((y_1)_1 = (z_1)_1\) for every \(1 \leq i \leq n - 2\) with \((y_{n-2})_1 = p\).

**Subcase 3.1:** \(p = q\). By Lemma 3, there is a hamiltonian path \(T_i\) of \(P_{n}^{(y_1)_n}\) joining \((y_i)_n\) to \((z_i)_n\) for every \(i \in \langle n \rangle\), and there is a hamiltonian path \(T_{n-2}\) of \(P_{n}^{(p)}\) joining \((u)_n\) to \((v)_n\). We set

\[
Q_i = (u, R'_i, y_1, (y_i)_n, T_i, (z_i)_n, H'_i, v) \quad \text{for } 1 \leq i \leq n - 3,
\]
\[
Q_{n-2} = (u, R'_{n-2}, y_{n-2}, y, z, z_{n-2}, H'_{n-2}, v), \quad \text{and}
\]
\[
Q_{n-1} = (u, (u)_n, T_{n-2}, (z_{n-2})_n, H'_{n-2}, v).
\]

Then \(\{Q_1, Q_2, \ldots, Q_{n-1}\}\) forms an \((n-1)^*\)-container of \(P_n\) joining \(u\) to \(v\). See Fig. 4c for illustration.

**Subcase 3.2:** \(p \neq q\). Without loss of generality, we assume that \((y_{n-3})_1 = q\). By Theorem 1, there is a hamiltonian path \(T_i\) of \(P_{n}^{(y_1)_n}\) joining \((y_i)_n\) to \((z_i)_n\) for every \(1 \leq i \leq n - 4\), there is a hamiltonian path \(T_{n-3}\) of \(P_{n}^{(q)}\) joining \((y_{n-3})_n\) to \((v)_n\), and there is a hamiltonian path \(T_{n-2}\) of \(P_{n}^{(p)}\) joining \((u)_n\) to \((z_{n-2})_n\). We set

\[
Q_i = (u, R'_i, y_1, (y_i)_n, T_i, (z_i)_n, H'_i, v) \quad \text{for } 1 \leq i \leq n - 3,
\]
\[
Q_{n-3} = (u, R'_{n-3}, y_{n-3}, (y_{n-3})_n, T_{n-3}, (v)_n, v),
\]
\[
Q_{n-2} = (u, (u)_n, T_{n-2}, (z_{n-2})_n, z_{n-2}, H'_{n-2}, v), \quad \text{and}
\]
\[
Q_{n-1} = (u, R'_{n-2}, y_{n-2}, y, z, z_{n-3}, H'_{n-3}, v).
\]

It is easy to check that \(\{Q_1, Q_2, \ldots, Q_{n-1}\}\) is an \((n-1)^*\)-container of \(P_n\) from \(u\) to \(v\). See Fig. 4d for illustration.

Thus, the theorem is proved. □

**Theorem 3.** \(P_n\) is super connected if and only if \(n \neq 3\).

**Proof.** We prove this theorem by induction. Obviously, this theorem is true for \(P_1\) and \(P_2\). Since \(P_3\) is isomorphic to a cycle with six vertices, \(P_3\) is not \(1^*\)-connected. Thus, \(P_3\) is
not super connected. By Theorems 1 and 2, this theorem holds on $P_4$. Assume that $P_4$ is super connected for every $4 \leq k \leq n - 1$. By Theorems 1 and 2, $P_n$ is $k^*$-connected for any $k \in \{1, 2, n - 1\}$. Thus, we still need to construct a $k^*$-container of $P_n$ between any two distinct vertices $u \in P_n^{(i)}$ and $v \in P_n^{(j)}$ for every $3 \leq k \leq n - 2$.

Suppose that $s = t$. By induction, $P_{n - 1}$ is $(k - 1)^*$-connected. By Lemma 4, there is a $k^*$-container of $P_n$ joining $u$ to $v$. Suppose that $s \neq t$. By induction, $P_{n - 1}$ is $k^*$-connected. By Lemma 5, there is a $k^*$-container of $P_n$ joining $u$ to $v$.

Hence, the theorem is proved. □

4. The star graphs

The $n$-dimensional star graph, denoted by $S_n$, is a graph with the vertex set $V(S_n) = \{u_1u_2 \ldots u_n \mid u_i \in \langle n \rangle$ and $u_i \neq u_j$ for $i \neq j\}$. The adjacency is defined as follows: $u_1u_2 \ldots u_i \ldots u_n$ is adjacent to $v_1v_2 \ldots v_i \ldots v_n$ through an edge of dimension $i$ with $2 \leq i \leq n$ if $v_j = u_j$ for $j \notin \{1, i\}$, $v_1 = u_1$, and $v_i = u_i$. Again, we use bold face to denote a vertex of $S_n$. Hence, $u_1, u_2, \ldots, u_n$ denote a sequence of vertices of $S_n$. In particular, $e$ denotes the vertex $12 \ldots n$. By definition, $S_n$ is an $(n - 1)$-regular graph with $n!$ vertices.

It is known that $S_n$ is a bipartite graph with one partite set containing all odd permutations and the other partite set containing all even permutations. For convenience, we refer an even permutation as a white vertex, and refer an odd permutation as a black vertex. Let $u = u_1u_2 \ldots u_n$ be any vertex of $S_n$. We use $(u)_i$ to denote the $i$th component of $u$ and $S_n^{(i)}$ to denote the $i$th subgraph of $S_n$ induced by those vertices $u$ with $(u)_n = i$. Obviously, $S_n$ can be decomposed into $n$ vertex disjoint subgraphs $S_n^{(i)}$ for $1 \leq i \leq n$, such that each $S_n^{(i)}$ is isomorphic to $S_{n - 1}$. Thus, the star graph can be constructed recursively. Let $H \subseteq \langle n \rangle$. We use $S_n^H$ to denote the subgraph of $S_n$ induced by $\cup_{i \in H} V(S_n^{(i)})$. By the definition of $S_n$, there is exactly one neighbor $v$ of $u$ such that $u$ and $v$ are adjacent through an $i$-dimensional edge with $2 \leq i \leq n$. For this reason, we use $(u)^i$ to denote the unique $i$-neighbor of $u$. We have $(u)^i_j = u$ and $(u)^n \in S_n^{(n^{-1})}$. For $1 \leq i, j \leq n$ and $i \neq j$, we use $E^{i, j}$ to denote the set of edges between $S_n^{(i)}$ and $S_n^{(j)}$. The star graphs $S_2$, $S_3$, and $S_4$ are shown in Fig. 5 for illustration.

The following theorem is proved by Hsieh et al. [12].

**Theorem 4** (Hsieh et al. [12]). $S_n$ is $1^*$-laceable if $n \neq 3$, and $S_n$ is $2^*$-connected if $n \geq 3$.

**Lemma 6.** Assume that $n \geq 3$. $|E^{i, j}| = (n - 2)!$ for any $1 \leq i \neq j \leq n$. Moreover, there are $\frac{(n - 2)!}{2}$ edges joining black vertices of $S_n^{(i)}$ to white vertices of $S_n^{(j)}$.

**Lemma 7.** Let $u$ and $v$ be any two distinct vertices of $S_n$ with $d(u, v) \leq 2$. Then $(u)_1 \neq (v)_1$. Moreover, $\{(u)^i_j \mid 2 \leq i \leq n - 1\} = \langle n \rangle - \{(u)_1, (u)_n\}$ if $n \geq 3$.

**Lemma 8.** Let $n \geq 5$ and $H = \{i_1, i_2, \ldots, i_m\}$ be any nonempty subset of $\langle n \rangle$. There is a hamiltonian path of $S_n^H$ joining any white vertex $u \in S_n^{(i_1)}$ to any black vertex $v \in S_n^{(i_m)}$. 


Proof. Note that $S_{n}^{(j)}$ is isomorphic to $S_{n-1}$ for every $1 \leq j \leq m$. We set $x_1 = u$ and $y_m = v$. By Theorem 4, this theorem holds for $m = 1$. Assume that $m \geq 2$. By Lemma 6, we choose $(y_j, x_{j+1}) \in E_{i,j}^{(j+1)}$ with $y_j$ is a black vertex of $S_{n}^{(j)}$ and $x_{j+1}$ is a white vertex of $S_{n}^{(j+1)}$ for every $1 \leq j \leq m - 1$. By Theorem 4, there is a hamiltonian path $Q_j$ of $S_{n}^{(j)}$ joining $x_j$ to $y_j$. The path $(x_1, Q_1, y_1, x_2, Q_2, y_2, \ldots, Q_m, y_m)$ forms a desired path. \qed

5. The super laceability of the star graphs

In this section, we are going to prove that $S_n$ is super laceable if and only if $n \neq 3$. As you will observe, the proof is very similar to the proof that $P_n$ is super connected if and only if $n \neq 3$.

Lemma 9. Let $n \geq 5$ and $k$ be any positive integer with $3 \leq k \leq n - 1$. Let $u$ be any white vertex and $v$ be any black vertex of $S_n$. Suppose that $S_{n-1}$ is $k^*$-laceable. Then there is a $k^*$-container of $S_n$ between $u$ and $v$ not using the edge $(u, v)$ if $(u, v) \in E(S_n)$.

Proof. Since $S_n$ is edge transitive, we may assume that $u \in S_n^{[n]}$ and $v \in S_n^{[n-1]}$. By Lemma 6, there are $(n-2)!(n-1)! > 3$ edges joining black vertices of $S_n^{[n]}$ to white vertices of $S_n^{[n-1]}$. We can choose an edge $(y, z) \in E_n^{n-1,n}$ where $y$ is a black vertex in $S_n^{[n]}$ and $z$ is a white vertex in $S_n^{[n-1]}$. By induction, there is a $k^*$-container $\{R_1, R_2, \ldots, R_k\}$ of $S_n^{[n]}$ joining $u$ to $y$, and there is a $k^*$-container $\{H_1, H_2, \ldots, H_k\}$ of $S_n^{[n-1]}$ joining $z$ to $v$. We write $R_i = (u, R_i, y_i, y)$ and $H_i = (z, z_i, H_i, v)$. Note that $y_i$ is a white vertex and $z_i$ is a black vertex for every $1 \leq i \leq k$. Let $I = \{y_1, (y_i, y) \in E(R_i) \text{ and } 1 \leq i \leq k\}$, and $J = \{z_i \mid (z_i, x) \in E(H_i) \text{ and } 1 \leq i \leq k\}$. Note that $(y_1)_1 = (y)_j$ for some $j \in \{2, 3, \ldots, n - 1\}$, and $(y)_i \neq (y)_m$ if $i \neq m$. By Lemma 7, $\{y_1\}_{1 \leq i \leq k} \cap \{n - 1, n\} = \emptyset$. Similarly, $\{z_i\}_1 \cap \{n - 1, n\} = \emptyset$. Let $A = \{y_i \mid y_i \in I \text{ and there exists an element } z_j \in J\}$
such that \((y_i)_1 = (z_j)_1\). Then we relabel the indices of \(I\) and \(J\) such that \((y_i)_1 = (z_j)_1\) for \(1 \leq i \leq |A|\). We set \(X\) as \(\{(y_i)_1 \mid 1 \leq i \leq k - 2\} \cup \{(z_j)_1 \mid 1 \leq i \leq k - 2\} \cup \{n - 1, n\}\). By Lemma 8, there is a hamiltonian path \(T_0 of S(n)_{Q(1), (z_1)}\) joining the black vertex \((y_i)_1\) to the white vertex \((z_j)_1\) for every \(1 \leq i \leq k - 2\), and there is a hamiltonian path \(T_{k-1} of S(n)_{Q(1), (z_1)}\) joining the black vertex \((y_{k-1})_1\) to the white vertex \((z_k)_1\). (Note that \(\{(y_i)_1, (z_j)_1\} = \{(y_i)_1\} if (y_i)_1 = (z_j)_1\). We set

\[
Q_i = \langle u, R'_i, y_i, (y_i)_1, T_i, (z_j)_1, z_i, H'_i, v \rangle \text{ for } 1 \leq i \leq k - 2,
\]

\[
Q_{k-1} = \langle u, R'_{k-1}, y_{k-1}, (y_{k-1})_1, T_{k-1}, (z_k)_1, z_k, H'_k, v \rangle,
\]

and

\[
Q_k = \langle u, R'_k, y_k, y, z, z_{k-1}, H'_{k-1}, v \rangle.
\]

It is easy to check that \([Q_1, Q_2, \ldots, Q_k]\) forms a \(k^*\)-container of \(S_n\) joining \(u\) to \(v\) not using the edge \((u, v)\) if \((u, v) \in E(S_n)\). \(\square\)

**Theorem 5.** \(S_n\) is \((n - 1)^*\)-laceable if \(n \geq 2\).

**Proof.** It is easy to see that \(S_2\) is \(1^*\)-laceable and \(S_3\) is \(2^*\)-laceable. Since the \(S_i\) is vertex transitive, we claim that \(S_3\) is \(3^*\)-laceable by listing all \(3^*\)-containers from the white vertex 1234 to any black vertex as follows:

\[
\begin{align*}
(1234, & (2134)) \\
(1234, & (2314), (4312), (3412), (2341), (3241), (2143), (3142), (4123), (4132), (2134)) \\
(1234, & (2341), (2413), (3421), (1243), (1342), (1423), (1432), (3214), (2142), (3124), (2314)) \\
(1234, & (2341), (2413), (2314), (4312), (3412), (2341), (3421), (1243), (1342), (1423), (1432), (3214), (2142), (3124)) \\
(1234, & (2341), (2413), (3241), (1243), (3142), (1423), (1432), (3214), (2142), (3124), (2314)) \\
(1234, & (2341), (2143), (4312), (3412), (2341), (3421), (1243), (1342), (1423), (1432), (3214), (2142), (3124)) \\
(1234, & (2341), (2143), (3241), (1243), (3142), (1423), (1432), (3214), (2142), (3124), (2314)) \\
(1234, & (2341), (2143), (3241), (1243), (3142), (1423), (1432), (3214), (2142), (3124), (2314)) \\
(1234, & (2341), (2143), (3241), (1243), (3142), (1423), (1432), (3214), (2142), (3124), (2314)) \\
(1234, & (2341), (2143), (3241), (1243), (3142), (1423), (1432), (3214), (2142), (3124), (2314)) \\
(1234, & (2341), (2143), (3241), (1243), (3142), (1423), (1432), (3214), (2142), (3124), (2314)) \\
(1234, & (2341), (2143), (3241), (1243), (3142), (1423), (1432), (3214), (2142), (3124), (2314)) \end{align*}
\]
Assume that $S_k$ is $(k - 1)^*$-laceable for every $4 \leq k \leq n - 1$. We need to construct an $(n - 1)^*$-container of $S_n$ between any white vertex $u$ to any black vertex $v$.

Case 1: $d(u, v) = 1$. We have $(u, v) \in E(S_n)$. By induction, $S_{n-1}$ is $(n-2)^*$-laceable. By Lemma 9, there exists a $(n-2)^*$-container $\{Q_1, Q_2, \ldots, Q_{n-2}\}$ of $S_n$ joining $u$ to $v$ not using the edge $(u, v)$. We set $Q_{n-1} = \langle u, v \rangle$. Then $\{Q_1, Q_2, \ldots, Q_{n-1}\}$ forms an $(n - 1)^*$-container of $S_n$ joining $u$ to $v$.

Case 2: $d(u, v) \geq 3$. We have star graph is edge transitive. Without loss of generality, we may assume that $u \in S_n^n$ and $v \in S_{n-1}^{n-1}$ with $(u)_1 \neq n - 1$ and $(v)_1 \neq n$. By Lemma 6, there are $\frac{(n-2)!}{(n-2)} \geq 3$ edges joining black vertices of $S_n^n$ to white vertices of $S_{n-1}^{n-1}$. We can choose an edge $(y, z) \in E^{n-1,n}$ where $y$ is a black vertex in $S_n^n$ and $z$ is a white vertex in $S_{n-1}^{n-1}$. Let $\{R_1, R_2, \ldots, R_{n-2}\}$ be an $(n-2)^*$-container of $S_n^n$ joining $u$ to $y$, and let $\{H_1, H_2, \ldots, H_{n-2}\}$ be an $(n-2)^*$-container of $S_{n-1}^{n-1}$ joining $z$ to $v$. We write $R_i = \langle u, R'_i, y_i, y \rangle$ and $H_i = \langle z_i, H'_i, v \rangle$. Note that $y_i$ is a white vertex and $z_i$ is a black vertex for every $1 \leq i \leq n - 2$. We have $\{(y_i)_1 | 1 \leq i \leq n - 2\} = \{(z_i)_1 | 1 \leq i \leq n - 2\} = (n - 2)$, Without loss of generality, we assume that $(y_i)_1 = (z_i)_1$ for every $1 \leq i \leq n - 2$ with $(y_{n-2})_1 = (u)_1$.

Subcase 2.1: $(u)_1 = (v)_1$. By Theorem 4, there is a hamiltonian path $T_i$ of $S_n^{(y_1)_1}$ joining the black vertex $(y_1)_n$ to the white vertex $(z_1)_n$ for every $i \in (n-3)$, and there is a hamiltonian path $H$ of $S_n^{(y_{n-2})_1}$ joining the black vertex $(u)_n$ to the white vertex $(v)_n$. We set
\[
Q_i = (u, R'_i, y_i, (y_i)_n, T_i, (z_i)_n, z_i, H'_i, v) \text{ for } 1 \leq i \leq n - 3, \\
Q_{n-1} = (u, R'_{n-2}, y_{n-2}, y, z, z_{n-2}, H'_{n-2}, v), \text{ and} \\
Q_{n-2} = (u, (u)_n, H, (v)_n, v, ).
\]
Then $\{Q_1, Q_2, \ldots, Q_{n-1}\}$ forms an $(n - 1)^*$-container of $S_n$ joining $u$ and $v$.

Subcase 2.2: $(u)_1 \neq (v)_1$. Without loss of generality, we assume that $(y_{n-3})_1 = (v)_1$. By Theorem 4, there is a hamiltonian path $T_i$ of $S_n^{(y_1)_1}$ joining $(y_1)_n$ to $(z_1)_n$ for every $i \in (n-4)$, there is a hamiltonian path $H$ of $S_n^{(y_{n-3})_1}$ joining the black vertex $(y_{n-3})_n$ to the white vertex $(v)_n$, and there is a hamiltonian path $P$ of $S_n^{(y_{n-2})_1}$ joining the black vertex $(u)_n$ to the white vertex $(z_{n-2})_n$. We set
\[
Q_i = (u, R'_i, y_i, (y_i)_n, T_i, (z_i)_n, z_i, H'_i, v) \text{ for } 1 \leq i \leq n - 4, \\
Q_{n-3} = (u, R'_{n-3}, y_{n-3}, (y_{n-3})_n, H, (v)_n, v, ), \\
Q_{n-2} = (u, (u)_n, P, (z_{n-2})_n, z_{n-2}, H'_{n-2}, v), \text{ and} \\
Q_{n-1} = (u, R'_{n-2}, y_{n-2}, y, z, z_{n-3}, H'_{n-3}, v).
\]
It is easy to check that $\{Q_1, Q_2, \ldots, Q_{n-1}\}$ is an $(n - 1)^*$-container of $S_n$ joining $u$ to $v$.

Thus, this theorem is proved. 

\textbf{Theorem 6.} $S_n$ is super laceable if and only if $n \neq 3$.

\textbf{Proof.} It is easy to see that this theorem is true for $S_1$ and $S_2$. Since $S_3$ is isomorphic to a cycle with six vertices, $S_3$ is not $1^*$-laceable. Thus, $S_3$ is not super laceable. By Theorems 4 and 5, this theorem holds on $S_4$. Assume that $S_k$ is super laceable for every $4 \leq k \leq n - 1$. By Theorems 4 and 5, $S_n$ is $k^*$-laceable for any $k \in \{1, 2, n - 1\}$. Thus, we still need to
construct a $k^*$-container of $S_n$ between any white vertex $u$ and any black vertex $v$ for every $3 \leq k \leq n - 2$. By induction, $S_{n-1}$ is $k^*$-laceable. By Lemma 9, there is a $k^*$-container of $S_n$ joining $u$ to $v$. □

6. Further study

In this paper, we prove that the pancake graph $P_n$ is super connected for $n \neq 3$ and the star graphs $S_n$ is super laceable for $n \neq 3$. We believe that there are other super connected and super laceable graphs. It would be very interesting to classify such graphs.

We may also study the fault tolerant $k^*$-connectivity for any super connected graph. For example, let $F \subseteq V(P_n) \cup E(P_n)$ with $|F| = f \leq n - 3$. Obviously, $P_n - F$ is $(n - 1 - f)^*$-connected. Similarly, we can study the fault tolerant $k^*$-laceability for any super laceable graph. For example, let $F \subseteq E(S_n)$ with $|F| = f \leq n - 3$. Obviously, $S_n - F$ is $(n - 1 - f)^*$-connected. However, we believe that $S_n - F$ is $(n - 1 - f)^*$-connected.

Assume that $G$ is $k^*$-connected. We may also define the $k^*$-connected distance between any two vertices $u$ and $v$, denoted by $d_k^*(u, v)$, which is the minimum length among all $k^*$-containers between $u$ and $v$. The $k^*$-diameter of $G$, denoted by $D_k^*(G)$, is $\max\{d_k^*(u, v) | u$ and $v$ are two different vertices of $G\}$. In particular, we are intrigued in $D_{k^*(G)}(G)$ and $D_2^*(G)$. Similarly, we define the $k^*L$-laceable distance on bipartite graph between any two vertices $u$ and $v$ from different partite sets, denoted by $d_k^{*L}(u, v)$, which is the minimum length among all $k^*$-containers between $u$ and $v$. The $k^*L$-diameter of $G$, denoted by $D_k^{*L}(G)$, is $\max\{d_k^{*L}(u, v) | u$ and $v$ are vertices from different partite sets}. Again, we are intrigued in $D_{k^*(G)}^{*L}(G)$ and $D_2^{*L}(G)$.

References